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DELAY EFFECT IN LASER AMPLIFIERS

F. Hopf
Optical Science Center, University of Arizona, U.S.A.

Résumé :

Abstract:
In the Stanford Free Electron Laser it is necessary to desynchronize the electron pulses from the accelerator with the light pulses. This effect is a major complication in the data analysis of the FEL. In this presentation, the desynchronism effect is considered for a Laser amplifier. The equations are substantially simpler than the FEL, and most of the results can be obtained analytically. Both the gain (linear) and saturation (nonlinear) regimes are investigated.

BASIC MODEL

Excited two level atoms are injected transversely into a channel in which the light builds up. The gain bandwidth of such an amplifier is caused by the transit time of the atoms, which is the same broadening mechanism as in the FEL in the "cold beam" limit.

Fig. 1. Light propagating down a channel is amplified by scanned atomic beam. The atoms are injected in the excited state and move transversely to the channel. The gain bandwidth of this amplifier is determined by the transit-time to the atoms in the channel.
EQUATIONS

Define slowly-varying amplitudes $e(t,z)$, $p(t,z)$ of the electric field $E(t,z)$ and the macroscopic polarization $P(t,z)$ induced in the atoms by the field as:

$$E(t,z) = \frac{1}{2} (e(t,z) \exp[i(kz - vt)] + c.c.)$$

$$P(t,z) = \frac{1}{2} (p(t,z) \exp[i(kz - vt)] + c.c.).$$

The equations of motion for these variables are simplified by using a frame which moves with the speed of light $c$. Define a retarded time $\tau = t - \frac{z}{c}$. Then from any standard text\textsuperscript{1,2} one obtains equations that read,

$$\frac{\partial e}{\partial \tau} = \alpha' p - \kappa e$$

which is the form taken by Maxwell's equations. Here $\alpha'$ is a constant related to the gain coefficient and $\kappa$ describes losses (e.g. the mirror losses in a laser). Schrodinger's equation reduces to

$$\frac{\partial p}{\partial \tau} = en$$

$$\frac{\partial n}{\partial \tau} = ep.$$  

Where $n(\tau,z)$ is the population inversion. A simple trick of defining

$$\Theta(\tau,z) = \int_0^\tau d\tau' e(\tau',z)$$

allows one to show that $p = \sin \Theta$, $n = \cos \Theta$, so that Eqs. (11-13) combine to give the damped sine-Gordon equation\textsuperscript{2}

$$\frac{\partial^2 \Theta}{\partial \tau \partial z} = \alpha \sin \Theta - \kappa \frac{\partial \Theta}{\partial \tau}. $$

The entire trick in solving this equation is to learn how to handle the function $\tau_0(z)$, which brings us to the question of boundary conditions.

INVERSION PULSE

The inversion defined by Fig. 1 propagates as a square pulse moving at a velocity $v$. This is illustrated in Fig. 2. At any one position $z$, the atoms enter the channel over a time interval that is much smaller than the transit time $\tau_0$. The width of the inversion pulse in space is then $vt_0$. This corresponds to giving a very large value to the "slippage"\textsuperscript{3}. 

Fig. 2. Illustration of the propagation of the inversion $n(t,z)$ in the gain channel in the absence of a light field.
The pulse described in Fig. 2 is nonzero over an interval \( \frac{Z}{v} < t < \frac{Z}{v} + t_0 \), where \( t_0 \) is the transit time of the atoms across the channel described in Fig. 1. In the retarded frame, it is convenient to define the velocity \( \beta \), where \( \frac{1}{\beta} = 1 - \frac{1}{c} \). In Colson's terminology, \( \frac{1}{\beta} \) is proportional to the desynchronism \( \alpha \) of the pulse. In the retarded frame the inversion is nonzero in the interval \( T_0(z) = \frac{Z}{\beta} < \tau < \frac{Z}{\beta} + t_0 \). The nonzero region of gain is illustrated in Fig. 3.

![Fig. 3. Regions of retarded time and space in which the fields are nonzero.](image)

The boundary conditions for this problem are applied at the moment the atoms enter the channel, i.e., at 

\[ \tau = \frac{Z}{\beta}, \]

and at the moment they leave the channel,

\[ \tau = \frac{Z}{\beta} + t_0. \]

The first is both easy and standard. It requires that the inversion and the polarization have the values they had just prior to entering the channel in Fig. 1. These values are \( n = 1 \), and \( p = 0 \) respectively. In terms of the variable \( \theta \), these read \( p = \sin \theta, \) \( n = \cos \theta \), so the Eq. (4) with the choice \( \tau_0(z) = \frac{Z}{\beta} \), guarantees that these conditions are met. The second is much more tricky. First we must choose an incident electric field at \( z = 0 \). A natural choice is that the field starts from spontaneous emission, which means that the initial field is zero in space-time regions in which the atoms are absent. We choose not to specify yet the nonzero part of the field, but it is vital to use

\[ e(\tau, z) = 0, \quad z = 0, \quad \tau > t_0. \]

Now use the equation for the field in the space-time region \( \tau \geq \)

\[ \frac{\partial e}{\partial z} = -ke. \]

This equation follows immediately from Eq. (11) since there are no gain-producing atoms in that region \( (\alpha' = 0) \), only losses. This equation is immediately soluble, giving the second boundary condition

\[ e(\tau, z) = 0, \quad \tau = \frac{Z}{\beta} + t_0, \text{ all } z. \]

This boundary condition can be expressed in terms of \( \theta \) as
These peculiar boundary conditions are the secret of finding analytic solutions to this problem.

