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SUM RULE APPROACH TO GIANT RESONANCE STATES

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Abstract - The structure of giant resonance states are studied with the aid of sum rules. The sum rule for the nuclear current yields an irrotational velocity field for the doorway states. The relationship between zero-range forces and separable forces in the random phase approximation is derived using a sum rule for transition densities. Tomanaga theory provides us with the reason why zero-range forces and the Q-Q force predict the same excitation energy for the quadrupole mode. This is because both the effective forces do not yield the restoring force. The restoring force coming from the increasing of the kinetic energy well reproduces the mass number dependence of the experimental values of the excitation energy. Tomanaga theory explains also the narrow damping width of the collective quadrupole state as a result that its collective momentum commutes with zero-range forces.

I. INTRODUCTION

Since the strength for excitation of giant resonance states exhausts most of the sum rule value for photoexcitation /1/, characteristic features of the resonance states would be considered to appear as a result of the related model-independent sum rules. In this paper we would like to discuss the structure of giant resonance states on the basis of sum rules. In the next section we shall derive the sum rules for the nuclear four-current. Using those sum rules, the fluid dynamical structure of giant resonance states will be investigated in section III. It will be shown that an irrotational flow model is applicable to the doorway state of giant resonance states. In section IV we shall show the relationship between zero-range forces and separable forces which are known to explain well the systematics of excitation energies. In section V we shall discuss in more details results of sections, III and IV. With the aid of Tomanaga theory for collective motions, we will answer the following questions for the quadrupole mode. First, how do we obtain the mass-number dependence of $A^{-1/3}$ in an irrotational and incompressible flow model? Second, why do zero-range forces like

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Skyrme force and the Q-Q force provide a similar excitation energy, despite of their different dependences on the nuclear coordinates? Last, why is the observed damping width narrow?

II. SUM RULES

For the past ten years, various kinds of sum rules for the nuclear four-current were derived by many authors /2-4/. We would like to summarize briefly those sum rules which will be employed in following sections. In a point particle model, the space- and the time-component of the nuclear four-current are written, respectively, as:

\[ \hat{J}(\vec{r}) = \frac{1}{2m} \sum_{k=1}^{A} \{ \vec{p}_k, \delta(\vec{r} - \vec{r}_k) \}, \]

\[ \hat{\beta}(\vec{r}) = \sum_{k=1}^{A} \delta(\vec{r} - \vec{r}_k). \]

For the nuclear four-current, the following sum rule holds /4/:

\[ \sum_{n} <0 | \hat{J}(\vec{r})|n><n|\hat{\beta}(\vec{r'})|0> = -\frac{i}{2m} \rho_0(r) \nabla \cdot \delta(\vec{r} - \vec{r'}), \]

where the subscript \( \vec{r} \) of \( \nabla \) indicates the differentiation with respect to \( \vec{r} \) and \( \rho_0(r) \) stands for the ground state density,

\[ \rho_0(r) = <0|\hat{\beta}(\vec{r})|0>. \]

The above sum rule is easily derived by calculating a commutator

\[ \sum_{n} <0 | \hat{J}(\vec{r})|n><n|\hat{\beta}(\vec{r'})|0> = \frac{1}{2} <0 | [\hat{J}(\vec{r}),\hat{\beta}(\vec{r'})]|0>. \]

When we use the identity for an arbitrary function, \( f(\vec{r}) \),

\[ \int \beta(\vec{r}) f(\vec{r}) d\vec{r} = \sum_{k=1}^{A} f(\vec{r}_k), \]

eq.(3) gives the sum rule for the transition currents as /3/,

\[ \sum_{n} <0 | \hat{J}(\vec{r})|n><n|\sum_{k=1}^{A} f(\vec{r}_k)|0> = -\frac{i}{2m} \rho_0(r) \nabla \cdot \sum_{k=1}^{A} f(\vec{r}_k). \]

The nuclear four-current satisfies the continuity equation:

\[ \nabla \cdot \hat{J}(\vec{r}) = i [ \hat{\beta}(\vec{r}), \hat{H} ] \]

in assuming that the density operator commutes with the nuclear forces. When we apply the continuity equation to the last sum rule, eq.(7), we obtain the following sum rule for the transition densities,

\[ \sum_{n} \omega_n <0|\hat{\beta}(\vec{r})|n><n|\sum_{k=1}^{A} f(\vec{r}_k)|0> = -\frac{i}{2m} \nabla \cdot \sum_{k=1}^{A} \rho_0(r) \nabla \cdot \sum_{k=1}^{A} f(\vec{r}_k). \]
where $\omega_n$ denotes the excitation energy of the nuclear state, $|n\rangle$. This sum rule has been derived by Fallieros, Noble, Ui et al./2/. If we utilize the identity, eq.(6), in the above sum rule, we obtain $f$-sum rule which is well known in a wide range of physics /5/.

