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PULSE PROPAGATION IN FREE ELECTRON LASERS

W.B. Colson and A. Renieri*

Quantum Institute, University of California, Santa Barbara, CA 93106, U.S.A.
*E.N.E.A., Dip. TIB, Divisione Fisica Applicata, C.R.E. Frascati (Roma), Italy

Résumé - Sont discutées ici les propriétés pertinentes de la propagation d'impulsions dans un laser à électrons libres en même temps qu'une comparaison avec les données obtenues à Stanford sur l'expérience d'oscillation.

Abstract - The main features relevant to pulse propagation in free electron lasers are discussed together with a comparison to the Stanford oscillator experimental data.

A. INTRODUCTION

Free electron lasers (FELs) use a beam of relativistic electrons to amplify coherent radiation stored in a resonant optical cavity /1/. When the electron beam is produced by an RF accelerating cavity such as in a linac, microtron or storage ring, it consists of a train of pulses whose length depends on the accelerator properties. The first operation of a free electron laser oscillator at Stanford /2-5/ used a 1 mm long electron pulse from a superconducting linac /6/. Most proposed FEL oscillators will also use short pulses, and many scientific applications find the picosecond time-scale an advantage. For these reasons the exotic short pulse aspects of the FEL have been experimentally and theoretically explored almost as early as the fundamental theory.

Figure 1 shows the FEL oscillator configuration. The optical pulse is much smaller than the resonator cavity and slowly passes over the freshly injected electron pulse on each bounce. The repetition time of the electron beam source must be closely synchronized with the bounce time of photons between the resonator mirrors. At the end of the N period undulator magnet, the resonance condition determines that the photons have slipped ahead of the electrons by an amount \( N \hbar \) where \( \hbar = 2 \pi /k \) is the optical carrier wavelength. When \( N \hbar \) is comparable to

![Diagram of FEL oscillator configuration](image)

Fig. 1 - A succession of electron pulses are injected into the resonator to overlap a rebounding optical pulse. The synchronism of the pulses is adjusted by moving the end-mirror an amount \( \Delta \zeta \). While "wiggling" through the transverse periodic magnet in the presence of the superimposed light, the electrons bunch to drive the radiation field (from ref. /16/).
the electron pulse length $\delta$, the dimensionless slippage is $s=\frac{N\lambda}{\delta} \approx 1$ and short pulse effects will be observed. Every point in the optical pulse envelope sees a varying electron density during the gain process and resulting amplification, while every point in the electron pulse sees a varying electric field strength.

The original Stanford experiments /2/ are described by $s \approx 1.2$, and the electron-optical pulse synchronism was finally adjusted by moving one of the mirrors. The effect of the short pulses caused the observed resonator synchronism to be an extremely sensitive parameter in determining the laser output power /3-5/. A description of "lethargic" behaviour in the gain medium of proposed atomic x-ray lasers had been predicted only a year before by F.A. Hopf et al. /7/ but had not been observed. Almost before the FEL was widely known as a workable device, the laser lethargy had been predicted, observed and explained /8-9/. The lethargic response of the FEL gain medium (the electrons responding to the optical wave envelope) is on the same time-scale as the slippage process and causes gain to only occur on the trailing edge of the light pulse.

There are now several papers dealing with short pulse lethargy in FELs which have different points of view and attributes. The Stanford experimental papers are in references /3-5/. The original theory has been followed by other papers from the same group using the quasi-Bloch equations to describe the interaction /10-12/. This approach is useful in establishing the relationship of the FEL to atomic laser theory but suffers from approximations when the optical field becomes strong. Another group of papers uses the Maxwell's wave equation coupled to the electron's Lorentz force equation /13-17/. This method has improved accuracy and is essentially a numerical simulation of the optical pulse shape evolution. A third group /18-21/ has coupled the electron Lorentz force equation to the multimode evolution of the optical field. This presents a comprehensive picture of the complex mode coupling in a non-relativistic reference frame and agrees with the spatial theory. While /3-21/ represents the main body of work on FEL pulse propagation, other contributions have been made /22-30/ and will be described more fully in later sections.

The goal of this paper is to review the physics of optical pulse effects in FELs which have been published to date. The structure of this paper is designed to emphasize the physical results while bringing in the contributions of many researchers. We must suppress the presentation of involved analytical results and excessive notation which would quickly "slog" the discussion. Actually, the theoretical methods used are quite similar; the biggest difference occurs in the description of the electron dynamics. The next two sections develop the non-linear self-consistent wave equation and the electron dynamics. Then several aspects of pulse evolution are presented.

B. THE PULSE WAVE EQUATION

All theories naturally start with the wave equation, but they also make essentially the same assumptions and approximations on the characteristics of the optical vector potential $\hat{A}$. These are listed below:

(1) The polarization of $\hat{A}$ is assumed to be the same as that emitted spontaneously from electrons passing through the undulator magnet. A helical undulator causes constant acceleration of the electrons and produces cylindrically polarized light. This is the easiest case analytically because the electron acceleration is constant. Polarization effects can be explored without much difficulty /31/, and this is a good topic for future research.

