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LATTICE GAUGE THEORIES

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Résumé. - Les développements les plus récents dans les théories de jauge sur réseau sont discutés. Les résultats numériques obtenus avec la méthode de Monte Carlo sont analysés en détail.

Abstract. - Recent research on lattice gauge theories is reviewed, with particular emphasis being placed on numerical results from Monte Carlo simulations.

1. Introduction. - During the last few years the lattice formulation of quantum gauge field theories\(^1\) has played a fundamental role in the understanding of these systems. In particular, the lattice regularization coupled with numerical techniques known as Monte Carlo (MC) simulations has allowed to derive some impressive results for the physics of strong interactions. It is useful to review briefly the features of the lattice formulation which makes it so valuable for the analysis of strong coupling phenomena.

By defining the theory on a lattice in Euclidean space-time (normally the lattice is taken to be hypercubical, with lattice spacing \(a\)) one regularizes the ultra-violet divergences. The dynamical variables are given by finite group elements \(U_{ij}\) (belonging to the gauge group \(G\)), which are associated with the oriented links between neighboring lattice sites \(i, j\). The \(U_{ij}\)'s play the role of the expressions

\[
igA_a u \ dx^v
\]

which define the transport between neighboring points \(x^u\) and \(x^u + dx^u\) in the continuum theory (\(g\) and \(\alpha\) being the bare coupling constant and the infinitesimal generators of \(G\)). The product of the link variables associated with the sides of an elementary square of the lattice (or plaquette) \(ijkl\) plays the role of the field strength of the continuum theory: the quantity

\[
U_{ijkl} = U_{ik} U_{jk} U_{kj} U_{ji}
\]

reduces to

\[
igF_{\mu \nu} \ dx^u \wedge dx^v
\]

if a suitable continuum limit is taken. There is then some freedom in defining a lattice gauge action: a commonly used form is obtained by defining as plaquette action

\[
S_{\square} = c \ Tr (I - U_{\square}), \tag{1}
\]

where \(I\) is the identity in group space and \(c\) is an appropriate constant of normalization. The expression \(E_{\square} = Tr (I - U_{\square})\) is often called the internal energy of the plaquette. In the continuum limit \(S_{\square}\) reduces to the standard density

\[
\frac{1}{2} F_{\mu \nu} F^{\mu \nu} \ dx^u \wedge dx^v \quad \text{(no sum over \(\mu, \nu\))}
\]

appearing in the continuum action. Indeed, one can establish the correspondence
as $a \to 0$ (the former factor of $1/2$ has become $1/4$ in the last expression because the sum over $\nu, \nu$ counts now every plaquette twice). Finally, the lattice regularized quantum averages of the observables are given by

$$\langle \phi \rangle = \frac{\int \prod U_{ij} \, \exp^{-S(U,g)}}{Z}$$

(2)

$$Z = \int \prod U_{ij} \, \exp^{-S(U,g)}$$

(3)

The regularization given by Eqs. (2) and (3) breaks Poincare's invariance but offers two extremely important advantages:

i) it is explicitly gauge invariant;

ii) does not rely on a weak coupling (perturbative) expansion for its definition.

Thus, the r.h.s. of Eq. (2) can be expanded for small values of $g$ as well as for large values of $g$ (strong coupling expansion). In the former case, recalling the explicit dependence on $g$ given by Eq. (1), one starts from a configuration where all $U_{ij}$ reduce to the identity (normally one would take $U_{ij} = I$) and builds up a weak coupling expansion by adding quantum fluctuations onto this classical vacuum configuration. The expansion one so produces is the standard perturbative expansion, made more complicated by the loss of Lorentz covariance and by the higher order couplings induced by the lattice action. For weak coupling computations the lattice regularization is indeed not convenient, and it is used only when one needs to compare the results obtained by the lattice formulation with those obtained by more standard perturbative regularizations, in order to obtain information on the relative renormalization scales. But the lattice regularization is unique in allowing expansions also for large values of $g$ and, as we shall see later, a special kind of numerical computations.

It is customary to define $\beta = c/g^2$ (in analogy with $\beta = 1/KT$ of thermodynamics). Then the measure in Eq. (2) takes the form

$$\exp^{-S(U,g)} \propto \sqrt{E_F}$$

and a strong coupling expansion consists in an expansion of the measure in powers of $\beta$. To leading order the whole measure is replaced by one and the quantum averages are just averages over totally random quantum fluctuations. Terms of higher order in $\beta$ can be easily incorporated.

The possibility of performing calculations at large coupling might appear, at first sight, the solution to all those problems in the physics of strong interactions which, because of the effective dependence of $g$ on the scale of momenta considered (running coupling constant), are known to be dominated by strong coupling effects. Actually, the situation is not that simple. One must remember that the parameter $g$ appearing in the lattice formulation is a bare coupling constant. Eventually one wants to remove the regularization by letting the lattice spacing $a$ approach zero. As $a$ is made smaller $g$ must also be readjusted, so as to keep physical quantities unchanged. A physical correlation length $z$, for instance, would be given by a formula

$$z = \lambda(g)a$$

(5)

where $\lambda$ is the dimensionless function of $g$ which expresses the correlation length in number of lattice spacings. Clearly, a continuum limit will only exist if there is a critical value of $g$ such that at $g = g_{cr}$ $\lambda(g) \to \infty$. Then one can correlate $a$ and $g$ in such a way that $g$ remains fixed as $a \to 0$ and $g + g_{cr}$. For the physically interesting asymptotic free theories one expects $g_{cr} = 0$ and a behavior (which can be inferred from perturbative considerations if $g_{cr} = 0$).
for $g = 0$ (where $A$ is the lattice scale parameter and $c_0$, $c_1$ and $c_2$ are constants which depend on the gauge group considered). Equivalently

$$a(g) = \frac{1}{A} \left( \frac{c_0}{g^2} \right)^{c_1} e^{-\frac{c_0}{g^2}} \left( 1 + O(g^2) \right)$$

for $g \to 0$. Thus we see that, although the lattice regularization allows one to perform expansions for small values of $g$, these must be extrapolated all the way to $g = \infty$ to infer information about the continuum limit.