$$\sum_n \omega_n \left| \sum_{k=1}^A f(\vec{r}_k) \right|^2 = S_f$$

(10)

where $S_f$ is usually called the sum rule value,

$$S_f = \frac{1}{2m} \int \rho_o(r) \nabla f(\vec{r}) \cdot \nabla f(\vec{r}) \, d\vec{r}.$$  

(11)

Thus the above sum rules are derived rather model-independently.

Now let us assume in the last three sum rules, eqs.(7), (9) and (10) that a single collective state, $|\omega_f\rangle$, only can be excited by

$$\sum_{k=1}^A f(\vec{r}_k).$$

Then, those are written as follows,

$$\langle 0 | \hat{J}(\vec{r}) | \omega_f \rangle |\omega_f \rangle = - \frac{i}{2m} \rho_o(r) \nabla f(\vec{r}).$$

(12)

$$\omega_f \langle 0 | \hat{\beta}(\vec{r}) | \omega_f \rangle |\omega_f \rangle = - \frac{1}{2m} \nabla \cdot (\rho_o(r) \nabla f(\vec{r})).$$

(13)

$$\omega_f |\omega_f \rangle |\omega_f \rangle ^2 = S_f,$$  

(14)

$\omega_f$ being the excitation energy of $|\omega_f\rangle$. Therefore, the transition current and density for the collective state, $|\omega_f\rangle$, are uniquely obtained as /2-4/

$$\langle 0 | \hat{J}(\vec{r}) | \omega_f \rangle = - \frac{i}{2m} \frac{\omega_f}{S_f} \sum_{k=1}^A f(\vec{r}_k) \rho_o(r) \nabla f(\vec{r}).$$

(15)

$$\langle 0 | \hat{\beta}(\vec{r}) | \omega_f \rangle = - \frac{1}{2m} \frac{\omega_f}{S_f} \sum_{k=1}^A f(\vec{r}_k) \nabla \cdot (\rho_o(r) \nabla f(\vec{r})).$$

(16)

III. FLUID DYNAMICAL STRUCTURE

Using the above result for the transition current, we explore the velocity field of the collective state, $|\omega_f\rangle$. The velocity field denoted by $\nabla \cdot f(\vec{r})$ is defined on the analogy of the classical model

$$\langle 0 | \hat{\beta}(\vec{r}) | \omega_f \rangle = \rho_o(r) \nabla \cdot f(\vec{r}).$$

(17)

Then, from the definition and eq.(15), the velocity field for the collective state, $|\omega_f\rangle$, is obtained as
Owing to the identity $\mathbf{\nabla} \times \mathbf{\nabla} f(\mathbf{r}) = 0$, this velocity field satisfies

$$\mathbf{\nabla} \times \mathbf{\nabla} f(\mathbf{r}) = 0. \quad (19)$$

Hence, we can conclude that the velocity field of $|\omega_f\rangle$ whose strength exhausts the sum rule value for $f$ is irrotational.

Now let us discuss the velocity field of giant resonance states. It is known experimentally that most of the strength for photoexcitation is exhausted by the giant resonance states. According to this fact, we assume the doorway state, $|\psi_X\rangle$, of the giant $\lambda$-pole resonance state to exhaust the sum rule value for $r^\lambda Y_{\lambda\mu}(\mathbf{r}) \ (\lambda \geq 1)$. Then the velocity field, $\mathbf{\nabla}_X(f)$, of the doorway state is given by setting $f(\mathbf{r}) = r^\lambda Y_{\lambda\mu}(\mathbf{r})$ in eq. (18),

$$\mathbf{\nabla}_X(f(\mathbf{r})) \sim \mathbf{\nabla} r^\lambda Y_{\lambda\mu}(\mathbf{r}) \quad (\lambda \geq 1). \quad (20)$$

One can find that the above velocity field satisfies

$$\mathbf{\nabla} \cdot \mathbf{\nabla}_X(\mathbf{r}) = 0, \quad (21)$$

as a result of the identity, $\nabla^2 r^\lambda Y_{\lambda\mu}(\mathbf{r}) = 0 \ (\lambda \geq 1)$. Eq. (21) implies in fluid dynamical terms that this velocity field is incompressible.