(2) The transverse diffraction of light along the undulator length is usually neglected so that $\hat{A}$ is taken to be only a function of $(z,t)$. This is justified when the scale of transverse field variations are much less than $\sqrt{2L/\lambda}$, where $L$ is the length of the undulator. In many cases, $\sqrt{2L/\lambda} \approx 1$ mm is roughly the same size as the electron beam diameter and the optical mode waist. Some aspects of the diffraction effects have been included in the pulse theories /12,14,17/, but the main features of short pulse propagation remained unchanged in the examples explored. This is another important problem for the futu-
re, but will lead to significant computer time. Often a mismatch between the optical wavefront and electron pulse cross section is approximately described with a "filling factor" equal to the ratio of their areas. This is inadequate, but does decrease the beam/mode coupling and is simple.

(3) The complex electric field envelope \( |E(z,t)| \ e^{i \phi(z,t)} \) is assumed to have some coherence so that a slowly varying amplitude and phase approximation is appropriate. The form of the light wave is then

\[
\hat{A}(z,t) = \left| \frac{E(z,t)}{k} \right| \sin \psi, \cos \psi, 0 \tag{1}
\]

where \( \psi = k_z - \omega t + \phi(z,t) \) and \( \omega = kc \) is the carrier frequency. When (1) is inserted in the wave equation, and second derivatives are dropped compared to first derivatives

\[
2i k \left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \hat{A}(z,t) = \frac{-4\pi}{c} \hat{J}_\perp(z,t) \tag{2}
\]

where \( \hat{J}_\perp \) is the transverse part of the total electron current

\[ -e \mathbf{c} \delta(z - \mathbf{z}_j(t)), \quad (e = |e| \text{ is the electron charge magnitude}) \]

\( \mathbf{v}_c \) is the velocity of the \( j \)-th electron, and \( c \) is the speed of light. Equation (2) occurs in nearly all free electron laser pulse papers in essentially the same form. Even in the multimode analysis, it is necessary to retreat to the spatial form (2) in order to develop the full pulse problem.

Applying the coordinate change \( z = \frac{2}{c} + ct \) and \( t = \tau L/c \), we normalize the propagation time through the undulator length \( L \) to unity and travel along with the light pulse. The operator \( \frac{\partial}{\partial z} + c^{-1} \frac{\partial}{\partial t} \) in (1) may be rewritten with the replacement

\[
\mathbf{z}_j + c^{-1} \mathbf{v}_j \mathbf{t} \]

The left side of (2) has been simplified in much the same way as in conventional atomic laser theory. The distinction of the FEL is found in the driving current \( \hat{J}_\perp \). Probably most important is that the electrons passing through the undulator are travelling at nearly the speed of light along with the radiation. As they oscillate through one periodic section of the undulator, \( \mathbf{v}_c \) rotates through one cycle. The condition of resonance between the optical field and electrons means that one wavelength of \( A(z,t) \) must pass over an electron while the electron passes through one wavelength of the undulator and \( \mathbf{v}_c \) rotates once \( /13,14/ \). Both sides of (2) oscillate at roughly the carrier wave frequency \( \omega \) when the electron \( z \)-velocity \( \mathbf{v}_c \) and the undulator wavelength \( \lambda_o = 2\pi / k_o \) are chosen so that \( \omega (1-\mathbf{v}_c^2) \approx k_o c \) which is the resonance condition. The fast oscillation of (2) is uninteresting, however; the slower wave envelope evolution \( |E_n(z,t)| e^{i \phi_n(z, t)} \) is of more concern. In fact, we want to follow the wave evolution over many bounces (\( n \to 0 \)) between the mirrors of the resonator. The slow components of (2) are projected out simply by multiplying by the orthogonal unit vectors \( \sin \psi, \cos \psi, 0 \) and \( \cos \psi, -\sin \psi, 0 \). The remaining derivation is now on the right of (2) where \( \hat{J}_\perp \) must be determined by an equation of motion coupling electrons to the optical wave. The feedback loop for laser amplification is then complete.

C. THE CURRENT'S RESPONSE TO LIGHT

The clearest description of the electron beam evolution in FEL involves a hierarchy of scales. Most crudely, all electrons are travelling along the undulator at some average speed \( \mathbf{v}_c \). Over many undulator wavelengths, the \( e \)-beam (electron beam) envelope is modulated by the betatron motion due to the focusing properties of the magnet. Usually, the experimenter is successful in making these betatron oscillations negligible, with a suitable matching of the \( e \)-beam
parameters, and we will ignore them here, assuming that the emittance of the e-beam is small enough to neglect the correspondent inhomogeneous broadening. On the next scale, we find that they all execute small transverse oscillations along the undulator wavelength $\lambda_0$ of magnitude $K/\gamma k_\perp$, where $K = eB \lambda_0 / 2\pi m c^2$, $m$ is the electron mass, $\gamma mc^2$ is the electron energy, and $B$ is the undulator's peak field strength. Typically the electron pulse length is the next smallest scale, and at this level, the electron at the front and trailing edges can evolve differently (it might be considered that Coulomb repulsion will expand the pulse and push some electrons ahead and others behind, but it has been shown that for relevant electron densities, this is a small effect /25,26/). More important is that the optical pulse is also on the same scale as the electron pulse, and when $s \approx 1$, the pulses noticeably pass through each other. This means that electrons on the front edge of the electron pulse evolve differently than those on the trailing edge, since they each are influenced by different optical fields. This is at the root of the lethargy or short pulse effects in FELs and is shown in Figure 1.