As a matter of fact, the results obtained at finite $\beta$ need not be extrapolated up to $\beta = \infty$: it is enough to extrapolate to a value of $\beta$ sufficiently large so that the scaling behavior expressed by Eq. (7) be observed (a quantity $u(\beta)$, for instance, expressing a mass $m$ in lattice units should behave as $u(\beta) = u_0 \ a(\beta)$ in the scaling domain; this would mean that $m = u(\beta)a(\beta)$ remains constant when the lattice spacing is further reduced, and indeed that the physical value of $m$ is given by $u_0 A$). The question then arises: are the strong coupling expansions that one can presently derive adequate to see the onset of the scaling behavior? Figures 1 and 2, (from Ref. 2) illustrate the state of affairs for two typical quantities of interest: the string tension $\sigma$ and the mass-gap $m_g$ in the SU(2) lattice gauge theory.

\[ \text{Fig. 1: Strong coupling results for the SU(2) string tension.} \]
The dashed-dotted lines represent the results obtained for the dimensionless quantities \( \kappa(\beta) = a^2 \) and \( M(\beta) = m_a a \) from straightforward expansions in \( \beta \) carried to the highest or next-to-highest available terms. The dashed and solid lines represent results of resummations performed to improve the properties of convergence. The dotted lines represent the expected scaling behavior with the constant of proportionality determined from Monte Carlo simulations (the three lines represent the fit to the numerical results and estimated errors). We see that the results of the strong coupling expansions are not inconsistent with the scaling behavior (as numerically determined), but they fail to give any evidence for it: in other words they appear to break down precisely where the scaling towards the continuum limit seems to set it.

The great success of Monte Carlo numerical simulations, as applied to lattice gauge theories* comes from the fact that these computations provide information on several observables for a range of values of \( \beta \) which extends beyond the typical values where the strong coupling series become inadequate: indeed, the domain of validity of the computations is large enough that one can see the onset of scaling to the continuum limit and estimate the value of several physical quantities. Figures 3 and 4 illustrate for instance the MC determination of the string tension in the SU(2) gauge theory.

*For an explanation of the Monte Carlo technique as applied to lattice gauge theories see, for instance, Ref. 3 or Ref. 4.
One determines numerically the same quantity $\kappa = g a^2$ as in Fig. 1, however, whereas the string tension is defined as the constant force between two static sources in the fundamental representation at infinite separation, the maximum separation between the sources in the numerical computation is limited by statistical fluctuations. Thus one measures the force (in lattice units) $\kappa(\beta, I)$ at separations of $I = 1$ to 5 lattice spacings. $\kappa(\beta) \equiv \kappa(\beta, \infty)$ should emerge as the envelope of all the curves at finite $I$. The tendency of the curves to determine an envelope with the correct scaling behavior (represented by the three parallel lines) is apparent in Figure 3.
During the past year remarkable progress has been made in the study of lattice gauge theories. Several of the new results have been obtained by Monte Carlo simulations; quite noteworthy analytical developments have also taken place. In this talk I shall concentrate on the numerical results, but not because I consider the other ones less relevant. On the contrary, I am convinced that our ultimate understanding of quantum gauge theories will derive from an intelligent combination of analytical and numerical investigations. I shall put more emphasis on the results from Monte Carlo simulations because some of the most important recent analytical breakthroughs will be reviewed in Brezin's talk and because I do not have the time to discuss both lines of development adequately.

Insofar as Monte Carlo simulations are concerned progress has been made both in the analysis of pure gauge systems and in the analysis of systems with fermionic fields. To illustrate the high degree of information that some of these investigations now provide, let me reproduce in Fig. 5, taken from a paper of Stack, contributed to this Conference, the determination of the potential between static, \( I = 1/2 \), sources in the SU(2) gauge theory.

Separations and potential are expressed in units of the appropriate power of the string tension. The continuous line represents a fit in terms of a superposition of a linear term and a Coulombic term with running coupling constant. The accuracy achieved in the determination of the potential is impressive.

A problem which has been addressed is how well the rotational invariance of the system is recovered as one proceeds toward the continuum limit. Figures 6 and 7 illustrate a determination of equipotential surfaces (from Ref. 8) in the SU(2) model at values of the coupling parameter \( \beta = 2 \) and 2.25, respectively, which place the system just before the observed onset of scaling and at the beginning, but already inside, of the scaling region. (The dots do not represent lattice points, but rather points in space where the interpolated potential assumes fixed values, and the curves represent fits in terms of a circular shape plus the lowest harmonic distortion compatible with the symmetry.)
The restoration of rotational symmetry as one enters the scaling region is evident.

One of the very interesting analytical developments which should be mentioned at this point consists in the formulation of a field theory, in general, and more specifically of a gauge field theory, on a random lattice, given by Christ, Friedberg and Lee. Such a formulation achieves rotational symmetry independently of any continuum limit. The results of Ref 8 indicate that, for computational purposes only, one need not worry about rotational symmetry: this appears to be recovered well enough as one enters the scaling region, which must anyway be attained to obtain physical observables. But this does not detract from the extreme elegance and
conceptual interest of the formulation proposed in Ref. 9.* Another problem which has been studied numerically and analytically concerns the dependence of scaling on the assumed form of the lattice action. I would like to recall some early results on this problem which were published some time ago in Ref. 11. It was found there that if one changed from Wilson's form of the SU(2) lattice action to Manton's form or to the heat kernel form (which amounts to changing the scheme of regularization) the results for the string tension exhibited rather similar patterns of scaling, although shifted in the values of the coupling parameter $\beta$: this shift can be interpreted as due to the expected variation of the scale parameter $\Lambda$ as one changes scheme of regularization.