It is well known that the irrotational and incompressible flow has been assumed in Bohr's liquid drop model /6/. According to the above discussions, we can say that his assumption is applicable to the doorway state of giant resonances rather than to the low lying collective states /3/.

Recently, detailed investigation /3,7/ on the velocity field has been done within the framework of RPA. It has been shown in those microscopic calculations that high lying collective states whose strength exhausts most of the sum rule value are well described as an irrotational and incompressible mode. This result is simply understood from a sum rule point of view as mentioned in the above.

For the doorway state of the giant monopole resonance state, the function $f$ is chosen to be $f = r^2$, because this state is known experimentally to exhaust most of the sum rule value for $f = r^2$. Then eq. (18) gives the velocity field, $\mathbf{\nabla}_0(\mathbf{r})$, for the monopole state,

$$\mathbf{\nabla}_0(\mathbf{r}) \sim \mathbf{\nabla} r^2. \quad (22)$$

This is the same velocity field as that of an irrotational but compressible flow assumed in Wernz-Uberall's liquid model /8/.

IV. ZERO-RANGE FORCES AND SEPARABLE FORCES

Next we would like to discuss effective nuclear forces used in RPA for the giant resonance states. For the past several years, on one hand, it was shown that zero-range forces like Skyrme force are very useful for explaining the excitation energy of the giant resonance states /9/. On the other hand, separable forces like the Q-Q force are also known to work well for estimate of the excitation energies /10,11/. Although the two effective forces have a different dependence on the nuclear coordinates, there must be some relationship between them. In this section, we would like to explore a relationship between the two forces in RPA. In the next section, we will discuss in more detail the reason why we obtain a similar excitation energy from them.
Let us take the following Skyrme force as an example of zero-range forces \cite{12},
\[
V = t_0 \sum_{i>j} (1 + \chi_0 \ p^0) \delta(\vec{r}_i - \vec{r}_j) + t_3 \sum_{i>j<k} \delta(\vec{r}_i - \vec{r}_j) \delta(\vec{r}_j - \vec{r}_k) \tag{23}
\]
where \( p^0 \) is a spin-exchange operator and \( t_0, \chi_0 \) and \( t_3 \) are constant. After a little manipulation, one can describe the RPA dispersion equation for spin- and isospin-independent modes as \cite{13}
\[
\frac{\varepsilon_{mi}}{1} = 2 \sum_{m} K_n \frac{\omega_n^2 - \varepsilon_{mi}^2}{\omega_n^2 - \varepsilon_{mi}^2}
\tag{24}
\]
where \( m \) and \( i \) denote the particle and the hole state, respectively and \( \varepsilon_{mi} \) and \( \omega_n \) stand for the unperturbed and the perturbed energy. Moreover we have defined \( K_n \) and \( G_n(\vec{r}) \) by
\[
K_n^{-1} = \int \left\{ \frac{3}{4} t_0 + \frac{3}{8} t_3 \rho_0(\vec{r}) \right\} g_n^*(\vec{r}) g_n(\vec{r}) \ d\vec{r}
\tag{25}
\]
\[
G_n(\vec{r}) = \left\{ \frac{3}{4} t_0 + \frac{3}{8} t_3 \rho_0(\vec{r}) \right\} g_n(\vec{r}) \tag{26}
\]
with the transition density,
\[
g_n(\vec{r}) = \langle 0 | \beta(\vec{r}) | \omega_n \rangle. \tag{27}
\]

Once we write the dispersion equation as in eq.(24), we can see that the zero-range force is equivalent to the separable force of the Hamiltonian
\[
H = T + V_{H,F}(\vec{r}) + \frac{1}{2} K_n \left( \sum_{m} G_n^*(\vec{r}_i) \sum_{j=1} G_n(\vec{r}_j) \right), \tag{28}
\]
where \( T \) is the kinetic part and \( V_{H,F}(\vec{r}) \) represents the Hartree-Fock Hamiltonian obtained from the Skyrme force,
\[
V_{H,F}(\vec{r}) = \left\{ \frac{3}{4} t_0 + \frac{3}{16} t_3 \rho_0(\vec{r}) \right\} \rho_0(\vec{r}). \tag{29}
\]
Thus, the zero-range force in RPA can be expressed as a separable force. It should be noted, however, that the separable force depends on the transition density for the perturbed state, \( |\omega_n\rangle \), so that eq.(28) has the meaning for the state, \( |\omega_n\rangle \), only.