ELECTROMAGNETIC PULSES; SCHEMATIC

A sufficient condition to have exponential growth in the unsaturated regime and to have a steady-state pulse in the saturated regime is to have a pulse whose shape in time is invariant in the rest frame of the inversion. These have been given the name 'supermodes' by Renieri and Dattoli. This situation is illustrated in fig. 4.

![Diagram of an electromagnetic pulse propagating in the gain channel](image)

**Fig. 4.** Illustration of an electromagnetic pulse propagating in the gain channel. It is assumed to have an invariant shape in the rest frame of the gain.

ANALYTIC FORM OF THE ELECTROMAGNETIC PULSES

A pulse with constant shape in the rest frame of the gain reads

$$ e(\tau, z) = f(\tau - \frac{z}{\beta}) \exp(\gamma z) $$

or, in terms of $\Theta$,

$$ \Theta(\tau, z) = \Theta(\tau - \frac{z}{\beta}) \exp(\gamma z) $$

Upon substitution into Eq. (5), this gives

$$ \frac{\theta''}{\beta} + (\kappa - \gamma) \theta' + \alpha' \exp(-\gamma z) \sin(\theta \exp(\gamma z)) = 0 $$

where the prime (') signifies the ordinary derivative with respect to the variable $\tau - \frac{z}{\beta}$.

Normally this equation is worthless since it is an explicit function of $z$. However, in the cases of interest the factors involving $\exp(\gamma z)$ cancel. We wish to solve first for the weak signal limit, in order to find out whether the gain exceeds the losses. The gain $g$ is given by $\gamma = g - k$. In that limit we expand the sine function as $\sin x = x$ and the exponentials cancel. In the saturation limit, the gain exactly balances the loss to give a steady state. In that case $\gamma = 0$ and the exponentials go away. We can therefore throw away the exponential terms in which case this reduces to a damped pendulum equation. There are a number of awkward features about this equation, e.g. the damping is negative, that are removed by a change of variables which involves a time shift and a time reversal. Set $\mu = t_0 - \tau + \frac{z}{\beta}$. Then

$$ \frac{1}{\beta} \frac{d^2 \theta}{d\mu^2} + (\kappa - \gamma) \frac{d\theta}{d\mu} + \alpha \sin\theta = 0 $$

In the equation for a damped pendulum, the desynchronism (proportional to $\frac{1}{\beta}$) plays the role of the inertial term. The gain coefficient $g = \gamma - k$, $\gamma = g - k$ plays the role of the damping, and the various coefficients that enter into the gain enter as the force field.
METHOD OF SOLUTION

The problem has now been reduced to the solution of a damped pendulum equation, which solves for the atoms and for the electric field (the pendulum equation in the FEL describes only one electron, and must be both averaged over many electrons and then coupled to Maxwell's equations). The boundary conditions are switched in the process of time reversal. The boundary at $t = z/v + t_0$ is the initial condition. The condition $e = 0$, i.e. $\theta' = 0$, means that the pendulum is at rest. The polarization is unknown, and is written as $p = \sin\theta_F$, where $\theta_F$ is the initial tipping angle of the pendulum in the time-reversed frame. The boundary condition at $t = \frac{z}{\beta}$ becomes the final condition of the pendulum. At that boundary, the polarization $p$ vanishes, in which case $\theta = 0$, i.e. the pendulum is vertical. The key to obtaining the correct solution, and from it the gain and saturation conditions, is that the pendulum must fall to vertical in a time $t_0$. The conditions on the pendulum are illustrated below.

\[
\begin{align*}
\text{INITIAL CONDITION} & \quad \mu = 0 \\
\text{FINAL CONDITION} & \quad \mu = t_0
\end{align*}
\]

Fig. 5. Illustration of the eigenvalue problem posed by the damped pendulum.