The simplest modification in the SU(2) model consists in adding to Wilson's action (which is basically the sum of the characters of the plaquette transport operators in the fundamental representation, multiplied by a coupling parameter $B_F$) a term containing the sum of the characters in the adjoint representation, multiplied by another coupling parameter $B_A$. The universality property of the ensuing system (with respect to the continuum limit) have been studied in Ref. 11-14 and have been discussed in one of the parallel sessions by Korthals-Altes. One can determine numerically the curves where the string tension $\kappa$ takes constant values (in lattice units) within the $B_F-B_A$ plane and compare with the lines of constant $\kappa$ that one would infer from a perturbative analysis coupled with renormalization group arguments. There are regions of the plane where the values of $B_F$ and $B_A$ are large enough that the system should scale yet the two sets of curves are in marked disagreement. This appears to occur, however, as one approaches the ratio between $B_F$ and $B_A$ where the action, as a function of the SU(2) invariant rotation angle $\phi$, develops a doubled-welled shape, with minima at values of $\phi$ different from the perturbative value $\phi = 0$. The discrepancy is then not surprising and should be attributed to a failure of the perturbative analysis. Indeed, it has been shown in Ref. 14 that a self-consistent non-perturbative treatment of the adjoint term in the action (in a way reminiscent of the Hartree-Fock approximation) can lead to a resolution of the discrepancy.

As a last illustration of a topic involving pure gauge systems, but which is of high theoretical and experimental interest and which has been the subject of several Monte Carlo studies, let me mention the determination of the masses of the pure gauge quantum excitations (the so-called glueballs). The large amount of work done in this direction has been reviewed by Berg and by Shierholz, in one of the parallel sessions. A widely used procedure to obtain information on the masses in the spectrum (which, as we shall see later, has also been employed to study numerically the quark-model spectrum) consists in evaluating the connected Green's function

$$G(t) = \sum_{x<0} <\phi(x,t)\phi(0)>_{\text{conn}} \equiv \sum_{x<0} \left[<\phi(x,t)\phi(0)> - <\phi(0)>^2\right],$$

where $\phi$ is a suitable operator, coupling the vacuum to the sought for states. $x$ and $t$ label coordinates of lattice sites in Eq. (8) and the summation over space positions is made to project over states with vanishing lattice momentum. By insertion of a complete set of intermediate states $|n\rangle$ in Eq. (8), and denoting by $|0\rangle$ the vacuum state, one obtains

$$G(t) = \sum_{n \neq 0} \left< n, \hat{F}_n = 0 \right| \phi \left| 0 \right>^2 e^{-m_n t},$$

In Euclidean time the propagation factor for a physical state is a decreasing exponential $e^{-m_n t}$ (rather than the phase factor $e^{-iE_n t}$) and the projection over $\hat{F}_n = 0$ allows one to identify $E_n$ with $m_n$. We see therefore that the mass $m_n$ of the lightest state excited by the operator $\phi$ can be determined by the asymptotic rate of decay

* A departure from the standard hypercubical lattice, but which still breaks rotational symmetry, is to be found in a recent paper by Celmaster, where a body centered lattice, which has a larger symmetry group than the hypercubical lattice, is considered.
Unfortunately, the Monte Carlo measurements used in estimating the $G(t)$ glueball spectrum are unable to evaluate $G(t)$ for separations exceeding 2 or 3 lattice spacings. The reason is that for practically all values of $\beta$ $G(t)$ decreases very fast with $t$: for small $\beta$ because in the strong coupling regime the dimensionless correlation length $\lambda = 1/\lambda_m$ is quite short, less than one lattice unit. For large $\beta$, $\lambda$ is expected to increase (like $1/a$, to produce a meaningful physical value for $m_g$ in the continuum limit); but the behavior of $G(t)$ for separations of few lattice spacings is then determined by perturbation theory (because one is then still at short distances), which again predicts a very fast (although power-like) fall-off for $G$. This leaves just a small window of values of $\beta$, at the onset of scaling, where one may hope to obtain some meaningful information about the mass gap. In practice one defines effective (dimensionless) masses

$$m(t,\beta) = -a \ln \frac{G(t)}{G(t-1)}$$

(sometimes a definition

$$m(t,\beta) = -\frac{a^2}{t} \ln \frac{G(t)}{G(0)}$$

is used instead to reduce statistical errors), which are plotted as functions of $\beta$. The curves thus obtained should tend, as $t \to \infty$, to an enveloping curve $m(\beta)$, which in turn should exhibit the scaling behavior $m(\beta) = a(\beta) m_g$. One tries then to see evidence of a scaling envelope in the curves measured with limited time separation and fitting the expected scaling behavior to the points which appear closer to the envelope one estimates $m_g$. Figure 8, taken from Ref. 2, illustrates the results of such procedure for the SU(2) theory and using the plaquette action itself as the operator $\phi$ used to excite the glueball states. ($\phi$ is, more precisely, given by a sum over all possible orientations of the space-like plaquettes, which projects over the states of lowest angular momentum.) The points represent Monte Carlo measurements of the effective masses at time separations of 1, 2, and 3 lattice spacings. The solid lines denote the estimated scaling behavior and error, the dashed line represents the lowest order in the strong coupling expansion.

Fig. 8: Determination of the SU(2) mass-gap from a measurement of plaquette-plaquette correlations.
The whole analysis can be refined by using a suitable linear superposition of operators in the expression for $\mathcal{O}$

$$\mathcal{O} = \sum_i c_i \mathcal{O}_i$$

and considering the $c_i$ as variational coefficients. By maximizing the probability $\mathcal{P} = |\langle 1 | \mathcal{O} | 0 \rangle|^2$ for a transition to the lowest excitation one achieves a rate of fall-off for $G(t)$ which approaches faster the asymptotic form $e^{-m g t}$ (if one could make $\langle 1 | \mathcal{O} | 0 \rangle$ the only non-vanishing amplitude, the exact value for $m_g$ could be obtained comparing $G(0)$ and $G(t = a)$). In practice, one measures the correlation matrix between the $\mathcal{O}_i$ operators and adjusts then the variational coefficients to obtain the minimum values of $m(t, \beta)$. The results of this variational procedure, using as $\mathcal{O}_i$ the lowest characters of transport operators along paths of length four and six, are illustrated in Fig. 9 (from Ref. 2).