Suppose that one of the perturbed states, \( |\omega_\lambda\rangle \), exhausts the sum rule value for photoexcitation, that is, for \( f(\vec{r}) = r^\lambda Y_{\lambda\mu}(\vec{r}) \). Then, according to the sum rule result, eq.(16), the transition density for the collective state is written as
\[
g_\lambda(\vec{r}) = \langle 0 | \beta(\vec{r}) | \omega_\lambda \rangle = C_\lambda \ r^{\lambda-1} \frac{d\rho_0(\vec{r})}{dr} Y_{\lambda\mu}(\vec{r}), \tag{30}
\]
\( C_\lambda \) being constant. Inserting this into eqs.(25) and (26) and using the Hartree-Fock field in eq.(29), therefore, we obtain the relationship between the Hartree-Fock field and the separable force for the collective state \cite{13},
\[
K_\lambda^{-1} = C_\lambda^2 \int r^{\lambda-1} \frac{dV_{H,F}(\vec{r})}{dr} Y_{\lambda\mu}^*(\vec{r}) \ r^{\lambda-1} \frac{d\rho_0(\vec{r})}{dr} Y_{\lambda\mu}(\vec{r}) \ d\vec{r}, \tag{31}
\]
This relationship has been obtained from eq. (23), but is derived also from more general density-dependent zero-range forces.

Now let us impose the above relationship on a harmonic oscillator potential model. By replacing \( V_{HF}(r) \) in eqs. (31) and (32) by a harmonic oscillator potential we obtain the following separable force,

\[
H = T + \frac{\hbar \omega}{2} \sum_{\lambda} r^\lambda - \frac{1}{2} \frac{\hbar \omega}{2} \sum_{\lambda} Y^\lambda_{\lambda \mu} \sum_{i \neq j} Y^\lambda_{\lambda \mu} (\hat{r}) ,
\]

where \( S_\lambda \) denotes the sum rule value for \( f(\hat{r}) = r^\lambda Y^\lambda_{\lambda \mu}(\hat{r}) \),

\[
S_\lambda = \frac{(2\lambda + 1)}{8\pi m} A \langle r^{2\lambda - 2} \rangle
\]

with \( A \langle r^{2\lambda - 2} \rangle = \int r^{2\lambda - 2} \rho_0(r) \, dr \). This is nothing but the separable force obtained by Bohr and Mottelson et al./11,14/. Thus, the harmonic oscillator potential and Bohr-Mottelson's separable force satisfy eqs. (31) and (32) obtained from the zero-range force in a sum rule limit. As will be shown later, this relationship, eqs. (31) and (32), between the Hartree-Fock field and the separable force plays an essential role in estimating the excitation energy of the collective state, rather than the fact that a harmonic oscillator potential is used as \( V_{HF}(r) \).

Before going to the next section it would be worthwhile noting the following comment. Bohr and Mottelson et al./11,14/ have derived the separable force, eq. (33), on the basis of two assumptions: volume conservation, and the self-consistency between deformations of the density and the harmonic oscillator potential. The former is related to the assumption of the sum rule limit in the present model, as seen from the arguments in the previous section, while the latter is the assumption assured in zero-range forces.

V. FURTHER DISCUSSIONS

For the arguments in the last two sections, it would be natural to ask the following three questions. First, we have concluded that the doorway state of the giant resonance states is described in terms of irrotational and incompressible flow. Bohr's liquid drop model assuming this velocity field, however, predicts the excitation energy of the quadrupole mode to have \( A^{-1/2} \) -dependence /11/, whereas experimental values show rather \( A^{-1/3} \) dependence /1/. Can we solve this discrepancy? Second, we have shown some relationship between zero-range forces and Bohr-Mottelson's separable force. It is not enough, however, to answer the question why the both effective interactions provide a similar excitation energy, despite of their different dependences on the nuclear coordinates. Last, all arguments in the previous sections are based on the assumption that the sum rule value for photoexcitation is exhausted by a single collective state. Hence, we should ask how the damping of this state is or if the coupling of this state with other degrees of freedom is weak. In order to answer qualitatively these questions, let us discuss in more detail the doorway state, in particular, of the quadrupole mode according to Tomonaga theory /15/ for collective motions.

