HYDROGEN ATOM IN STRONG MAGNETIC FIELDS: REGULAR AND IRREGULAR MOTIONS

M. Robnik

To cite this version:
M. Robnik. HYDROGEN ATOM IN STRONG MAGNETIC FIELDS: REGULAR AND IRREGULAR MOTIONS. Journal de Physique Colloques, 1982, 43 (C2), pp.C2-45-C2-61. <10.1051/jphyscol:1982205>. <jpa-00221814>

HAL Id: jpa-00221814
https://hal.archives-ouvertes.fr/jpa-00221814
Submitted on 1 Jan 1982

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
M. Robnik

Institut für Astrophysik, Universität Bonn, Auf dem Hügel 71, D-5300 Bonn, F.R.G.

Résumé.- La transition stochastic dans le système hamiltonien classique associé à l'atome d'hydrogène dans un champ magnétique intense est mise en évidence au moyen d'applications de Poincaré. Le résultat est discuté dans le contexte de la théorie générale de la destruction des tores. La découverte d'une troisième intégrale du mouvement lorsque l'énergie est inférieure à l'énergie critique peut permettre d'expliquer l'existence d'anticroisements exponentiellement petits des niveaux d'énergie lorsqu'ils mouvement est régulier. Au-dessus de l'énergie critique, le mouvement stochastique est responsable de la structure irrégulière du spectre dans le régime d'inter-n-mixing. La régularité du spectre dans le régime des résonances quasi-Landau est associée à l'existence d'invariants adiabatiques au-dessus de la limite d'ionisation.

Abstract.— The stochastic transition in the classical Hamilton system describing the hydrogen atom in a strong magnetic field is observed by means of Poincaré mappings, and discussed in the framework of the general theory of tori destruction. The existence of the third integral in the regular region below the critical energy can explain the exponentially small separations at avoided crossings of the energy levels, while the stochastic motion above the critical energy accounts for the irregular structure of the spectrum in the inter-n-mixing regime. The regularity of the quasi-Landau resonances corresponds to the existence of the adiabatic invariants above the escape energy.

1. Introduction.— There are many reasons that so much attention is being paid to the problem of the hydrogen atom in strong magnetic fields. For example, in astrophysics we have firm evidence that strongly magnetic white dwarfs can have magnetic fields up to $5 \times 10^8$ G [1,2]. This has been inferred from polarization measurements in the optical continuum initiated by Kemp et al. [3]. On the polar caps of the neutron stars the field can be as high as $5 \times 10^{13}$ G. This value is in agreement with the observed pulsar spin-down rates, assuming that they are isolated rotating neutron stars with known moment of inertia [4]. Neutron stars in binary systems can accrete material flowing from its normal companion star. The material falling down onto the polar cap of the neutron star gains the gravitational potential energy of the order of 10% of its rest mass. This will be ultimately dissipated and radiated away. The high temperature of several $10^6$ K then means that we see the star in X-rays [5], and gives us a chance of a direct measurement of the surface magnetic field. Indeed, Trümper et al [6] discovered an electron cyclotron line in the X-ray spectrum of the binary X-ray source Her X-1, from which they inferred $B=5 \times 10^{12}$ G for the magnetic field. Other lines, e.g. FeXXVI, have been observed in X-ray binaries [7]. It should be noted that the observed magnetic fields in white dwarfs and in neutron stars are consistent with the predicted values, when these objects are assumed to be collapsed normal stars. Their high magnetic field is simply explained as a compressed initial magnetic field of some 10G, which occurs in some main-sequence stars. We are thus safe in believing that the magnetic fields in astrophysical objects can have strengths by a factor of $10^7$ in excess of those available in laboratory.

A quantitative theoretical understanding of the physical and chemical properties of matter in strong magnetic fields is thus of great importance in astrophysics.

Note that in the astrophysical situations sketched above the magnetic field is so strong that the spacing between the Landau levels $\hbar \omega, \omega$ being the cyclotron fre-
frequency, becomes comparable or even greater than the spacings between the unperturbed atomic levels. The magnetic interaction is competing with the electrostatic interaction of comparable strength, reflecting also the approximate equality of the Landau and atomic radii. That part of the spectrum, where these conditions are met, has been called the "strong field mixing regime" [8] and is believed to have some quite general features common to all situations, where two equally strong forces of different symmetries are acting [9]. For the hydrogen atom the critical magnetic field \( B_0 \) is defined by the equality of the ionization energy in vacuum of 1\( R_y \) and of the Landau energy \( \hbar^2/2mL^2 \), so that \( B_0 = a^3 e^2/r_0 = 2.35 \times 10^6 G (\alpha = \text{the fine structure constant}, \, r_0 = e^2/mc^2 = \text{the classical electron radius}) \). For hydrogen-like ions with the nuclear charge \( Z \) the likewise defined critical magnetic field is \( B_0 Z^2 \). For FeXXVI it equals \( 1.59 \times 10^{12} G \). \( B_0 \) is a natural unit of the magnetic field and in the following the dimensionless strength \( \gamma = B/B_0 \) will be used. For hydrogen atoms in strong cosmic magnetic fields, all levels, including the ground level, can lie within the strong field mixing regime. Moreover, in case of neutron stars, this is marginally true even for one-electron iron ions.