![Fig. 9: Variational determination of $m_g$.](image)

One estimates $m_g = (170 \pm 30) A$, which, using the Monte Carlo result, $A = 0.013 \sqrt{e} \text{ and } \sqrt{e} = 450 \text{ MeV}$, translates into $m_g = 1000 \pm 180 \text{ MeV}$, where only the error in the measurement of $m_g$ has been included.

Several groups have presented, during the last year, estimates of the SU(2) and also the SU(3) mass-gap, using the variational procedure described above, or other methods, based on finite-size scaling, change in boundary conditions, thermodynamical effects etc. I refer to the discussions presented by Berg and Shierholz for a detailed analysis of the various results as well as the appropriate references. Let me just say here that all results for the SU(2) system are compatible with the value quoted above and that the number $m_g = 800 \text{ MeV}$ for the SU(3) mass gap, always with a possible error of about 20%, seems by now pretty well established. In the parallel sections we have heard reports on Monte Carlo estimates of the masses of higher-spin glueballs. One tends to find values quite larger than those obtained for the $0^{++}$ state. While these may not be in disagreement with recent experimental findings, the fact that the corresponding correlation lengths appear to be much shorter than the one for the $0^{++}$ state leaves open the possibility that contributions from higher mass excitations be still relevant. Thus I would tend to consider the results for glueballs of higher spin still preliminary.

There is one more general comment which I would like to make here. In the MC
determinations of the quark model spectrum one also studies the rate of decay of suitable correlation functions. Yet this can be followed over the entire time extent of the lattice rather than over 2 or 3 lattice spacings only. Fig. 10 (from Ref. 15) illustrates the point. It exhibits the behavior of the Green's function \( A_n(t) \) for a product of operators which excite and then annihilate states with the quantum numbers of the pion.

![Fig. 10: Behavior with time separation of the \( n-n \) correlation function.](image)

Periodic boundary conditions are assumed with a total extent in time equal to 16 lattice spacings, so eventually \( A_n(t) \) must increase again. But there is a range of values of \( t \) over which an almost purely exponential rate of decay is clearly observed. It would be invaluable for the estimates of the mass-gap if one could determine the value of the corresponding Green's functions over comparable ranges and with comparable accuracy.

The reason for the much higher precision in the measurements relative to quark model states is that one deals in this case with two separate type of fields: the gauge field and the fermionic fields. The latter propagate in a background given by a definite MC configuration of the gauge field (I shall consider more explicitly these computations later) and their propagation is determined by a linear operator. Finding the Green's functions is tantamount to finding the inverse of this operator, which can be done by accurate numerical techniques. An average over all possible background fields must still be performed, but one does not rely on this average to produce an exponential rate of fall-off. The rapid decrease of the Green's functions is, rather, a consequence of the elliptic nature of the operator being inverted and occurs within each background field configuration. The separation between a background field and a propagating field is not available in the case of the mass-gap: both are given by the same gluonic gauge field and the fall-off of the Green's functions is entirely a consequence of cancellations among contributions from different configurations occurring in the Monte Carlo averaging procedure. Statistical fluctuations make then the measurement of small values of \( G(t) \) impossible.

The distinction between propagating gluons and gluons which provide either a confining force or an effective mass is nevertheless common to all models (bag model, potential models) which deal with glueball states phenomenologically and it appears to me that the same separation could be performed, albeit introducing some approximation, also in the framework of MC simulations. I shall indicate very concisely how one might proceed, considering for the exemplification the case where one wants to evaluate a Green's function \( G(x,y) = \langle \Phi(x)\Phi(y) \rangle_{\text{connected}} \) in a theory
with fields $\phi$ and action $S(\phi)$. $G(x,y)$ can be expressed as

$$\frac{\delta}{\delta J} \langle \phi^+_x \rangle \bigg|_{J=0}$$

where

$$\langle \phi^+_x \rangle = \frac{\int \delta \phi \phi_x e^{-S(\phi) + J\phi_y}}{\int \delta \phi e^{-S(\phi) + J\phi_y}}$$

represents the expectation value of $\phi_x$ in the presence of a source of strength $J$ located at $y$. (Indices other than those labelling positions are left implicit.) A crucial step in the Monte Carlo algorithm consists in evaluating the variation in action induced by a variation $\delta \phi$ of the fields. The evolution of $\phi$ in the stochastic process is determined by such a variation. If we assume $\delta \phi$ to be small (one would always expect small variations to become dominant as one approaches the continuum limit) we can expand $S$, finding a total variation of action (in the presence of the source)

$$\delta S_{tot} = \delta S - \delta \phi_z - J \delta \phi_y$$

(a sum over the index $z$ is implied). The idea is now to associate to every configuration $\phi$ a shifted configuration $\phi' = \phi - JX$ such that $\delta S_{tot}$ becomes equal, up to terms of order $J^2$, to the variation of the original action $S$ without source, when expressed in terms of $\phi'$

$$\delta S_{tot} = \delta S - \delta \phi_z = \delta \phi'$$

If we treat the shift $X$ as if it were independent of $\phi$, Eqs. (12) and (13) give us

$$\frac{\delta S}{\delta \phi_z} (\phi = \phi - JX) \delta \phi_z = \frac{\delta S}{\delta \phi_z} (\phi) \delta \phi_z - \frac{\delta^2 S}{\delta \phi_z \delta \phi_w} JX_\phi \delta \phi_z = \frac{\delta S}{\delta \phi_z} \delta \phi_z - J \delta \phi_y$$

i.e.