V.1 Tomonaga theory

In his last paper /15/ published in 1955, Tomonaga proposed a quantum mechanical theory of collective motions. He applied his theory to an irrotational and incompressible nuclear vibration and the plasma oscillation. In particular he showed that the frequency of the plasmon is well reproduced and that it is qualitatively understood why the plasmon appears as a sharp resonance state. In this subsection, let us briefly review the framework of his theory.
In short, when we find a collective coordinate and a momentum, $\xi$ and $\pi$, satisfying the canonical commutation relation, $[\xi, \pi] = i$, Tomonaga theory tells us how one can separate the total Hamiltonian:

$$H = T + V$$

(35)

into three parts:

$$H = H_{\text{coll.}} + H_{\text{coup.}} + H_{\text{int}}$$

(36)

Here, $H_{\text{coll.}}$ and $H_{\text{int.}}$ stand for the Hamiltonians for the collective motion and for the other degrees of freedom, respectively, while $H_{\text{coup.}}$ denotes the coupling term between them and induces the damping of the collective state.

First the kinetic part of the collective Hamiltonian, $T_C$, is supposed to come from the kinetic part of the total Hamiltonian,

$$T = t + T_C$$

(37)

where $t$ and $T_C$ is given by

$$T_C = \frac{\pi^2}{2I}, \quad t = T - \frac{\pi^2}{2I},$$

(38)

$I$ being the momentum inertia*. Since we are interested in up to the quadratic term of $\pi$, $H - T_C$ is required to commute with $\xi$:

$$[t + V, \xi] = [t, \xi] = 0.$$  

(39)

This requirement will provide the value of the momentum inertia for the collective motion.

Next, $V$ and $t$ are expanded in terms of $\xi$. For example, $V$ is written as

$$V = V_0 + \xi V_1 + \frac{1}{2} \xi^2 V_2 + \ldots,$$

(40)

where the coefficients are given by

$$V_0 = \langle V \rangle_{\xi=0},$$

(41)

$$V_1 = \langle \frac{\partial V}{\partial \xi} \rangle_{\xi=0} = \langle i [\pi, V] \rangle_{\xi=0},$$

(42)

$$V_2 = \langle \frac{\partial^2 V}{\partial \xi^2} \rangle_{\xi=0} = \langle - [\pi, [\pi, V]] \rangle_{\xi=0}.$$

(43)

In the above equation, we need the expression of $V$, $\partial V / \partial \xi$ and $\partial^2 V / \partial \xi^2$ at $\xi=0$. Up to second order of $\xi$, these coefficients can be written as

$$V_0 = V - i \xi [\pi, V] - \frac{1}{2} \xi^2 [\pi, [\pi, V]],$$

(44)

$$V_1 = i [\pi, V] + \xi [\pi, [\pi, V]],$$

(45)

$$V_2 = - [\pi, [\pi, V]].$$

(46)

Thus if we have expressions of $\xi$ and $\pi$ in terms of the nuclear coordinates and momenta, these coefficients are also described in terms of the nuclear coordinates and momenta.

*For isovector modes or $\xi$ being momentum dependent, $T_C$ stems from $T + V$ rather than $T$ only, because of $[V, \xi] \neq 0$. In the following discussions, we will assume $[V, \xi]=0$. 

In the same way as for the interaction $V$, we can expand the kinetic part, $t$, where the coefficients are given by

$$t = T_0 + \xi T_1 + \frac{1}{2} \xi^2 T_2,$$

where

$$T_0 = -it \xi [\pi, \xi] - \frac{1}{2} \xi^2 [\pi, [\pi, \xi]],$$

$$T_1 = i [\pi, \xi] + \xi [\pi, [\pi, \xi]],$$

$$T_2 = - [\pi, [\pi, \xi]].$$

Finally, we obtain the expression of the Hamiltonian as

$$H = \frac{\pi^2}{2 I} + \frac{1}{2} \xi^2 (T_2 + V_2) + \xi (T_1 + V_1) + (T_0 + V_0).$$

The first two terms provide us with the Hamiltonian for the collective motion, while the last term describes the other degrees of freedom,

$$H_{\text{coll.}} = \frac{\pi^2}{2 I} + \frac{1}{2} \xi^2 (T_2 + V_2),$$

$$H_{\text{int.}} = T_0 + V_0.$$

The third term in eq.(51) is interpreted as the coupling Hamiltonian between the collective mode and others,

$$H_{\text{coup.}} = \xi (T_1 + V_1).$$

All the terms are expressed self-consistently using the original force and the kinetic energy. The excitation energy of the collective state is given by $\{(T_2 + V_2)/I\}^{1/2}$.