Now the naive question may be asked: Why is this spectral problem so difficult? We know the potential, we have the Schrödinger equation, so in principle we should be able to calculate the energy levels. However, the problem is not separable, not even approximately, so that the wonderful world of the perturbation theories appears inefficient. Nevertheless, quite accurate results on the lower levels of the spectrum have been obtained using various numerical methods, so that a lot is known now on the hydrogen spectrum in strong magnetic fields. (The reader should consult the review by Garstang [10], his and the contributions by McDowell and by Wunner to the proceedings of this conference.)

The basic method is to represent the Hamilton operator in some complete basis. The basis should be simple to handle, and should provide a rapid convergence of the eigenvalues, so that one can control the error in spite of the necessary truncation. When the hydrogen basis is chosen, the following difficulty arises: The basis is not complete in the \( L_2(R^3) \) space of integrable functions, and therefore suitable only for calculating the low levels. The basis could be completed by adding the continuum hydrogenic functions, but this leads to the practical divergence difficulties in calculating the matrix elements. A solution to this problem has been recently obtained by Clark et al [14, 15, 16]. (See also his contribution to this meeting.) Following a suggestion by Edmonds [17] they use a basis of Sturmian radial functions, allowing partial representation of the continuous spectrum. A rapid convergence of even the highly excited levels in intermediate magnetic fields (typically \( 50kG \)) has been achieved. But the method is equally well suited in cases of astrophysical interest.

The strong field mixing regime exists of course also at laboratory field strengths (a preferred value being \( \gamma = 2 \times 10^{-5} \)), but then only for highly excited levels, which are now observable thanks to the new experimental techniques (see an extensive review by Gay [18]). To be specific, there are several different regimes, defined by the following apriori qualitative criteria: (i) inter-1-mixing, when the diamagnetic perturbation term \( H_0 \) of the Hamiltonian is much smaller than the spacing between the unperturbed Coulomb levels \( 2R_y/n^3 \); (ii) inter-n-mixing when they are comparable; (iii) strong field mixing regime already defined; (iv) the Landau regime, when \( 2R_y/n^3 \) is much smaller than the Landau spacing. While the nature and the origin of the equally spaced quasi-Landau resonances observed in the strong field mixing regime discovered by Garton and Tomkins [19] can be explained semiclassically [8, 9, 20-23], a lot of the structure in the inter-n-mixing regime must be explained, e.g. the observed precursors of the quasi-Landau spectrum reported by Delande and Gay [24]. Undoubtedly, the most remarkable subtlety of the spectrum is the existence of close anticrossings, which are revealed when the spectrum as a function of the magnetic field is calculated by high precision numerical methods [25, 26, 16]. The existence alone of avoided crossings is not sufficient to draw any conclusions, since the level repulsions are typical of almost all systems, even of ergodic systems without any integral at all. Berry [27] gave the important example of the Sinai billiard rigorously known to be ergodic and he calculated the exact spectrum, showing many avoided
crossings. What matters is therefore the fact that the separation of these near-degeneracies decreases exponentially with the main quantum number \( n \). This led to the conjecture that a hidden dynamical symmetry of the Hamiltonian exists, approximate but more and more accurate with increasing \( n \). Bearing in mind that degeneracies are typical of separable systems, there was a hope that the Schrödinger equation can be separated, although approximately, for the highly excited levels. Since the general understanding of the correspondence between the fine structure of the spectra and the global features of the Hamiltonian systems is still incomplete, it seems appropriate to study the classical system [28,21]: the hidden dynamical symmetry must show up as an additional integral of motion. The result of the numerical experiment, in which the Hamiltonian mappings were inspected, was not as dramatic as the conjectures. The hydrogen atom in a magnetic field is a system of the generic class. It displays all features of an integrable system for energies below some critical energy, becoming stochastic very rapidly upon increasing the energy above the critical energy. The present paper is concerned mainly with the consequences of this fact.

In any case, the genericity of the hydrogen atom in a magnetic field is a fortunate circumstance, since many features are expected to occur also in more complex atoms.

The sequence of the sections is as follows: a brief review of the classical and quantum mechanical regular and irregular motion; presentation of the Poincaré mappings; possibilities to predict the critical energy; approximate integral of motion below the critical energy; correspondence between the classical and the quantum system.

2. A brief review of the regular and irregular motion

Most of the material of this section (and much more) can be found in the recent reviews of Berry [29] and by Zaslavsky [30]. Other reviews might be appreciated by interested readers [31,32,33]. Those familiar with the subject can skip this section without any loss of continuity.

The classical Kepler problem is a good text-book example of an integrable system. Owing to its supersymmetry there are many more integrals of motion than needed for integrability. By integrable we mean that there are at least \( N \) (\( N = \) dimension of the configuration space) global, analytical, single-valued and independent integrals of motion, being in involution, i.e. such that all Poisson brackets vanish. By the important theorem stated explicitly by Arnold [34] integrability implies that each invariant surface in phase space (of dimension \( 2N \)) must have the topology of the \( N \)-dimensional torus (sphere with one handle). The phase space of an integrable system is thus filled everywhere with invariant tori. In absence of perturbations the motion is confined to an invariant torus, which justifies the attribute "stable system". It is then possible to use the topologically most natural canonically conjugate variables, namely the action-angle variables. The actions \( I_j \), \( 1 \leq j \leq N \), are defined as the integrals