SMALL SIGNAL GAIN

In this case the sine is expanded to linearize the pendulum equation. The details of this solution are found in Ref. 4. The period of the pendulum is independent of the initial angle. However the period must be an half-integral submultiple of, i.e. if $T$ is the period, then $T = (n + \frac{1}{2})t_0$, $n = 1,2,3,...$ To achieve this condition, one must adjust $\gamma$, which determines the small signal gain $g$ through $\gamma = g - \kappa$. This is a straightforward eigenvalue problem, whose details are found in Ref. (5). The gain vs. $\beta$ is given in Fig. 6 below.

\[
\begin{align*}
\text{POSITIVE BRANCHES} \\
\text{NEGATIVE BRANCHES}
\end{align*}
\]

Fig. 6. Graph of gain plotted as a function of $\sqrt{\beta}$ (the choice of independent variable makes the plot somewhat more compact). The largest value achieved by the gain is \(~.43\) \(\alpha t_0\).
ELECTROMAGNETIC PULSES

In the problem posed in fig. 5, there are actually an infinite number of solutions. All but a few involve complex $\gamma$, which turn out to have negative gain. There may also be more than one $\gamma$ with positive gain. These involve cases in which the pendulum is allowed to swing through vertical one or more times. These solutions provide a complete set of functions in which an arbitrary initial function can be expanded. As the pulse propagates, the component which has the largest gain will dominate the solution. The component with highest gain is the one in which the pendulum falls to vertical only once. The pulses associated with the highest gain are shown in Fig. 7.

![Fig. 7](image_url)

Fig. 7. The electro magnetic pulse amplitudes $e(\tau,z)$ plotted vs $\tau$ for four values of $\beta$ (see Fig. 6).

SATURATION REGIME

In this case the appropriate solution is one in which the pulse is in steady state, i.e., it neither grows nor attenuates. For this we want $\gamma = 0$. To find the condition under which the the pendulum falls to vertical in a time $t_0$, it is necessary to adjust $\theta$. In so doing, the degree of saturation is determined. The energy extracted per atom is $\frac{h \Delta n}{2}$, where $\Delta n = 1 - \cos \theta$ is the total change in the inversion (the factor of two comes from the fact that $n = 1$ when the atom is excited, and $n = -1$ when a photon of energy $h \nu$ is extracted). In this model the energy is dissipated in the losses. In a laser it is extracted as useful power. In fig. 8 the extracted energy is shown as a function of $\beta$. Note that since the pendulum falls most rapidly when linear. A nonlinear solution exists only if a linear solution exists with $\gamma \neq 1$. Hence energy is extracted if an only if gain exceeds losses.

![Fig. 8](image_url)

Fig. 8. Extracted energy as a function of $\beta$ for two values of $\alpha'$ and $\kappa$. The parameter that defines the curve is the ratio of the loss to the maximum achieved gain (i.e. $\kappa/43\alpha'^2t_0$) in the lower curve this is .9, in the upper it is .75.

COMPARISON WITH THE FEL

In fig. 9 below, the results of the laser theory are compared with the FEL. The gain can be compared directly, since the amplifier theory predicts that the gain curve is universal$^5$, and
all cases can be obtained from all others by scaling. In the top row, the gain curves
obtained from fig. 6 are on the right, and the corresponding curves, obtained by numerical
calculation of the FEL equations is shown on the left. The independent variable in the FEL
is the "delay time" which is proportional to the resonator length, and is directly
proportional to $\frac{1}{\beta}$. The agreement is remarkable. In the bottom row, a comparison of the
output powers is given. In this case, there is no universal law available, so detailed
comparison will have to wait until after it is learned how to convert FEL numbers into Laser
amplifier numbers. Nonetheless the similarities in the curves in apparent. Similar qualitative
agreement in found if pulse shapes are compared.

CONCLUSION

The purpose of this presentation is severalfold. Perhaps the most important message is
that, to a large extent, the FEL is well described by its laser analogue. The major
conclusions of the amplifier study, which need to be verified in the FEL are:

1) The gain is a universal function of the boundary velocity (i.e. the delay or
desynchronism). All cases can by determined from a single function through scaling.
   For each case there is an optimum gain that can be used to compute saturation.
2) The saturation is given by a family of curves. To determine which curve to use, it
   is necessary to determine the gain through scaling. Then the saturation is given by a
   function that is universal for a given ratio of optimum gain to loss.

There are two minor conclusions of the study that have been verified in FEL
calculations.

3) The optimum velocity (i.e. delay) for gain and saturation are very different.
4) The amplifier emits coherent radiation only if gain exceeds losses, i.e. it has a
   conventional threshold condition. There is no bistability.

There is a minor conclusion that may not have been verified in the FEL.

5) The saturation condition cannot be related to any saturation formula found in
   conventional laser physics. It must be determined from fig. 8.

Finally there is a speculative conclusion with respect to fluctuations.

6) The fluctuations in the FEL will be laser like, i.e. small in the region in which
   gain exceeds losses. There are no macroscopic fluctuations in this regime such as
   are found in other superradiant systems.

The third and the last conclusion however, apply only to the cases discussed here in
which noise is excluded. If noise is included the power-on curves will be widened,
and in cases where $\beta < 0$ but the power out is large, there will be large
fluctuations.
Acknowledgment

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References


2. The original derivation of these equations is found in F.T. Arecchi and R. Bonifacio, IEEE J. Quantum Electron. QE-1, 169 (1965).

3. See W. Colson, this volume.


6. G. Dattoli and Al Renieri, Technical report 79.37/p, CNEN, Frascati, Rome (1979). See also articles in this volume reported here have been found in this earlier work.