V.2 Quadrupole excitation

a) Collective variables

Now let us apply Tomonaga theory to the giant quadrupole state. First we must find the collective variables to describe the doorway state. In discussing the collective state to exhaust the sum rule value for photoexcitations, we can take the collective coordinate to be

$$\xi = C \sum_{k=1}^{A} (2 z_k^2 - x_k^2 - y_k^2) = :C Q_0:,$$

where $C$ being constant. The canonical momentum would be chosen as

$$\pi = \sum_{k=1}^{A} (2 z_k p_{zk} - x_k p_{xk} - y_k p_{yk}),$$

because, as mentioned in section III, the velocity field of the collective state is irrotational and incompressible and this $\pi$ induces such a motion. When we calculate the commutator between $\xi$ and $\pi$, we have

$$[\xi, \pi] = i C \sum_{k=1}^{A} \{4 r_k^2 + 2(2 z_k^2 - x_k^2 - y_k^2)\} = i \hat{\theta}.$$
Unfortunately, this does not have a canonical form. However, if we replace r.h.s. by its ground state expectation value, \( \langle 0 | \Delta | 0 \rangle \), and choose \( C \) to be \( \frac{1}{4} A \langle r^2 \rangle \), we can obtain the canonical commutation relation,

\[ [\xi, \pi] = i. \]  

(58)

The order of this approximation is estimated by calculating the mean deviation, \( \langle 0 | \Delta^2 | 0 \rangle - \langle 0 | \Delta | 0 \rangle^2 \). In using a harmonic oscillator potential model, we find

\[ \frac{(\langle 0 | \Delta^2 | 0 \rangle - \langle 0 | \Delta | 0 \rangle^2)^{1/2}}{\langle 0 | \Delta | 0 \rangle} \approx \frac{3}{2} \frac{1.3}{A^{2/3}}, \]  

(59)

which is small enough to be neglected for heavy nuclei. Therefore, let us take the above \( \xi \) and \( \pi \) as the collective variables for the quadrupole mode.

For latter discussions, it is worthwhile noting that \( Q_0 \) and \( \pi \) satisfy the relationship:

\[ \pi = i \frac{m}{2} [\pi, Q_0]. \]  

(60)

eq(60) and the canonical commutation relation between \( \xi \) and \( \pi \) yields

\[ i = i \frac{m}{2} C \langle 0 | [Q_0, [\pi, Q_0]] | 0 \rangle = i m C S(1), \]  

(61)

where \( S(1) \) is written as

\[ S(1) = \frac{1}{2} \langle 0 | [Q_0, [\pi, Q_0]] | 0 \rangle \]

\[ = \sum_n \omega_n \langle n | [Q_0, \pi] | n \rangle^2 = \frac{4}{\pi} \frac{A \langle r^2 \rangle}{m}. \]  

(62)

Thus, the constant \( C \) in \( \xi = C Q_0 \) is related to the linearly energy-weighted sum rule value, \( S(1) \), for \( Q_0 \),

\[ C = \frac{1}{m S(1)}. \]  

(63)

b) The momentum inertia

The momentum inertia is obtained by requiring eq.(39). Inserting the expressions of \( \xi \) and \( \pi \) into eq.(39), we obtain

\[ I = \frac{1}{2} m^2 S(1) = 2 mA \langle r^2 \rangle. \]  

(64)

This is nothing but the well-known momentum inertia obtained in the irrotational and incompressible liquid drop model /11/.

c) Excitation energy

Let us estimate the excitation energy of the collective state by using two kinds of effective nuclear forces mentioned before : zero-range forces and the Q.Q force.