\[
I_j := \frac{1}{2\pi} \oint \frac{d\theta}{\sin \theta} \text{ along irreducible circuits on the torus (these are loops, which cannot be shrunk to a point). They label a given torus, while the position of a phase point on the torus is specified by the angles \( \theta_j \), \( 1 \leq j \leq N \), canonically conjugate to the actions \( I_j \). The angle \( \theta_j \) changes by \( 2\pi \) along the closed irreducible cycle \( C_j \). The Hamiltonian reads } H = \tilde{H}(I), \text{ so that the angles are cyclic variables by the construction. The equations of motion } \]

\[
\dot{I}_j = - \frac{\partial \tilde{H}}{\partial \theta_j} = 0, \quad \dot{\theta}_j = \frac{\partial \tilde{H}}{\partial I_j} =: \omega(I) = \text{ constant,}
\]

can be immediately integrated to yield: \( I_j = \text{ constant}, \ \theta_j = \omega_j t + \text{ constant}, \) showing that the motion on an invariant torus is quasiperiodic. A torus is called rational (also resonant) or irrational according to whether the frequencies \( \omega_j = \partial H/\partial I_j \) are rationally connected or not. In the resonant case the motion can be periodic (see Fig. 1).

Incidentally, for the Kepler problem all tori are resonant, since all frequencies depend only on the value of the total energy, and are thus equal, as is well known.

What happens to tori of an integrable system under a small perturbation has been a long standing and most important question of classical mechanics. Some people like Landau thought that all systems are integrable, but that we are unable to find the integrals. The opposite extreme was favoured by Fermi, who believed that integrable
systems are exceptions (which is correct), and become ergodic\(^1\) when slightly perturbed (which is incorrect).

\(^1\)There are several equivalent definitions of an ergodic system [35-37]: (a) each invariant set has either measure zero or one; the measure is the normalized Liouville measure, and the property is called metrical transitivity; (b) the Liouville measure is the only invariant absolutely continuous measure; (c) the microcanonical distribution is the only invariant distribution; (d) the time average of each integrable function is almost everywhere equal to the phase average; it can be most clearly seen from (c) that an ergodic Hamilton system can have no integrals of motion except of the Hamiltonian itself (Jeans Theorem!). Roughly speaking, in an ergodic Hamilton system almost every point of the \(2N-1\)-dim energy surface will visit each neighbourhood of each point on this surface as time goes on.

As a surprise came then a partial answer due to Kolmogorov, Arnold and Moser (KAM Theorem): All tori of the unperturbed integrable system \(H_0(1)\) outside the resonant gaps survive a perturbation \(H_0 + \epsilon V\). The resonant gaps are centered around the rational tori, and their width is determined by the inequality [35]:

\[
\sum_{j=1}^{j=N} k_j \omega_j < K(\epsilon) \left( \sum_{j=1}^{j=N} |k_j| \right)^{-a},
\]

where \(K(\epsilon)\) is some constant vanishing when \(\epsilon \to 0\), being the same for all tori, while \(k_j\) are integer numbers. The volume of each gap is small with \(\epsilon\), and summing up the volumes of all gaps we still reach the same conclusion, since the series is convergent \((a > N-1)\). This is the basis for the condensed versions of the KAM Theorem, such as: "most tori survive", or "all sufficiently irrational tori survive", although they may be slightly distorted. However, this theorem says nothing about the motion within the resonant gaps. No general predictions are known, but there are examples for the persistence of even the rational tori, the first one having been demonstrated by Henon and Heiles [38]. But usually tori within the resonant gaps are destructed, resulting in a hierarchy of smaller and smaller tori imbedded in the chaotic region.

---

**Figure 1**

\[\text{2-dim nested tori}\]

\[\text{SOS of the integrable system}\]

\[H_0 \to \tilde{H} = H_0 + \epsilon V\]

\[\text{SOS of the perturbed system with resonant gaps of destructed tori}\]

\[\text{Magnification of an elliptic island: an infinite hierarchy}\]
It is extremely difficult to predict what is actually going to happen when a perturbation is applied. One important concept for such an analysis is the Poincaré mapping of a Surface of Section. It is a mapping of some surface \( \Sigma \) in phase space onto itself, generated by the phase flow. If the energy is the only integral of motion we know, SOS should be \((2N-2)\)-dimensional. When each orbit of given energy passes through the SOS, then the Poincaré mapping will give full information on the stability of motion. For example, when \( N=2 \), the irrational invariant torii will appear in the SOS as invariant curves, while the resonant tori will appear as a curve consisting of periodic points. Simple periodic orbits correspond to fixed points, while a chain of \( n \) periodic points is called \( n \)-cycle, and is a fixed point of the \( n \)-th iterate of the Poincaré mapping (see Fig. 1). One important property of the Poincaré mappings is that they are area preserving.