The final scale length is the optical wavelength $\lambda$. The resonance condition insures that $\lambda \approx \lambda'_o / \gamma^2$ so that $\lambda \ll \lambda'_o$ when electrons are relativistic $\gamma \gg 1$. Typically the slippage distance $N \lambda \approx N^O \lambda'_o / \gamma^2 \ll \lambda'_o$ because $\gamma' > N$, the number of undulator periods. And since $N \gg 1$ for FEL undulators, the hierarchy of scales just described relies completely on the relativistic nature of the FEL and $\gamma > 1$. This was not the case for the non-relativistic FEL predecessor, the microwave electron tubes, and is responsible for the short wavelength capabilities of FELs.

On the optical wavelength scale, we can understand the FEL gain mechanism as electron bunching within each wavelength.

The undulator field consistent with the polarization of (1) is $
abla \mathbf{B} = B (\cos k \perp z, \sin k \perp z, \beta_\perp)$. With perfect injection, the motion is simply helical:

$$\vec{\beta} = \left( (K/\gamma) \cos k_\perp z, (K/\gamma) \sin k_\perp z, \beta_\perp \right);$$

with less than perfect injection, the previously mentioned betatron motion will occur. The assumption of perfect injection in the equation of motion /14/ is equivalent to assuming zero transverse canonical momentum /9,19/ when starting from an Hamiltonian picture. The full Lorentz force includes an optical transverse force not included in $\vec{\beta}$ above. This small force is proportional to $B(1-\beta_\perp)$ and is much smaller than the undulator field $B$ when electrons are relativistic ($\beta_\perp \approx 1$). Neglecting the transverse optical force is equivalent to neglect laser-laser interactions in the Hamiltonian picture /9,19/.

The fast oscillating $\vec{\beta}$ and the fast unit vectors ($\sin \psi$, $\cos \psi$, 0) and ($\cos \psi$, $\sin \psi$, 0) combine to make a slowly evolving electron phase $\zeta = (k + k_\perp)z - \omega t /13/$. The instantaneous current at a position $z$ is then a sum over all single particle currents $J_1 \propto \Sigma_i \left( e^{-i\zeta_j} \right)$. If the number of electrons in the pulse were small enough, this sum could be evaluated numerically /22/, but in most cases this is impractical. Typically, a slice of the electrons pulse which is small enough to be considered a point (where $E$ is substantially constant in $z$) still contains $N \approx 10^2$ electrons. Since all of the electrons in this small volume element evolve in the same way, the sum can be evaluated by sampling electrons (as few as ten may be adequate) and using the local electron density $\rho$ as a weight function /13, 14/:

$$J_1 \propto \left( \rho / N_e \right) \sum_{j=1}^{N_e} \left( e^{-i\zeta_j} \right) / \gamma_j$$

where $N_e$ is the number of sample electrons. This method can be made arbitrarily accurate by including more sample particles, but usually less than $10^2$ is adequate.

A traditional approach is to use the one-dimensional Boltzmann equation
which evaluates the electron distribution function $h(p)$. The beam current is then expressed as $J_{\text{beam}} = \int dp \, h(p) \, \rho(t)$ which sums over all electrons addressed by their $z$-momentum $p$. In weak optical fields, the distribution of $\zeta$ and $h$ are both simple and sinusoidal in structure. In strong optical fields, the structure of each $\zeta(p, t)$ is non-sinusoidal, but still simple, while the distribution function $h$ becomes complicated with discontinuities and infinities. It therefore becomes necessary to use approximate forms of $h$. As in conventional laser theory, $h$ is expanded in harmonics of the electron phase $\zeta$:

$$h = \sum_{m=0}^{\infty} i g_m(t) (e^{i m \zeta} + c.c.)$$

Gain and lethargy in weak and even in moderately strong fields can be described by only $m = 1$. But the laser will extend itself to higher powers where $m > 1$ terms are needed for an adequate description.

We emphasize again that the above discussion addresses only one point $z$ within the electron pulse structure at a time $T$ on pass number $n$ in the resonator. In a typical plasma (where the Boltzmann equation is often used), the distribution function or electron density is continually distorting due to the flow of electrons. In FELs, however, the electron beam quality must be sufficient to ensure that electrons bunch in phase on the optical wavelength scale. Since the electron pulse is much larger than an optical wavelength, the distortion of the pulse shape in a single pass due to emittance or energy spread must be small. In most cases, however, the electron pulse weight factor $\rho(z)$ and the $m=0$ term in the distribution function $h(z)$ are independent of $\tau$ during the gain process.