$$\frac{\delta^2 S}{\delta \phi_z \delta \phi_w} X_\phi = \delta_{z,y}$$

since the equality should be valid for all variations $\delta \phi_z$. If we let $\phi$ evolve stochastically according to the action $S$ (without source term) the equality between $\delta S$, as function of $\phi'$, and $\delta (S - J\phi)$, as function of $\phi$, will imply that averages over $\phi'$ with measure $e^{-S}$ will produce the same result as averages over $\phi$ with measure $e^{-S+J\phi}$. Thus we can evaluate

$$\langle \phi^+_x \rangle_J \text{ as } \frac{\int \delta \phi (\phi^+_x + J\phi_x) e^{-S(\phi')}}{\int \delta \phi e^{-S(\phi')}}$$

(up to order $J^2$) and

$$\langle \phi^+_x \phi_y \rangle = \frac{\int \delta \phi \phi_x e^{-S(\phi')}}{\int \delta \phi e^{-S(\phi')}}$$
\( \chi_x \) represents the results of a perturbation in the field configuration introduced by the source located at \( y \) and propagating to the point \( x \) according to the linearized Eq. (14). Eq. (15) instructs one to average this perturbation over all field configurations to obtain \( \langle \Phi_x \Phi_y \rangle \); however, in analogy with the case of fermionic fields, one would expect the fall-off of \( \langle \Phi_x \Phi_y \rangle \) for large separation to be determined by the elliptic nature of Eq. (14) and to occur configuration by configuration. A numerical solution of Eq. (14) would then allow estimates of the Green's functions at large separation as well.

The approximate nature of Eq. (15) derives from the assumption that \( \gamma \) may be treated as independent of \( \Phi \). This is correct only if the action is quadratic (and then Eq. (15), reducing to \( \langle \Phi_x \Phi_y \rangle = \chi_x \chi_y \), becomes of course exact). Taking into account the dependence of \( \gamma \) on \( \Phi \) would introduce modifications to the measure to be used in Eq. (15). I would tend to consider the scheme of approximation defined by Eqs. (14 and 15) as analogous to the quenched approximation, used in connection with fermions, to be discussed later. Applied to the self-interacting gauge field (with some care because of the constrained nature of the possible variations \( \delta \Phi \equiv 0 \)) it artificially separates a few propagating gluons from those that form the quantum vacuum, reproducing the analogous separation used in phenomenological models, but without free parameters.

Let me now consider the topic of lattice gauge theories with fermionic fields, another subject where analytical and numerical research endeavors have been prominent during the past year. Interesting conceptual problems arise in the very formulation of the theory: a straightforward adaptation of Dirac's Lagrangian to the lattice leads to a fermionic system which exhibits an unwanted degeneracy when one tries to recover the continuum limit. One finds that, beyond the obvious modes where the field varies slowly from point to point, the continuum limit spectrum also receives contributions from modes where a smooth variation from lattice point to lattice point is modulated by an alternation of signs. Equivalently, in momentum space one obtains a continuum action not only expanding the momenta about the origin, but also about the points in the Brillouin zone where any of the momentum components is shifted by \( \pi/a \) (a denotes the lattice spacing). The neighborhoods of all these points contribute equally to the continuum limit, which contains therefore \( 2^d \) (\( d \) being the dimensionality of space-time) degenerate fermions.

A simple way to see why the \( 2^d \) degeneracy is present consists in noticing that the equation for the propagation of fermionic fields involves first order derivatives (rather than second order derivatives). On the lattice these are expressed as central differences: \( \delta \Phi_{x_+} \), for instance is replaced by \( \langle \Phi_{x_+} - \Phi_{x_-} \rangle / 2a \). But this procedure effectively makes the size of the unit cell equal to two lattice spacings: the unit cell contains then \( 2^d \) points and this degeneracy persists in the continuum limit.

It might seem that one could circumvent the problem by less straightforward transcriptions of the fermionic action onto the lattice, and indeed the degeneracy can be reduced by suitable formulations of the lattice theory. There are however no-go theorems which state that not all the features of the continuum system can be carried over onto the lattice. These theorems are based either on considerations involving the anomaly in the U(1) axial current\(^{16} \) or on topological arguments\(^{17} \). The topological no-go theorems have been reviewed in the parallel sessions by Ninomiya.

The outcome is that it is impossible to formulate a theory of fermions on the lattice preserving a diagonal U(1) chiral symmetry and locality of the interactions (in the sense that these are of finite range): in particular, one cannot have single-handed neutrinos because chiralities must come in pairs.

Thus, the problems of degeneracy are seen to be of deeper origin than from the doubling in size of the unit cell: to remove the unwanted fermionic modes one must introduce Lagrangian interactions of infinite range (as in the SLAC formulation of lattice fermions\(^{18} \) or terms, which explicitly break chiral invariance and get rid of the degeneracy by giving to the undesired fermions masses of order \( 1/a \), as in the formulation proposed by Wilson\(^{19} \). Another method of reducing the fermionic degeneracy, introduced by Susskind in a slightly different context\(^{20} \) (spatial lattice with continuous time rather than Euclidean space-time lattice), consists in assigning different components of the fermionic fields to the different points on the lattice. The degeneracy, in four dimensions, is then lowered to four. A very interesting recent development, illustrated by Horn in one of the parallel sessions, has been the...
recognition that the Susskind formulation is the natural generalization to the lattice of a version of the Dirac equation based on differential forms rather than spinors. I refer to Horn's contribution for details and literature.

A remarkable advantage of Susskind's formulation is that, when the bare mass parameter of the fermions is set to zero, the system becomes invariant under a one-parameter continuous group of chirality transformations. These chiral rotations do not correspond to the diagonal (singlet) chiral transformations of the continuous theory (which would violate the no-go theorems), but rather, to a non-diagonal chiral rotation mixing the various degenerate St

In one of the parallel sessions Morel has given an account of the symmetry properties of lattice gauge theories with Susskind fermions as well as of results obtained for the spectrum in the framework of strong coupling expansions. Here let me mention only that the strong coupling results neatly confirm the spontaneous breakdown of chiral symmetry and the presence of a massless pseudoscalar boson in the limit where the fermion (quark) mass goes to zero. As a matter of fact, the leading order of the strong coupling expansion gives reasonably good results also for the ratios of masses among the other low lying states in the spectrum. Of course, what one cannot obtain in strong coupling is an absolute normalization for these masses, relating them for instance to the string tension. For this purpose, as in the case of other observables, one must be able to perform computations at values of the coupling constant where the scaling towards the continuum limit has already begun. Such computations have been attempted by the Monte Carlo technique.