The stability analysis of periodic orbits is then reduced to the stability investigation of the Poincaré mapping or of its iterates, i.e. the stability of fixed points of some area preserving mapping \( T \). A fixed point \( x_{0} \) of an area preserving mapping \( T \) is said to be stable if for each neighbourhood \( U \) of \( x_{0} \) there exists a sub-neighbourhood \( V \subseteq U \) such that all iterates \( T^{k}(V) \) lie in \( U \). Putting the coordinate origin to \( x_{0} \), and linearizing \( T \), we find that its linear part \( L \) is a 2x2 unimodular matrix with constant real coefficients. Its eigenvalues are thus either complex conjugates \( \lambda, \bar{\lambda} \) on the unit circle, or reciprocals \( \lambda, 1/\lambda \) on the real axis. They are determined by \( \lambda^{2} - \lambda \text{Tr}(L) - 1 = 0 \). In the former case \( L \) describes elliptic rotation, the discrete motion being confined to ellipses; \( x_{0} \) is called an elliptic fixed point of \( T \). In the latter case we have hyperbolic rotation, the motion being confined to hyperbolae on the same or interchanging branches, depending on whether the eigenvalues are positive or negative, respectively. A hyperbolic fixed point of \( T \) is unstable, and nearby orbits separate exponentially. An elliptic fixed point is stable (and nearby orbits separate linearly) except in cases of low-order resonances, i.e., in cases that the rotation angle of \( L \) is \( 2m/m_{0}, m=1,2,3,4 \), where the linear stability analysis is not sufficient (Moser Twist Theorem \[35\]; \( T \) is assumed to be of class \( C^{2} \)).

The stable elliptic islands surrounding the elliptic fixed points within the otherwise chaotic resonant gaps are sketched in Fig. 1. How they come about upon a small perturbation of an integrable system is explained by the Poincaré-Birkhoff theorem: when an area preserving mapping has a simple closed invariant curve consisting of fixed points, then due to the KAM Theorem and the area preserving property, an even number of these fixed points is shown to survive a small perturbation. Half of them are elliptic, and the others are hyperbolic. The almost, self-similar infinite hierarchy is explained by the fact that all considerations above are valid for any iterate of the Poincaré mapping.

It remains to understand how the chaotic motion can arise. This is connected with the hyperbolic points. All what has been done so far is the linearization around a hyperbolic fixed point. We shall concentrate on the asymptotes of the hyperbolae and the motion on them. We call them incoming or outgoing strand, according to whether \( L \) acts as a contraction or expansion, respectively (Fig. 2). Actually, these linear parts can be extended into the nonlinear region of \( T \) \[40\], and are then called stable and unstable manifold, respectively. Now, the most important thing is how they meet each other. If they join smoothly, nothing special happens. But such a smooth joining is exceptional, since they will generically (i.e., in almost all cases) meet transversally at a point which is not a fixed point. When they do so, they will do it infinitely many times, as a little reflection shows. Moreover, due to the area preserving property of the mapping the amplitude of the oscillations will become larger and larger, so that an extremely complex motion arises, which has little in common with the laminar integrable behaviour. Such an oscillatory motion is called homoclinic oscillation. It is a prototype of chaotic motion. Indeed, it can be shown that no smooth integral of motion exists for homoclinic oscillation (by imbedding the Smale horseshoe mapping \[40\]). We see that hyperbolicity + transversality imply chaotic motion. Since both occur in almost all cases upon a perturbation, if not otherwise, we get a feeling that the type of motion within the resonant gaps, as shown in Fig. 1, is of the generic class.

We can now understand the nature of the extreme of completely unstable systems, as opposed to the stable integrable system: They must be full of hyperbolic periodic orbits. Indeed, an ergodic system can be and must be full of periodic hyperbolic
orbits. Their measure is zero, so that this does not contradict the property (a). In a certain sense the periodic orbits, being all unstable, span the flow of even an ergodic system. This fact is not less surprising than the approximation of real numbers by the rationals.

However, a generic Hamilton system is neither integrable nor ergodic, both extremes being exceptional. (We consider only systems with a few degrees of freedom. It should be noted that the ergodic systems are less exceptional than the integrable ones, since they are structurally stable: a small perturbation does not change the two fundamental properties, hyperbolicity and transversality.) To be specific, a typical system displays all features of an integrable system at low energies. At some critical energy a rather sharp transition, the so-called stochastic transition occurs: the invariant tori are observed to disappear rapidly as the energy is increased, and the motion becomes irregular. When the motion is numerically integrated the best way to observe the transition from the regular to the irregular motion is by means of Poincaré mappings. But it is very difficult to predict the critical energy. We shall return to this question in a later section.

Figure 2: When the stable and unstable manifolds of an hyperbolic point H meet transversally at a homoclinic point P_0, then there is infinity of homoclinic points P_1, P_2, ... No smooth integral of motion exists in homoclinic oscillation. It is a prototype of irregular motion.

While a lot is known on the classical irregular motion, the study of its quantum mechanical aspects is now in progress [29,30, and references therein]. The fact that the Schrödinger equation is known and in principle solvable for any system helps as little as the analogous fact of the Hamilton-Jacobi equation in the classical case did. What we need are more explicit quantization conditions, and this is the reason for a considerable revitalization of the semiclassical mechanics. There are two major problems to be solved: firstly, the time evolution of nonstationary wavefunctions, which is ignored here; secondly, the properties of the energy spectra. The latter problem has no analogy in classical mechanics.

Before I start reviewing some general but fragmentary results, I should like to emphasize the fact that nobody so far has clearly formulated what is the quantum analogy of the KAM Theorem, which must exist, and should be a natural basis for the study of stochasticity in quantum mechanics. I believe that it can be expressed in terms of integrals of motion, i.e. in terms of commutators.