Even though $\rho(z)$ is fixed in shape, the electrons do move with respect to the light in free space, since they travel slower. The local weight function $\rho(z)$ and the sum over sample electrons

$$\sum_j \frac{(e^{-i \zeta_j})/\gamma_j}{\gamma_j}$$

can be combined at each site $z$ to form the driving current. The wave equation can now be written as an evolution equation for the optical field at each site $z$:

$$\frac{dE(z)}{d\tau} = 2 \pi e K L \left\langle \rho \frac{1}{\gamma} \frac{e^{-i \zeta}}{\gamma} \right\rangle_{\gamma+\tau}$$

where $<\frac{e^{-i \zeta}}{\gamma+\tau}$ is the average over sample electron in the volume element at $(z+\tau)$. Sometimes a "filling factor" (defined as the ratio of electron beam area to the optical mode area) is included on the right side of (3) in order to estimate the mode coupling. We define $\rho$ above as the number of electrons driving the optical wavefront at $z$ so the "filling factor" is already included. Note that $\rho$ is not the actual electron density. In many cases, the electron energy is nearly constant so that $\gamma$ is constant and can be removed from the average /13/.

The self-consistent change is $\zeta_j$ determined by the optical field and undulator together. The work done on an electron is proportional to $\mathcal{E}_j \cdot \mathbf{B}$. This product is slowly varying, as already shown above, and for jth electron gives

$$\frac{d\gamma_j}{d\tau} = \frac{eK |\mathcal{E}| L}{\gamma_j m c^2} \cos (\zeta_j + \phi)$$

The electron energy $\gamma_j$ changes the phase $\zeta_j$ through $d \zeta_j / d\tau = \nu_j$ where

$$\nu_j = L \left[ (k_0 + \nu_j) \left( \beta_j \mathbf{e}_z \right) - k \right] \approx L \left[ k_0 - k (1 + k^2) \gamma_j^2 \right]$$

since $\beta_j \approx 1 - (1+k^2)/2 \gamma_j^2$ for relativistic electrons in the helical undulator. The dimensionless electron velocity or energy $\nu_j$ measures the degree of resonance between the optical wave, electron, and undulator. The initial value $\nu_j$ is called the "resonance parameter". In the cases where the electron energy is nearly constant, it can be eliminated from (4) and (5) to give the pendulum equation in $\zeta_j/\gamma_j$. The initial phases of sample electrons are either random or uni-
form to characterize an unbunched beam. Little difference has been found between these choices /32/; this is reasonable since the large number of randomly positioned electrons in the relevant volume element form a nearly uniform distribution. The range of sampling is only one 2π range in ζ, since /4/ is periodic. The electron energy spread can often be represented as monoenergetic in v. A sufficiently wide spread randomizes the phases in <> and amplification decreases. In weak optical fields, a monoenergetic beam requires that the spread in v be much less than π, or in practice less than unity. A moderate spread in energies of the order of unity can have a small effect on pulse propagation results, but not much work has been done in this area.

The coupled equations (3), (4), and (5) or their equivalent forms provide the theory for pulse propagation studies. A modal expansion of /4/ is also possible either as a method of solution /12/ or as an alternative viewpoint /19/. Undulators other than the conventional periodic design can be characterized by modifying /4/ and/or (5) with (3) remaining unchanged in many cases. Examples are the tapered undulator, the optical klystron, and the transverse gradient "gain-expanded" magnet.

D. LETHARGY AND DESYNCHRONISM

The effects of short pulses in FELs are intimately tied to the time development of gain along the laser. At τ = 0, the wave is not driven since <> = 0. As τ increases, the distribution of ζ's responds to the local radiation field E and <> becomes non-zero. In weak fields and low gain with no pulse structure, gain is g(τ) = α(1 - cos vτ - (vτ/2) sin vτ)/v² and g ≈ τ² for small τ. This delay in the gain medium has been called "lethargy" and is also present in conventional lasers. The phase space evolution of ten sample electrons is shown in Figure 2 where slight bunching is observed only at τ ≈ 1.

![Figure 2](image_url)

**Fig. 2** - In weak optical fields, the electrons are initially uniformly spread along the ζ-axis of phase-space at τ = 0. Near τ ≈ 1 bunching occurs (from ref./16/).

During the bunching time, N wavelengths of light have passed over the electrons. If s ≈ 1, the electron pulse and presumably the light pulse are passing through each other during this time. Therefore, gain only occurs on the trailing edge of the optical pulse. This is nicely shown for an idealized square pulse in Figure 3 at the beginning τ = 0 (or z = 0) and end τ ≈ 1 (z = L) of the undulator length. After many passes (c), the light pulse initially started in (a) moves out of the electron pulse.

![Figure 3](image_url)

**Figure 4** shows the same concept for an initially Gaussian optical pulse. Presumably, the initialized optical pulses in Figures 3 and 4 would never started since they have no final steady-state solution.