Zero-range force : the restoring force, \( T_2 + V_2 \), is obtained by calculating the double commutators in eqs.(46) and (50). As shown by Martorell et al./16/, zero-range forces commute with \( \pi \),

\[ [\pi, V] = 0. \]  

(65)
Therefore, the restoring force of the quadrupole mode comes from the kinetic part of
the Hamiltonian,

\[ T_2 = \frac{2}{m} \sum_k \{ 2 p_k^2 + (2 p_{2k}^2 - p_{xk}^2 - p_{yk}^2) \}. \]  

(66)

This is not constant, but r.h.s. is replaced by its expectation value of the ground state, in neglecting the mean deviation of order \( \lambda^{-2/3} \),

\[ T_2 = \langle 0 | T_2 | 0 \rangle = 8 \langle 0 | T | 0 \rangle, \]

(67)

\( \langle 0 | T | 0 \rangle \) being the total kinetic energy of the system. Therefore, from eqs. (64) and (67) we obtain the excitation energy of the quadrupole state as /16,17/

\[ \omega_2 = \sqrt{\frac{4 \langle 0 | T | 0 \rangle}{mA^2}}. \]

(68)

Q-Q force: let us write the nuclear interaction, \( V \), as

\[ V = V^{(1)} + V^{(2)}, \]

(69)

where \( V^{(1)} \) represents a harmonic oscillator potential, while \( V^{(2)} \) the Q-Q force given in eq. (33) for \( \lambda = 2 \). When we calculate the double commutator between \( V \) and \( \pi \), we have

\[ - [\pi, [\pi, V^{(1)}]] = 4 \omega^2 A \langle r^2 \rangle, \]

(70)

\[ - [\pi, [\pi, V^{(2)}]] = -4 \omega^2 A \langle r^2 \rangle, \]

(71)

neglecting the mean deviation of order \( \lambda^{-2/3} \) in both the right hand sides. The above two equations provide us with

\[ [\pi, [\pi, V]] = 0, \]

(72)

just as in the case of zero-range forces. Thus, if the strength of the Q-Q force is determined so as to satisfy its relationship to the harmonic oscillator potential, eqs. (31) and (32), we obtain the cancellation between the restoring forces from the Hartree-Fock potential and from the Q-Q force. Therefore, the restoring force in the present model stems from the kinetic energy as in the model with zero-range forces, and yields the same excitation energy for the quadrupole mode as in eq. (68).

According to the above discussions on the excitation energy, we are ready to answer the first two questions mentioned at the beginning of this section. First, the mass-number dependence of the excitation energy for the quadrupole mode is easily seen from eq. (68). The restoring force is proportional to the total kinetic energy, that is, to \( A \), while the momentum inertia to \( A^{2/3} \). As a result we obtain the excitation energy proportional to \( A^{-1/3} \). If we estimate the kinetic energy by a harmonic oscillator potential model with the oscillator parameter, \( \omega \), the virial theorem provides us with \( \langle 0 | T | 0 \rangle = \omega^2 A \langle r^2 \rangle / 2. \) Inserting this value into eq. (68) and taking the value \( \omega = 41/A^{1/3} \) (MeV) determined from the mean square radius of nuclei /11/, we obtain the excitation energy /10,11/,

\[ \omega_2 = \sqrt{2} \omega = 58/A^{1/3} \text{ (MeV)}, \]

(73)

which is in good agreement with experimental value \( \sim 65/A^{1/3} \) (MeV) /1/.

Since the momentum inertia of the present model is the same as that of Bohr's liquid drop model, the difference between the mass-number dependences of the two models is due to their restoring forces. The present arguments provide us with the restoring force coming from the increase of the kinetic energy due to the deformation of the
In contrast, the restoring force in the liquid drop model is assumed to stem mainly from the surface tension. This yields a $A^{2/3}$-dependence of the restoring force, so that one has $A^{-1/3}$-dependence of the excitation energy in the liquid drop model. Thus, if we take into account the contribution from the increasing of the kinetic energy to the restoring force, we can reproduce the systematics of the experimental values using the irrotational and incompressible flow model.

The second question was why zero-range forces and the Q.Q force yield a similar excitation energy. The answer is provided in eqs. (65) and (72). The both effective forces do not produce the restoring force. It should be noted that, in fact, eq. (72) is not peculiar to the model with a harmonic oscillator potential and the Q.Q force. For any Hartree-Fock field and separable force which satisfy the relationship, eqs. (31) and (32), there occurs such a cancellation between restoring forces from the Hartree-Fock potential and from the residual interaction. Thus, in order to obtain the excitation energy, eq. (58), it is not necessary that the Hartree-Fock field and the separable force show a nuclear-coordinate dependence similar to that from the zero-range forces. This cancellation is probably understood as follows. When the nucleus is deformed, the overlap between the nuclear density and the spherical Hartree-Fock field becomes bad. As a result, the energy of the system increases and we have a restoring force coming from the Hartree-Fock potential /4/. The separable force which satisfies eqs. (31) and (32), however, restores the self-consistency between the density and the potential. Indeed, the separable forces by Bohr and Mottelson et al. have been derived with requiring the self-consistency /11/. After the self-consistency is restored, therefore, we should have $V_2 = 0$.