Monte Carlo simulations of systems with fermions are made particularly difficult by the anticommuting nature of the fermionic fields. A typical Green's function used to estimate meson masses (see Eqs. 8, 9 and 10) may involve evaluating the expectation value of the product of 4 fermi fields $\langle \psi_\alpha \bar{\psi}_\beta \gamma^\mu \gamma^\nu \psi_\gamma \bar{\psi}_\delta \rangle$. Using a very concise notation this is given by

$$\langle \bar{\psi}_\delta \gamma^\mu \gamma^\nu \psi_\gamma \bar{\psi}_\beta \gamma^\epsilon \psi_\alpha \rangle = Z^{-1} \int d\mathbf{U} \int d\mathbf{\bar{U}} \exp \left\{ -S_G(\mathbf{U}) + \bar{\psi}_\delta \gamma^\mu \gamma^\nu \psi_\gamma \bar{\psi}_\beta \gamma^\epsilon \psi_\alpha \right\}$$

In Eq. (16) $\bar{\psi}$ and $\psi$ represent the fermionic fields, which are elements of a Grassmann algebra and are defined at the lattice sites; $S_G(\mathbf{U})$ is the pure gauge action; $\mathbf{U}(U) + \mathbf{m}$ represents the lattice generalization of Dirac's operator, with a dependence on $U$ introduced by the replacement of finite differences with covariant finite differences; $\mathbf{J}(U) + \mathbf{m}$ is the fermionic lattice action, bilinear in $\bar{\psi}$ and $\psi$; and finally $Z$ is given by an integral analogous to the one appearing in the r.h.s. of Eq. (16), but with an integrand given by the measure factor $\exp \{- S_G(U) \bar{\psi}(\mathbf{U}) + \mathbf{m}) \bar{\psi}(\mathbf{U}) + \mathbf{m}) \psi(\mathbf{U}) + \mathbf{m}) \psi(\mathbf{U})$ alone.

The integration over the anticommuting fields $\bar{\psi}$ and $\psi$ really serves to reproduce a summation over all possible occupations of the fermionic states. Once the system is discretized by the introduction of the lattice and made of finite volume by imposing, for instance, periodic boundary conditions, the number of fermionic states becomes also finite. It would appear, therefore, that it ought to be possible to reformulate the Green's functions in terms of sums over a finite set of binary variables, taking values 0 or 1, and that this sum should be amenable to the Monte Carlo procedure. Unfortunately, once the quantum averages are so re-expressed, the measure loses the character of positive definiteness, which is so crucial for its stochastic interpretation and the implementation of the Monte Carlo algorithm. Only in two dimensions (1 space and 1 time) has it been possible to express the sum over configurations as a sum over occupation numbers involving a positive definite measure, and to approximate it quite efficiently by the Monte Carlo method. For systems of higher dimensionality a different procedure has been followed.

Because of the Gaussian form of the measure for $\bar{\psi}$ and $\psi$, the integration over these variables can be performed explicitly and one obtains

$$\langle \bar{\psi}_\delta \gamma^\mu \gamma^\nu \psi_\gamma \bar{\psi}_\beta \gamma^\epsilon \psi_\alpha \rangle = Z^{-1} \int d\mathbf{U} \int d\mathbf{\bar{U}} \exp \left\{ -S_G(\mathbf{U}) + \bar{\psi}_\delta \gamma^\mu \gamma^\nu \psi_\gamma \bar{\psi}_\beta \gamma^\epsilon \psi_\alpha \right\} \times \left\{ \det(\mathbf{U}) + \text{Tr} \ln(\mathbf{U}) \right\}$$

(17)
In Eq. (17) \(<\Phi|\Phi>U\) stands for the expectation value of the product of fermionic fields in the background given by a fixed gauge field configuration U. If, for instance, \(\Phi_1\) and \(\Phi_2\) carry different quantum numbers (to reduce the number of contractions), this expectation value is given by

\[
<\Phi_1|\Phi_2>U = -<\Phi_1|\Phi_2>U = -<\Phi_1|\Phi_2>_U^{-1}\delta_{\alpha\beta} <\Phi_1|\Phi_2>_U^{-1}.
\]

We see therefore that the evaluation of expectation values \(<\phi>\) of fermionic observables \(\phi\) is amenable to a two stage process:

i) the expectation value of \(\phi\), \(<\phi>_U\), in the background given by a fixed gauge field configuration is calculated.

ii) \(<\phi>\) is then obtained by averaging \(<\phi>_U\) over all gauge field configurations with a measure

\[
S_{eff}(U) = S_G(U) - Tr \ln (\mathcal{M}(U)+m).
\]

A major difficulty for Monte Carlo simulations is to incorporate the effects of the factor \(\exp{Tr \ln (\mathcal{M}(U)+m)}\) in the integration over gauge field configurations. For the Monte Carlo procedure one needs to evaluate the variation of \(S_{eff}(U)\) when one of the link variables \(U_{ij}\) is modified. Whereas this is straightforward for the term \(S_G(U)\), the non-local nature of the second term in \(S_{eff}\) makes an exact evaluation of \(\Delta S_{eff}\) computationally quite demanding. A major effort has gone into finding efficient ways to evaluate \(\Delta S_{eff}\). The most promising techniques start from the observation that \(\Delta S_{eff}\) can be computed easily if the inverse of the matrix \((\mathcal{M}(U)+m)\), (i.e. the fermionic propagator with background gauge field U) is known. This is easy to see for small variations of U, since one can then expand

\[
\Delta \text{Tr} \ln (\mathcal{M}(U)+m) = \text{Tr}[\mathcal{M}(U)+m]^{-1} \Delta \mathcal{M}(U). \tag{20}
\]