In these examples, the electrons were synchronized precisely with the bounce time of the light pulse. The effect of lethargy is to distort the light pulse by only amplifying the trailing. In effect, over many passes the light pulse centroid travels slower than c. This is not a plasma effect (which is present but many orders of magnitude smaller). The cure is to desynchronize the FEL and shorten the
Fig. 3 - In (a) the initial light pulse sees no bunching at the beginning of the laser $z = 0$. The bunching shown as dots drives only the trailing edge of the pulse. After more passes (b) and (c), the pulse continues to shorten (from ref. /10/).

The amplitude of the optical pulse shape $a(z)$ is plotted at $\tau = 1$ every 150 passes through the resonator up to $n = 1500$ passes. The fields start at $a = 1$, and are driven by the parabolic current density $r(z') = r_0 (1 - \frac{1}{2} (z')^2)$ where $z' = z + s(\tau - 1)$. With no desynchronism $d = 0$, the light moves away from the electrons and eventually decays with the resonator $Q$. The fixed point solution is therefore zero (from ref. /16/).

The resonator length to advance the delayed light pulse after each pass. Figure 5(a) shows the log of the gain labeled $\ln(a)$ as a function of the delay time $\delta t$ (psec) for a 2 psec long electron pulse. The higher curves are for higher current densities and gain. With $\delta t = 0$ or exact synchronism, there is no gain and the resonator losses cause the light pulse to decay. Higher gains give a larger width in $\delta t$.

Fig. 5 - Plots showing the effects of desynchronism delay $\delta t$ (from ref. /8/).

Fig. 6 - Supermode (SM) picture of laser power vs desynchronism $\Theta$ with superimposed the laser spectrum for the first SM (from ref. /19/).
In Figure 5(b), the threshold current needed for gain to exceed the resonator losses is plotted for two different losses. Figure 5(c) is for longer pulses, and 5(d) is the steady-state power resulting from δt for high and low resonator losses.

In order to maintain weak fields in steady-state, the losses and gain must be close in value. If this is the case, the laser can saturate in weak fields. Figure 6 shows laser spectra in steady-state for high losses. The desynchronism labeled θ is increased to show less broad-band behaviour in ν. Since fields are weak, the electron distribution is narrow. Small desynchronism implies a narrow pulse.

E. DESYNCHRONISM AND FEL OPERATION

In most experimental situations the resonator length L is so much longer than the operating range of desynchronism distances that it is not likely than the absolute value of desynchronism d = δt c/δ = 2 δL/δ will be measured. Exact synchronism d ≈ 0 provides a cut-off on one side. At large d the optical pulse centroid can be advanced faster than the lethargy distortion so that FEL performance diminishes. In fact, the whole character of FEL performance changes as d moves through its full range of values.

In numerical simulations without practical or fundamental noise such as shot noise, quantum fluctuations, mirror vibrations, electron pulse jitter, etc. a tiny non-zero value of d (∼10⁻⁴) allows maximum steady-state power P in the resonator. Since a small difference in d makes a large difference in the steady state power, the experiment can be expected to be unstable to fluctuations in any of several physical quantities. For sufficiently large d (but still near synchronism) the simulations become stable.

Several properties follow from the strong fields in the peak of the desynchronism curve and different theoretical approaches agree on these qualitative points /14, 15, 16, 18, 19/. Figure 7 shows the evolution of several aspects of the simulation for small d. The strong optical fields produce a wide electron momentum distribution.

**OPTICAL PULSE IN FEL**

![Optical Pulse Diagram]

**POWER EVOLUTION**

**OPTICAL AMPLITUDE**

**ELECTRON DISTRIBUTION**

**LASER LINESHAPE**

**STEADY-STATE RESULTS**

**DRIVING CURRENT**

**OPTICAL PHASE**

*Fig. 7 - At small desynchronism d = .001, the Stanford parameters produce an optical pulse shape a(ν) whose length is four times shorter than the electron pulse and has a large peak field. This gives a broad power spectrum P(ν) centered at ν ~ 6, and a broad electron velocity distribution f(ν) due to the high field strength. The driving current continually reshapes the optical pulse to compensate for desynchronism, and the phase profile φ(ν) shifts P(ν). The pulse energy (or laser power) reaches steady state after n ~ 10⁷ passes, and the final results are shown on the lower left (from ref. /16/).*
The optical pulse length is narrow and the centroid is positioned just behind the middle of the electron pulse. The optical pulse is trying to move away from the electrons as in Figure 4 but the small $d > 0$ maintains some overlap. The narrow optical pulse shows a broad line shape with sideband structure. The sideband growth and the multiple peaked structure of optical pulse shape are due to the synchrotron instability associated with strong fields and will be discussed in more detail later. Theories using the harmonic expansion /10/ are insufficient in this regime because higher frequency components are not retained.