Before discussing the damping term to answer the third question, we would like to note one comment on the relationship between the excitation energy obtained in the above and the energy moments of the response function. As noted in b), the momentum inertia is proportional to the first moment for $Q_0$, that is, the linearly energy-weighted sum rule value, $S^{(1)}$. Using eq. (60) and $[V, \mathcal{Q}] = 0$, the restoring force, $T_2 + V_2$, is described as

$$T_2 + V_2 = -\left[H, [\mathcal{Q}, H]\right] = \frac{m^2}{4} \left[H_0, Q_0\right],$$

so that its ground state expectation value is proportional to the third moment, $S^{(3)}$,

$$\langle 0 | T_2 + V_2 | 0 \rangle = \frac{m^2}{2} S^{(3)}$$

where

$$S^{(3)} = \sum_n \omega_n^3 |\langle n | Q_0 | 0 \rangle|^2.$$  

Hence, the excitation energy obtained in eq. (68) is the ratio of $S^{(3)}$ to $S^{(1)}$,

$$\omega_2 = \sqrt{S^{(3)}/S^{(1)}}.$$

The fact that $(S^{(3)}/S^{(1)})^{1/2}$ yields eq. (68) has been already shown by Martorell et al. /16,18/ in another point of view. In particular, for the meaning of this ratio in RPA, we should refer to their work.

d) Coupling Hamiltonian

In order to say that the collective state mentioned above is well defined and observed in nuclei, we should explore its damping. If the coupling of the collective state with other intrinsic states were strong, the collective state would be destroyed so that one could not observe it as a sharp resonance state.

In Tomonaga theory the coupling term is provided by eq. (54) which is self-consistent with $H_{coll}$ and $H_{int}$. In the approximation neglecting the mean deviation of order
A \sim 2/3$, the coupling term coming from the kinetic part is written as

$$T_1 = - \frac{1}{m} \sum_{k=1}^{A} (2 p_{z_k}^2 - p_{x_k}^2 - p_{y_k}^2) - \frac{2}{A \langle r^2 \rangle} \sum_{k=1}^{A} (2 z_k^2 - x_k^2 - y_k^2). \quad (78)$$

This is a one-body operator and causes the coupling of the collective state with the intrinsic 1 particle-1 hole states, that is, Landau damping. When we assume the intrinsic Hartree-Fock field to be a harmonic oscillator potential, eq.(78) is written as

$$T_1 = - \mu \omega^2 \sum_{k=1}^{A} \left\{ (2 z_k^2 - x_k^2 - y_k^2) + b^* (2 p_{z_k}^2 - p_{x_k}^2 - p_{y_k}^2) \right\} \quad (79)$$

with using the abbreviation $b^2 = 1/m\omega$. This implies that $T_1$ can excite the only 1-particle-1 hole states within the same shell. Since the collective state considered are mainly composed of $2\hbar \omega$-excitations, $T_1$ seems not to be important for the damping of the high lying state. Of course, a part of the strength for $Q_0$ is absorbed by the low lying states through $T_1 \xi$.

Contribution of nuclear forces to the coupling term may be generally complicated. In assuming zero-range nuclear forces, however, we can obtain simply an interesting result. As mentioned before (16), the collective momentum for the quadrupole mode, $\pi$, commutes with zero-range forces, $[\pi, V] = 0$. Therefore, zero-range forces do not provide any coupling term, $V_1 = 0$. From this fact, we can expect that the collective state is well defined as the doorway state of the giant quadrupole resonance. The observed damping width of $3 \sim 4$ MeV (1/ would be mainly due to $\xi$, the finite-range part of nuclear interactions and higher order terms neglected in the present discussions.

The present results on the damping term are restricted to the quadrupole mode. For other modes, the collective momentum, $\pi$, does not commute even with zero-range forces. Therefore, it is not easy to give a qualitative answer to the question whether or not their damping widths are narrow. Since Tomonaga theory provides us with the coupling term self-consistently, however, it is interesting to explore in detail the damping of other collective modes, in particular of the monopole mode. This will be discussed elsewhere.

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