The local nature of \(\mathcal{M}(U)\) makes it immediate to evaluate \(\Delta \mathcal{M}(U)\), and therefore also \(\Delta \text{Tr} \ln (\mathcal{M}(U)+m)\) if \((\mathcal{M}(U)+m)^{-1}\) is known. (As a matter of fact, because of the sparse nature of the matrix \((\mathcal{M}(U)+m)\), if its inverse is known the exact variation of \(\text{Det}(\mathcal{M}(U)+m)\) can be computed with a rather limited number of arithmetic operations.) Various proposals differ then in the method followed to evaluate \((\mathcal{M}(U)+m)^{-1}\) (or, better, those matrix elements of \((\mathcal{M}(U)+m)^{-1}\) which are needed to compute \(\Delta S_{eff}\)). In Ref. 25 it was suggested that one evaluates the required matrix elements of \((\mathcal{M}(U)+m)^{-1}\), which in fact constitute the fermionic current coupled to the gauge field \(U_{ij}\), by applying a Monte Carlo algorithm to a set of complex fields \(\psi\) and \(\bar{\psi}\), called the pseudofermionic variables, coupled to U in the same was as \(\Phi\) and \(\bar{\Phi}\). The idea is to perform only small variations of U in the upgrading, to evaluate the whole of the fermionic reaction at U fixed, then to proceed to an upgrade of all \(U_{ij}\) variables. The method has been successfully applied to the 2-dimensional Schwinger model in Ref. 26. In Ref. 27 it was proposed that one should store all matrix elements of \((\mathcal{M}(U)+m)^{-1}\) in the memory of the computer, upgrading them together with the field \(U_{ij}\), which again is constrained to small variations. To evaluate the change of the propagator for a small variation of U again requires only a limited number of arithmetic operations. A limitation to the implementation of this proposal comes from the large amount of memory required for 4-dimensional systems of acceptable size. In Ref. 28 the suggestion was made to evaluate \((\mathcal{M}(U)+m)^{-1}\) by a stochastic technique, based on a random walk, originally due to Ulam and Von Neumann. This method has been successfully tested against some soluble models.

Finally, if Wilson's formulation of the fermionic action is adopted, it is possible to evaluate both \(<\Phi>_U\) and the effects of the fermionic determinant in the measure \(\exp[-S_{eff}]\), namely \(\exp[-S_G]\) and \(\exp[\text{Tr} \ln (\mathcal{M}+m)]\) = \(\text{Det}[\mathcal{M}+m]\), are responsible for two separate physical effects. Once the expectation value of a fermionic observable, such as \(<\Phi_1\Phi_2\>_U\), in the presence of a fixed background field U has
been computed, a further average over $U$ with measure $\exp\{-\frac{\beta}{T} \text{Tr} \ln(U^2)\}$ introduces the possibility of pair creation and annihilation, i.e., always with perturbative language, is responsible for the inclusion of inner fermionic loops. The approximation of performing the average over $U$ with a measure given by the factor $\exp\{-\frac{\beta}{T} \text{Tr} U\}$ alone suggests itself. The meaning of this approximation is that, while the quantum dynamics of the gauge field is considered exactly, the creation and subsequent annihilation of fermionic pairs is neglected. This approximation has been used successfully in several works aiming at estimating the masses in the quark model spectrum. It can be justified by the same arguments which have been put forth, in the context of QCD, to justify the dominance of planar diagrams without inner fermionic loops or related phenomenological (OZI) rules. The above approximation was also shown in Ref. 26 to produce reasonable results when applied to the Schwinger model. The name "quenched approximation" has been introduced for it in Ref. 26 (the term "valence approximation" has also been used). 

All numerical computations of quark model masses have been based on the procedure outlined for the glueball states. The masses are evaluated from the rate of decay of a suitable correlation function, as in Eqs. (8-10), where the operators are now bilinear (for mesons) or trilinear (for baryons) in the quark fields. The only parameters of the computation are the lattice cut-off parameter $\Lambda$ (see Eqs. 6 and 7) and the quark masses. But $\Lambda$ is really not a free parameter: by independent computations its value can be related to the value of other observables, such as the string tension $\sigma$. Thus, all masses can be expressed in terms of one physical quantity (typically $\sqrt{\sigma}$) and the masses of the quarks.

The quenched approximation has been used to estimate the meson masses in a simplified model with $SU(2)$ as color group. In Refs. 15 and 32 Susskind's formulation of the lattice fermions was followed, assuming a common mass $m_q$ for all quarks. It was found that the mass squared of the lowest pseudoscalar state ($\pi$-meson) approached zero linearly as $m_q \to 0$. This agrees nicely with the notion of spontaneous breaking of chiral symmetry. The experimental value of the pion mass can then be used to determine $m_q = 7$ MeV (this should be considered an average value for the masses of the up and down quarks). The masses of the lowest $1^-$ and $0^+$ ($\rho$ and $\phi$) mesons were found to be much less dependent on $m_q$ and to approach finite limits as $m_q \to 0$. Using the string tension to set the scale, values $m_\rho = 800 \pm 80$ MeV and $m_\phi = 950 \pm 100$ MeV were calculated in Ref. 15. In Ref. 32 the same analysis was extended to a study of the states in the $J/\psi$ family. An $SU(2)$ model was also investigated in Ref. 31, but with Wilson fermions. In this formalism the role of $m_q$, as a free parameter, is taken by the hopping parameter $\kappa$ and there is no value of $\kappa$ which will make the system explicitly chiral invariant. However, the numerical results did indicate that the mass of the lowest pseudoscalar meson vanishes as $\kappa$ approaches a value $\kappa_{cr}$ dependent on the coupling constant $g$. The difference $\Delta \kappa = \kappa - \kappa_{cr}$ plays then the same role as a bare quark mass parameter, and may be determined by fitting the pion mass to its correct value. The masses of the other low-lying mesonic states are found to be less dependent of $\kappa$ and, again, in satisfactory agreement with experimental values.