If we move to the lower power section of the desynchronism curve where $d$ is large, the operating characteristics of the FEL change considerably. In this operation the large desynchronism is advancing the optical pulse by a large amount with respect to the electron pulse on each pass. The gain is amplifying only the trailing edge of the pulse and the new light is advanced by desynchronism to "feed" the the front edge of the optical pulse. The optical pulse centroid is now in front of the electron pulse and much of the pulse energy is decoupled from electrons. In front of $j(z)$ the optical pulse shape is described by an exponential whose form is $a(z) = \exp(-z/4qd)$. The long pulses produce a narrow linewidth as seen in Figure 8. There is no sideband structure in weak fields and the resulting

Fig. 8 - With large desynchronisms $d = 0.042$, the steady-state pulse energy is small and the fields are weak. The optical pulse is now three times longer than the electron pulse. The electron velocity distribution $f(v)$ is narrow as is the power spectrum $P(v_k)$, which is centered about $v_k \approx 3 \epsilon$ (from ref. /16/).

electron energy spread is small. In fact, the electron energy spread is smaller than expected for the characteristic field strength in the optical pulse. This is because of the relative starting position of the pulses on each pass. Electrons start near the trailing edge of the optical pulse and drop back out of the fields at about half way down the undulator length. This decreases the interaction time in the weak optical fields and produces an anomalously narrow electron momentum spread. The theoretical pulses in the large $d$ regime are stable and not subject to the fluctuations at small $d$.

The Hamiltonian-supermode picture describes the same trends as shown in the simulations. Figure 9 shows the desynchronism curve for each of higher order supermodes $r = 1, 2, 3$. Superimposed are the steady-state modes at various selected va-
Fig. 9 - Desynchronims curves for the first three supermodes and spectra relevant to the second one (from ref. /19/).

Values of desynchronism labelled $\Theta$ in the figure. At peak power the mode shows sideband structure at small desynchronism. At larger desynchronism the sideband structure disappears indicating a longer smoother optical pulse.

Another feature predicted by theory is limit cycle behaviour /16/ in the optical pulse structure and power. This is shown in Figure 10. The power oscillates

Fig. 10 - With $d = .003$ we see clear limit-cycle behaviour of the laser pulse energy. Subpulses in $a(z)$ start at the trailing edge and pass through the pulse profile to continually modify its shape (from ref. /16/).
periodically over several hundred passes as does the optical pulse shape and line shape. This is observed to occur at moderate values of desynchronism where the synchrotron instability occurs but for large enough values of $d$ so that a stable configuration does not occur. The moderate field strengths produce about one synchrotron cycle in a distance comparable with the electron pulse width. The optical pulse continually reshaping itself from a shape with two peaks then one peak and so on. The subpulse structure initiates on the trailing edge and is pushed through the optical pulse envelope by the desynchronism mechanism.

F. EXPERIMENTAL REVIEW

The only experimental data available on pulse propagation in FELs is that obtained from the Stanford oscillator. It is worthwhile to point out that the comparison of the oscillator parameter with the theory suffers for the strong sensitivity of the FEL dynamics on the e-beam characteristics, like the electron density, which are not continuously measured during the experimental runs. Moreover, in the Stanford device, many effects were observed which were subsequently recognized to be simply due to the variation of the e-beam parameters during the macropulse. However, in spite of these difficulties, the qualitative and partially, quantitative behaviour of the Stanford oscillator is satisfactorily described by the theories outlined in the previous sections. Namely, let us start with the desynchronism curve, which is the most relevant feature of a FEL operating with a bunched e-beam.

The first measurement of this curve (Figure 11) was reported in ref. /3/. The most striking feature was the asymmetry, which was not expected on the basis of an oversimplified model which merely takes into account the time overlap between laser and electron pulses over many passes. In addition this model predicts a curve broader than the experimental one by a factor $10^5$. Accurate measurements /4/ confirmed this asymmetry and the shape of the experimental curves became more similar to the theoretical ones (this fact was very encouraging for the theory). In particular, the desynchronism width agrees better than a factor 2 with the experiment (see figure 12).

Fig. 11 - The desynchronism curve (b) of optical power vs delay $\delta t$ is first calculated with the harmonic expansion. The experimental curve (a) is $\sim 10^5$ times narrower in $\delta t$.

Fig. 12 - The desynchronism curve of optical pulse energy vs $d$ using the pendulum equation simulation agrees with experiment.
There is also qualitative agreement for the laser linewidth which decreases at large desynchronism as was shown in Figure 13.

However, in the experimental spectrum no sidebands appear. This fact could be explained by the oversimplified assumptions of the model (e.g. no transverse effects) or by the particular experimental conditions (indeed, for small gain, the theories too predict no sidebands).

The measurements of the laser pulse length with the autocorrelation method (see ref. /4/) show another discrepancy with the theory: the lack of a long tail for large $d$. Namely, at large desynchronism, the laser pulses appear to be shorter by a factor 3 with respect to the theoretical ones.

Finally, the last data on gain (see ref. /4/) confirms the qualitative behaviour given by the theory (see ref. /18/). The gain is minimum near the maximum amplitude output at small $d$, reaches a maximum at larger $d$, finally decreases for large desynchronism.

There is qualitative agreement for the electron spectrum during the FEL interaction. For large slippage, the electron momentum distribution shows two peaks of equal height indicating that all the particles in the beam feel practically the same field (see Figure 14).