In the following the terms "chaotic" and "integrable quantum system" are meant to imply that they are irregular or integrable in the classical limit, respectively. In the rare cases that a system is integrable we have the well defined semiclassical quantization conditions, initiated by Einstein, and rigorously developed by Maslov. The method is known under the names: tori quantization, topological quantization, Maslov quantization, EBK-quantization,... It is just the quantization of the actions,

\[ I_j = (m_j + a_j/4) \hbar \]  

(2)

where \( m_j = 0,1,2, \ldots \), while \( a_j \) is the number of caustics encountered in configuration
space upon traversing the irreducible cycle $C_j$, and is called Maslov index. The
spectrum is then given by $E_\mathfrak{H} = H(I_\mathfrak{H})$.

Unlike the classical case, the quantum spectral problem poses great difficulties
even in the integrable system: the exact quantization can yield an explicit solution
typically when the system belongs to the very special class of separable systems. Of
course this touches the purely classical problem of when is an integrable system also
separable. (Kalnins and Miller [41] have recently studied the separation problem in
nonorthogonal systems.) To my knowledge there is not even a single example of a non-
separable but integrable system, whose exact spectrum is known. So we do not know
the generic fine structure of their spectra: are the degeneracies typical just as in
the separable systems, or is the exponentially small separation of levels at close
anticrossings characteristic for them, as Berry is suggesting [29].

So far no quantization conditions as explicit as (2) are known for irregular sys-

tems. It is not unexpected that a search for them is directed towards the usage of
Feynman path integrals (ordinary and phase space path integrals), since these pre-
sent the most intimate link between the classical and quantum dynamics. The approach
has been largely developed by Gutzwiller [42].

To see how it works, observe that the Kernel $K(q_a, q_b, t)$ for the Schrödinger
equation with eigenvalues $E_j$ can be expanded in terms of the eigenfunctions $\psi_j(q)$,

$$K(q_a, q_b, t) = \sum_j \psi_j(q_a)\psi_j^*(q_b) e^{-iE_j t/\hbar} \quad ,$$

so that its Fourier transform (Green function) equals,

$$G(q_a, q_b, E) = \sum_j \psi_j(q_a)\psi_j^*(q_b) \delta(E - E_j) \quad ,$$

where $\delta(E - E_j)$ are the Dirac delta functions. Putting $q_a = q_b$, and integrating over $q_a$,
we find that the response $d(E)$ is equal to the sum of the delta spikes:

$$d(E) = \sum_j \delta(E - E_j) \quad (5)$$

Hence, knowing the kernel (3) we know the spectrum (5). But Feynman has shown that
the kernel $K(q_a, q_b, t)$ can be exactly given by the sum of all paths going from $q_a$ at
time $t = 0$ to $q_b$ at time $t$,

$$K(q_a, q_b, t) = \int e^{iS[a, b]/\hbar} \mathcal{D}\mathcal{q}(t) \quad (6)$$

where $S[a, b] = \int L(q, \dot{q}, t)dt$ is the action of a path. In the semiclassical limit
$\hbar \rightarrow 0$ two facts are important. Since the contributions to (6) will display rapid os-
cillations only the stationary ones will not cancel out, whence: (i) for fixed end-
points $a, b$ the stationary paths are exactly the classical paths; (ii) the identifica-
tion $q_a = q_b$ demands that only those classical paths, which are recurrent in configu-
ration space should be taken. But since they should also be stationary with respect
to variations of the endpoints $q_a, q_b$, due to integration over $q_a$ in passing from (4)
to (5), we reach the conclusion that: in the semiclassical limit only the closed (i.
e. periodic) classical paths contribute to the response $d(E)$. In this sense, the
semiclassical spectrum is spanned by the periodic orbits.

This seems paradoxical as the periodic orbits will have as a rule measure zero in
phase space, no matter whether the system is regular or irregular. Zaslavsky [30]
considers almost periodic orbits instead.

In an integrable system the periodic orbits lie on resonant tori, and these are
precisely those allowed to be destructed according to the KAM Theorem. We see that
destruction of tori connected with the onset of chaotic motion will have important
consequences for the (semiclassical) spectrum. However, not all resonant tori are
generically destroyed at the same time upon a small perturbation: the destruction
begins for high energy tori, appearing at lower and lower energies as the perturba-
tion gets stronger.

A warning is in order concerning the semiclassical quantization resulting in the response (5): the quantization of the action of a single periodic classical orbit can give only accidentally a correct energy eigenvalue. Berry [29] points out that closed orbits and energy levels correspond to each other only collectively.

Notwithstanding some major disagreements [29,30] as concerns the quantization of irregular systems, there is a generally accepted fact that their energy spectra are statistical in nature. This comes not only from the methodologically statistical approach to the response (5), as was the case in the early statistical theory of spectra of complex nuclei, developed by Wigner, Porter and Dyson. A justification for the statistical description is believed to be found in the fact, that this is the only meaningful description. Since the works by Percival [43] and Pomphrey [44], enough observational material has been gathered to be convinced that energy levels corresponding to the irregular motion are unstable with respect to small variations of some (family) parameters of the Hamiltonian. Since all real physical systems are actually ensembles with some scatter in external conditions, such as magnetic field, the observed spectra of irregular systems will be rather smooth, reminiscent more of the continuous than of the discrete spectra. Zaslavsky [30] links this instability of levels directly to the instability of the classical motion, via the K-entropy. (Roughly speaking, this is equal to the phase average of the largest Lyapunov exponent. It measures the rate of increase in time of the coarsened phase volume, i.e. the rate of entropy increase.) On these grounds the response formula (5) should be understood more as a distribution of energy levels rather than a quantization condition for individual levels.