The results obtained within the quenched approximation for the more realistic model based on the $SU(3)$ color group are, of course, of much higher interest. This system has been considered by several groups. I have only the time for a rather general discussion of the results: this line of investigation has been reviewed by Martinelli in one of the parallel sessions and I refer to his contribution for a more detailed account.

The vanishing of the pion mass, i.e., a dynamical realization of the chiral limit, as either $m_q \to 0$ (Susskind's formulation) or $\kappa \to \kappa_{cr}$ (Wilson's formulation) is supported by all numerical studies. The other states are less dependent on $\kappa$ or $m_q$ and their masses appear to approach finite limits as $m_q$ goes to zero. These masses can then be expressed in terms of the string tension. Since the relation between $\sqrt{\sigma}$ and $\Lambda$ is less precise in $SU(3)$, however, some groups prefer to use the mass of a definite state, e.g. $m_\rho$, to set the scale: the values for the other masses and $\sqrt{\sigma}$ follow and can be compared with experimental results. The
determination of the masses is affected by various sources of errors: the lattices are of rather small size (a typical lattice could extend for 6 sites in each of the spatial directions and 10 or 12 sites in the temporal direction, larger lattices require prohibitive amounts of computer time) and finite size effects can affect the results both because the hadron barely fits inside the spatial volume or because the rate of decay of $G(t)$ cannot be followed for a sufficiently long time; in Wilson's formulation the numerical methods used to evaluate $(\Delta m)^{-1}$ fail to converge before $k$ reaches $k_{cr}$ and a substantial extrapolation in $\kappa$ is often required; long range statistical fluctuations are present. If all these limitations are taken into consideration, then the Monte Carlo results for the quark model spectrum appear to be rather satisfactory. Beyond the evidence for the dynamical realization of chiral symmetry, the masses of most low lying states are found at values which are consistent, within estimated errors ranging from 10 to 20%, with the experimental spectroscopic data. Also, the quenched approximation has been used to compute the values of other observables, such as $f_{p}$, $f_{q}$, or baryonic magnetic moments again with results compatible with experiment.

A very substantial amount of work remains to be done. The results of the quenched approximation must be refined; in particular, it would be quite useful to extend the computations to large lattices. The possible sources of error must be better understood. Also, there are observables which cannot be calculated within the quenched approximation, because they are fundamentally dependent on closed fermionic loops: the mass splitting between the $\eta$ and $\eta'$ constitutes an example. To evaluate such quantities one needs to go beyond the quenched approximation and include effects of the fermionic dynamical reaction. Moreover, the validity of the quenched approximation itself ought to be numerically verified. Work to include the effects of the fermionic determinant in the measure is in progress. From preliminary results it appears that at least the methods proposed in Refs. 25 and 28 can be used in the context of a realistic SU(3) model without excessive requirements on computer time.

I would like to summarize, saying that Monte Carlo computations for lattice gauge theories have produced a wealth of numerical results, generally in good agreement with experimental data when these are available, sometimes constituting parameter free predictions of the theory, as in the case of the glueball mass or the deconfining temperature. In a sense it is almost surprising that such results could be obtained, since the four-dimensional nature of the systems, implying a very rapid increase of degrees of freedom with linear size, effectively limits the computations to rather small lattices. How can one see, on such coarse approximations to space-time, any detail of hadronic structure? The answer lies in the fact that the scaling toward the continuum limit appears to set in when the correlation length is still very small, on the order of 2 to 3 lattice spacings. This is most likely a reflection of the rather abrupt transition between confined quarks and almost free quarks occurring in the physical world over a small variation of scale. Monte Carlo computations verify and exploit this fact — do not explain it.

Investigations based on analytical rather than numerical techniques may be useful to develop a deeper understanding of the dynamics of strong interactions. This last year has witnessed remarkable progress also in the development of analytical methods, in particular in the use of mean field expansions and of an approximation based on the replacement of the SU(3) color group with a more general SU(N) group. Indeed, it has been shown that, as N becomes large, a set of trade-off occurs between ordinary space-time degrees of freedom and degrees of freedom within the group manifold, and, in the limit $N \rightarrow \infty$, the infinite volume system becomes equivalent to a system in which space-time has a single vertex, four links and six plaquettes. Both mean field techniques and large-N expansions will be reviewed in Brezin's talk. Yet, entirely analytical approaches are likely to remain inadequate for years to come: probably the most gratifying future results will derive from an intelligent coupling of analytical and numerical techniques.

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Bibliography

   H. Kluberg-Stern, A. Morel and B. Peterson, Phys. Lett., to be published and
Discussion

N.N. Khuri (Rockefeller Univ.). - I would like to make a remark:
1) It is possible to get exact results for four dimensional gauge theories not in the form of equalities but numerical inequalities. These results follow from simple analytical techniques.
2) Lattice gauge theorists should try to test their method by checking whether they satisfy these results. For the specific matrix element I mentioned yesterday to our speaker, he informed me that it was not possible even to get a definite result by lattice methods.
3) Some of the folklore about large $N$ behavior can be shown to be just that — and in fact wrong.
4) I hope young physicists will not be like lemmings and at least check if other methods are available before plunging into a lattice or $1/N$ calculation. With a little thought they might be surprised.

T. Truong (Ecole Polytechnique).- Would you like to comment on the status of the $n'$ or the $U_q(1)$ problem. The lattice gauge calculation should provide a natural answer to this problem which other phenomenological models are unable to do.

C. Rebbi.- The splitting between $\eta$ and $\eta'$ can be evaluated in the framework of lattice gauge theories. The computation requires however going beyond the quenched approximation, because the $\eta-\eta'$ splitting is induced by the effects of fermionic loops.