G. THE SYNCHROTRON INSTABILITY

The FEL pulse propagation studies have provided the most significant comparison between theory and experiment thus far. The non-trivial trends predicted /15/ and observed /4,5/ indicate a successful modeling of the FEL mechanism in so far as experimental details permit. An important new direction for the multimo-de methods developed is to characterize the synchrotron instability in the FEL. The effect is well-known in synchrotrons and storage-rings as the Robinson instability /34/ and originates from the longitudinal oscillations of electrons in an RF trap.
ping potential. It was first predicted /35/ for specially designed FELs where strong trapping in the optical-undulator potential well caused many synchrotron periods. It has now been shown that the FEL synchrotron instability can be exhibited in a wide range of undulator designs operating in strong fields/17/.

The origin of the synchrotron motion is most readily understood by considering the phase space evolution of electrons in strong optical fields. For now assume that we have only a single-mode field with strength a. The electron equation of motion can be written as $\xi = a \cos \zeta$. In strong fields a, a range of values of $\zeta$ about $\pi/2$ are trapped and perform harmonic synchrotron motion given by $\xi = a \zeta - \pi/2$. The synchrotron frequency is then $\omega = \sqrt{a}$ in these dimensionless units. The motion is described by $\cos \omega \tau$, where $0 < \tau < 1$ during one pass through the undulator. When $\omega \approx 2 \pi$, or $a \approx \frac{4 \pi^2}{30-40}$, there is about one synchrotron oscillation of harmonic electrons trapped near $\zeta \approx \pi/2$. Figure 15 shows the evolution of twenty sample electrons starting at $\nu = 2.6$ in strong fields $a = 30$. The electrons are monoenergetic at $\tau = 0$ and already develop strong bunching at $\tau = 1/3$. Maximum gain $g(\tau)$ is reached at $\tau = 2/3$, but then unbunching causes a decrease at $\tau = 1$. The harmonic electrons rotate around $\zeta \approx \pi/2$ by slightly less than one cycle since $\omega \approx \sqrt{30} < 2 \pi$. The additional quasi-periodic motion mixes with the carrier wave and gives rise to gain in the side-band spaced at

$$\pm \omega \approx \pm \sqrt{a} \approx \pm \sqrt{\frac{4 \pi^2}{30} \frac{Ne KLE/\gamma^2}{\gamma c^2}}$$

The gain in both side-bands need not be identical, but this is a detail for more involved analysis. In a spatial view of the process, we have the gain function $g(\tau)$ amplifying sections of the optical wave envelope as it passes over the electrons. The complicated temporal structure of $g(\tau)$ converts to a more complicated waveform after many passes; the side-band growth gives a modulated wave envelope.

We have already shown the results of the synchrotron instability in the small desynchronism range of Figure 7. The modulation of the pulse shows ringing after the leading spike in $a(t)$; the fields calculated ($a \approx 40$) are consistent with synchrotron motion in the electron pulse. The sharpness of the pulse actually cut-off the ringing so only few cycles are present, but this is a clear example of the instability. Note the side-band growth in $f(\nu)$. The limit cycle behaviour of $f(\nu)$. The limit cycle behaviour of $f(\nu)$.

Fig. 15 - Twenty sample electrons representing a current density $j=1$ start at $\tau = 0$ with dimensionless velocity $\nu$ in the self-consistent pendulum phase space $(\xi, \nu)$. The self-consistent separatrix is drawn for reference to the evolving phase space paths. For strong fields $a = 30$ there is bunching near $\xi$ at $\tau = \frac{1}{3}$. However, because the electrons perform a full synchrotron orbit about $\xi \approx \pi/2$ there is a decrease in gain $g(\tau)$ at $\tau = 1$. The optical phase shift $\phi(\tau)$ is also shown.
Figure 9 shows a periodic development of sideband structure with a perpetually changing pulse shape \( a(t) \). Here the presence of the side-band itself is not stable, probably because of large desynchronism \( d \). In figure 16 we show a new result for smaller \( s = 0.18 \) where many modulation peaks caused by the sideband growth. The modulation is large and the lineshape shows almost equal power in the sidebands as in the fundamental.

![Figure 9](image)

**Fig. 16** - A pulse with small slippage \( (s = 0.18) \) and high \( Q = 200 \) shows strong modulation (left) due to the sidebands forming about the central carrier wavelength (right). This is for \( j=1 \) and \( d=10 \).

The presence of side-bands in the FEL is a quality of mixed benefits. Generally, the experimenter would like a single-line high power laser. The presence of synchrotron side-bands means that strong fields and high power has been obtained and, in fact, more the sidebands develop, more power is attained. The suppression of the sideband must be tailored to individual needs and applications. Decreasing the sideband growth can be realized by degrading the FEL performance. This can take several forms. The resonator \( Q \) may be decreased so that the laser saturates at a moderate field strength decreasing the synchrotron frequency. Similarly, the current density, gain, or coupling constant \( j \) may be decreased just above the resonator losses to achieve the same effect. Decreasing the \( Q \) is probably a better option because it increases the output coupling of the FEL providing more power to the user. Another degradation is to increase the initial electron energy spread over many \( \nu ' s \). Although most simulations use a monoenergetic beam, a modest energy spread is not sufficient to remove the synchrotron instability; the spread must be comparable with the gain band-width or the trapping region in phase-space in order to suppress the growth of side-bands. Qualitatively, this is similar to lowering \( j \) and is probably an undesirable cure. Short electron pulses \( (s \gg 1) \) also suppress side-band growth because the modulation wavelength is also comparable to \( s \). The short pulse does not allow even one synchrotron oscillation before the electrons and light decouple. However, for larger enough \( j \) and \( Q \) fields can become sufficiently strong to increase \( \omega > 2\pi \) making the modulation wavelength shorter than any given electron pulse length. In fact, current densities providing something like 100\% gain are capable of generating a chaotic optical pulse envelope with a broad-band linewidth which destroys the FEL coherence.