It is then appropriate to define the probability density $P(S)$ that the separation of two neighbouring levels is equal to $S$ (normalized to the mean separation of levels). For integrable systems there is a generally accepted result that $P(S)=\exp(-S)$, reflecting the possible existence of level crossings. In this case the levels are clustered. In the irregular case, there is agreement that $P(S)$ is given by some power law, $P(S)=\text{const.} \times S^b$. Berry [29] suggests $b=1$, while Zaslavsky argues that $b=\text{const.}/\hbar$, $\hbar$ being the Kolmogorov K-entropy of the corresponding classical system in the energy shell around the levels considered. As $P(S)$ vanishes with $S$, this accounts for the fact that level repulsions are typical of the irregular systems. Therefore, no clustering occurs and the spectra appear more random.

3. The classical Hamiltonian and the stochastic transition

In the following I shall consider the "naked problem" of the hydrogen atom in a magnetic field, which remains after dropping all effects except the diamagnetic, which is essential for the transition from the regular to the irregular motion. Thus, the trivial paramagnetic term is omitted; the nucleus is assumed to have infinite mass, so that no coupling between the electronic states and the CM-motion is present; the CM is assumed to be at rest in a homogeneous magnetic field, accounting for the absence of the motional Stark effect. Then the classical electron motion is governed by the dimensionless Hamiltonian

$$H = \frac{1}{2} \rho^2 + \frac{1}{8} \gamma \rho^2 - \frac{1}{r},$$  

where $\rho^2 := x^2 + y^2$. The unit of length is equal to the Bohr radius, the energy is measured in atomic units (hartree), and $\gamma$ is the already defined dimensionless magnetic field strength, $\gamma = B/B_0$. The only continuous symmetries of (7) are time translation and rotation around the z-axis. The energy $E = H(p, r)$ and the angular momentum (i.e., its z-component) $L = x p_y - y p_x$ are two obvious integrals of motion. Since $N = 3$, a third integral is needed for integrability.

Accounting for the invariance of $L$ we can reduce (7) by using the cylindrical coordinates:

$$H = \frac{1}{2} (p_\rho^2 + p_\phi^2) + \frac{L^2}{2\rho} + \frac{1}{8} \gamma \rho^2 - \frac{1}{r}.$$  

The three-dimensional parameter space $(E, L, \gamma)$ in which the Hamilton flow is analyzed can be reduced by one by a simple stretching of variables: new coordinates $= \alpha \times$ old coordinates, new momenta $= \beta x$ old momenta. A simple calculation shows that it is im-
possible to eliminate two parameters, the remaining possibilities being:

(A) \( \alpha = \text{const. } y^{2/3}, \beta = \text{const. } y^{-1/3} \), when \( L = 0, y \neq 0 \), with the Hamiltonian

\[
H = \gamma^{2/3} \left\{ \frac{1}{2} (p^2 + p_z^2) + \frac{1}{8} \rho^2 - \frac{1}{r} \right\}
\]  

(9)

(B) \( \alpha = \text{const. } L^{-2}, \beta = \text{const. } L \), when \( L \neq 0 \), with the Hamiltonian

\[
H = L^{-2} \left\{ \frac{1}{2} (p^2 + p_z^2) + \frac{1}{2} \rho^2 + \frac{1}{8} (\gamma L^3)^2 \rho^2 - \frac{1}{r} \right\}
\]  

(10)

where all constants are independent of \( L \) and \( y \). We have taken them equal to unity. Here the same symbols are used for the new variables \( \rho, p, z, p_z \). The case (A) has been extensively considered in the works by Edmonds and Pullen [21]. I shall concentrate on case (B) of nonvanishing angular momentum, where we have the following explicit scaling property: If for \( L = 1 \) some feature of the flow (e.g. a bifurcation of a periodic orbit) occurs at energy \( E_L(y) \), then at any \( L \neq 0 \) the same feature appears at \( E = L^{-2} E_L(\gamma L^3) \). It is thus sufficient to explore the system for \( L = 1 \).

The equipotential lines of the potential \( U(\rho, z) := 1/2 \rho^2 + 1/8 \gamma^2 \rho^2 - (\rho^2 + z^2)^{-1/2} \) (see (8) and (10)) are plotted in Fig. 3. Its minimal value \( E_{\min} = \min U(\rho, z) \) as a function of \( y \) is given by

\[
E_{\min}(y) = \frac{(2-3\lambda_0)}{2\lambda_0^2},
\]

(11)

where \( \lambda_0 := \lambda(y/2) \), and the function \( \lambda(x) \) is defined as a positive root of the equation \( x^2 \lambda^4 = 1-\lambda \), so that \( \lambda \in [0,1] \) for non-negative \( y \). At \( y = 1 \) one has \( E_{\min} = -0.3943046 \).

The potential lines open at the escape energy \( E_{\text{esc}}(y) = \gamma/2 \), above which the electron can escape the Coulomb potential well, while it is still transversally bounded by the magnetic field. Note that \( E_{\text{esc}} \) is not equal to the quantum mechanical ionization energy \( E_{\text{ion}} = \gamma(L+1)/2 \), which is higher by exactly the Landau energy \( \gamma/2 \), and does not obey the scaling law. If the electron escapes in the direction of the magnetic field, then its kinetic energy must be asymptotically equal to or larger than the quantum mechanical zero point energy \( \gamma/2 \). For this reason the states with energies between \( E_{\text{esc}} \) and \( E_{\text{ion}} \) are true bound, localized states, and not resonances.

For small \( x \) we have \( \lambda(x) \approx 1-x^2 \), while \( \lambda(x) \approx 1/\sqrt{x} \) as \( x \gg 1 \). Therefore,

\[
E_{\min} = -\frac{1}{2} (1 - \frac{1}{4} \gamma^2 + \sigma(4^4)), \quad (E_{\text{esc}} - E_{\min}) - \frac{3}{2} \sqrt{\gamma/2} \quad \text{as } \gamma \to \infty.
\]

Later we shall use the relative energy \( g(E) \), defined by

\[
g(E) = \frac{(E - E_{\min}(\gamma))/(E_{\text{esc}}(\gamma) - E_{\min}(\gamma))}{2}.
\]

(12)

Figure 3: The equipotential lines of \( U(\rho, z) \) with \( y=L=1 \). The minimal energy is equal to \( E_{\min} = -0.3943046 \) and the escape energy \( E_{\text{esc}} = 0.5 \).
The dynamics of the system has been examined by investigating the Poincaré mappings on the SOS in the plane \((p, p_z)\) of the phase space. The bounding curve of the allowed region is determined through \(p_z^2 = 2E - p_z^2 - 2U(p, z = 0) = 0\), and the motion on this invariant curve corresponds to the bouncing of the electron along the gradient line \(z = 0\) in Fig. 3, i.e. to plane orbits in the 3-dimensional picture. A Poincaré mapping was examined by taking some typical initial conditions, and by plotting the sufficiently large number of its iterates to see whether the phase point is confined to some invariant curve. In Figure 4 we show SOS at ten different energies for \(\gamma = 1\). At low energies (Figs. 4a-b) the motion appears integrable, since SOS is filled with invariant curves, which are nested around an elliptic fixed point. This corresponds to the simple periodic bouncing between the potential lines of Fig. 3, but now approximately along the \(z\)-axis and exactly vertical at \(z = 0\) (because \(p_z = 0\) at the fixed point). At higher energy (Fig. 4c) another fixed point occurs, while at still higher energy (Fig. 4d) bifurcations resulting in new stable periodic orbits have occurred. Here the hyperbolic points can be also observed, and in vicinity of them there are first signs of unstable motion. By increasing energy this chain of hyperbolic points is seen to become a broad belt of stochastic motion (Fig. 4e). There are several definitions of the stochastic transition, all being almost equivalent due to the fact that the transition is fairly sharp. (The definition in terms of the K-entropy, which becomes positive, seems to be the best, since it refers to a global property of the flow \[30\]. Still other definition refers to the destruction of the "last large scale torus"). I have adopted the criterion, that the critical energy is reached when the stochastic motion becomes observable on the largest scale. According to this definition the critical energy \(E_{\text{crit}}(\gamma=1)\) lies between \(-0.04\) and \(-0.03\).

Figure 4(a-j): The Poincaré surfaces of section for \(\gamma=1\) at ten different energies, \(E=-0.3\) (a), \(-0.1\) (b), \(-0.05\) (c), \(-0.04\) (d), \(-0.03\) (e), \(-0.01\) (f), \(0.0\) (g), \(0.1\) (h), \(0.2\) (i), \(0.5\) (j). The irregular motion becomes "macroscopically visible" between \(E=-0.04\) and \(-0.03\), so that \(E_{\text{crit}} = -0.035\). At higher energies, when the "last" large scale invariant torus is destructed, the motion becomes completely stochastic.
Upon increasing the energy above the critical energy the irregular region becomes larger and covers the whole energy surface when the energy approaches the escape energy.

How can we understand the existence of invariant tori and the emergence of the irregular motion on the other hand? When the system is viewed as a perturbed Kepler system (with a rather large perturbation, since \( \gamma = 1 \)) the KAM Theorem allows all tori to be destroyed, because all tori of the Kepler problem are resonant tori. However, we can look at the system as being integrable at the bottom of the potential well, where the potential can be approximated by a two-dimensional harmonic oscillator, which is resonant only at some discrete values of \( \gamma \), for instance \( \gamma = 0, 7 \sqrt{2}/8, \ldots \). The energy itself can be treated as a perturbation parameter, and the KAM Theorem can be applied, predicting now the existence of tori filling large regions of the energy surface, the excluded volume being small with \( (E - E_{\text{min}}) \). Upon increasing the energy the resonant gaps become wider until the chaotic motion within the gaps becomes macroscopically visible. When the last large-scale invariant torus is destructed, the motion is irregular on the whole energy surface.