More constructive methods for suppressing side-band growth use selective resonators with Fabry-Perot plates or multilayer dielectric mirrors. These methods work by selectively increasing the losses (lowering \( Q \)) in the side-band frequency range. Since the side-band gain is negligible for \( \omega \ll 2\pi \), the selection must be narrow compared to the normal \( 1/N \) gain band-width. This is not always easy since the selective apparatus must work at high power levels to be effective.
and heating can be a problem. A somewhat more subtle approach may be effective in specialized machines where the FEL characteristics can be modified during the accelerator macro-pulse. For instance, when the undulator field is provided by an ironless electromagnet, there is the possibility of decreasing B slowly, over many passes, while the optical field strength E is increasing. If B decreases while E increases so that a constant $\sqrt{2}\pi$, so that more optical power can be extracted from the FEL while avoiding the synchrotron instability.

**PHASE SPACE EVOLUTION**

Fig. 17 - Twenty sample electrons representing a current density $j = 1$ start on resonance in the self-consistent phase space $(\xi, \nu)$ of a tapered wiggler with $\delta = 5\pi$. About half the electrons are trapped near resonance by strong optical fields $a = 30$. The wave is driven in a non-uniform way as shown by the gain $g(\tau)$ and shifting phase $\phi(\tau)$.

The tapered undulator design $/36,37/$ attempts to make specific use of particles trapping in order to increase the FEL efficiency or electron beam energy extraction. Imagine that just as electrons are decelerated into optical traps which normally decrease gain and cause FEL saturation, a longitudinal accelerating electric field $E$ replenished their energy to maintain energy exchange. This increases gain in strong fields and moves the saturation limit to higher power. The pendulum equation governing electron evolution acquires a constant accelerating term: $\xi = a \cos \xi$. The effect of re-acceleration can be reproduced by simply tapering the undulator wavelength $\lambda$ (and/or field strength B) along the undulator length (see Figure 17). This alters the resonance condition for fixed electron energy and optical wavelength producing the same kind of accelerating term $\delta$ and is practically more feasible. Roughly speaking, the effect of $\delta$ in strong fields where $a \approx \delta$ is to trap electrons near the phase $\xi \approx \pi$, at resonance $\nu = \xi \approx 0$ (see Figure 17). These electrons trapped at $\xi \approx \pi$ continue to drive the wave equation in fields stronger than normal saturation. Synchrotron oscillations still occur as can be seen with the substitution $x = \delta + a \xi$; then in the new coordinate we have $\omega = \sqrt{a}$.

In fact, the nominal tapered undulator seeks to have many synchrotron oscillations with larger fields $a /35/$. Some recent pulse propagation results have shown the synchrotron instability to be quite prominent $/16,17,30/$. Figure 18 shows the modulated pulse shape of a high gain tapered laser after only eleven passes and side-bands are observed in the lineshape. Similar results are obtained in Figure 19 in the simulation of future experiments at Los Alamos. However, just as side-band growth may be suppressed by short electron pulses, the actual efficiency enhancement of a tapered undulator
may be affected as well. In Figure 19 we show a modestly tapered undulator with a short pulse similar to the Stanford experiment. In fact, the parameters chosen here are equivalent to introducing a 1.5% taper to the Stanford undulator in order to attempt to extract more electron beam energy. The result however is less energy extraction because of the short pulse. The design of $\delta = 5\pi$ in the pendulum equation anticipates a single-mode amplitude $a$ for appropriate trapping and energy extraction.

But when the resulting multimode short pulse passes over electrons, $a$ is not constant and trapping is not efficient. Future experiments should beware of this potential problem.

Fig. 18 - The slippage distance is shown at the top between the arrows. The initial pulse amplitude becomes quickly distorted due to the synchrotron instability in this tapered undulator. The final spectrum with side-bands is shown at the bottom.

Fig. 19 - Tapered wiggler pulse evolution with $\delta = 5\pi$ and small desynchronism $d = .001$ results in a short optical pulse $a(z)$ with subpulse structure. Trapping is evident in the final electron distribution $f(v_e)$ when fields are strong. The power spectrum $P(v_k)$ is broad and its center smoothly evolves from $v_k = -2\pi$ to resonance $v_k = 0$ as strong fields begin to trap electrons. This example corresponds to the Stanford laser with a 1.5% taper of the magnet wavelength during $\tau = 0 \rightarrow 1$, and 25% more current ($j = 2$).
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