In Section 2 it was shown that the hyperbolic fixed points are "germs" of the chaotic motion (hyperbolicity + transversality = homoclinic oscillation). To explain the latter it is necessary to show the way how the system can become full of hyperbolic fixed points. In their beautiful work Greene, MacKay, Vivaldi and Feigenbaum [46] discover a universal behaviour of families of area preserving mappings, quite analogous to the properties of one-dimensional dissipative mappings. The generic case is as follows: as a parameter of an area preserving mapping is varied a stable, elliptic fixed point (or periodic point) may go unstable and at the same time stable periodic orbits of twice the original period are born out of it. With the parameter \( p \) varying the process can be repeated resulting in a sequence of period doubling bifurcations, occuring at \( p_0, p_1, \ldots, p_j, \ldots \). The sequence converges towards some value \( p_\infty \), and is asymptotically geometric as \( j \to \infty \), with the universal rate

\[
\frac{p_{j+1} - p_j}{p_j - p_{j-1}} \to \frac{1}{\delta}
\]

as \( j \to \infty \), where the "rescaling factor" \( \delta = 8.721097200 \ldots \). The explanation for this universal behaviour, independent of the particular map, is found in the existence of a universal map acting as an attractor in the function space, so that as a parameter of a specific family of maps is varied, the iterates of the maps will at least locally converge to the universal mapping. This universal map has other rescaling properties, such as universal convergence rates of the positions of the bifurcated fixed points. In any case, the map is full of hyperbolic orbits and does indeed explain irregular motion, which need not be ergodic, however. The discovery has practical implications, since the convergence is very rapid: If we observe, or estimate, two period doubling bifurcations of the same sequence appearing at the parameter values \( p_0 \) and \( p_1 \), then we can predict the limit \( p_\infty \), beyond which the motion becomes irregular. (Note that the limiting behaviour (13) is invariant with respect to the reparametrization.) In our case the parameter is the energy and this offers a possibility to predict the critical energy, discussed in the next section.

The stochastic transition has been observed by inspection of the Poincaré mappings for other magnetic field strengths, and the result is shown in Figure 5, where the critical energy is plotted as a function of the magnetic field. The relative critical energy (see (12)) \( g(E_{\text{crit}}(\gamma L^3)) = \mu(\gamma L^3) \), which depends only on \( \gamma L^3 \), is plotted in Figure 6. Here it is seen that the irregular region becomes smaller in the two integrable limiting cases \( \gamma = 0, \gamma = \infty \), while it extends down to almost \( E_{\text{min}} \) at \( \gamma L^3 = 2.7 \).

The question of the limit \( L \to 0 \) connects the cases (A) and (B) (see (9,10)), and is at the same time related to the limit of vanishing magnetic field. It is reasonable to assume that some topological feature of the flow obeys a power law \( E_f(\gamma) = \gamma^a \)
Figure 5: The critical energy as a function of $\gamma$ at $L = 1$.

Figure 6: The relative critical energy $\mu=g(E_{\text{crit}})=u(\gamma L^3)$, separating the regular and irregular region. The latter occupies the largest fraction of the phase space around $\gamma L^3 \approx 2.7$. 
as $\gamma \to 0$. The rescaling property of the energy then implies $E_f = L^{-2} (\gamma L^3)^a$, for any $L \neq 0$. If the limit $L \to 0$ should exist, then $a > 2/3$, or $a = 2/3$. In the former case the feature would approach the escape energy, while in the latter case it becomes independent of $L$, and scales with the magnetic field as $\gamma^{2/3}$. As regards the critical energy $E_{\text{crit}}$, Edmonds and Pullen [21] have shown that in case $L = 0$ it is given by $E_{\text{crit}} = -0.5 \gamma^{2/3}$. The limit as $L \to 0$ therefore exists, and does not vanish, whence $a_{\text{crit}} = 2/3$ as well. From this it follows $(E_{\text{esc}} - E_{\text{crit}}) = 0.5(\gamma + \gamma^{2/3})$ as $\gamma \to 0$, with $L = 1$. Since the observed slope is 0.87 (see Figure 5), the asymptotic region has not yet been reached completely. However, the theoretical slope $2/3 + \gamma^{1/3}$ at $\gamma = 0.1$ equals 0.82, which is not far from the observed value 0.87.

In the reference [28] the escape orbits have also been observed. Some facts are worth stressing: The motion near the plane $z = 0$ is very irregular. Among some typical randomly chosen initial conditions no trapped orbits were seen. It is conjectured that the measure of orbits which are trapped in a finite part of the phase space with energy above $E_{\text{esc}}$ is negligibly small. However, sufficiently far from the plane $z = 0$ the Coulomb potential becomes weaker, the motion becomes ordered and the adiabatic invariant $\phi p_\parallel$ does becomes more and more exact. In order to escape, the electron must have the appropriate value of the adiabatic invariant. If not, it will be reflected and will return to the plane $z = 0$, where the value of the adiabatic invariant can change due to the chaotic motion. After several reflections the electron will ultimately escape in direction of the magnetic field. This is illustrated in Figure 7.

4. Some possibilities to calculate the critical energy

The attempts to predict the critical energy, i.e. to reproduce the Figure 6, by using the approximate analytical methods have been inconclusive so far. For instance, the considerations based on the curvature $R$ (is negative $R$ a valid criterion for stochasticity?)

$$R = (E-U)^{-3} \left\{ (E-U) \, \text{Tr}(a_i a_j U) + \text{Tr}(a_i U a_j U) \right\}$$
in the Jacobi metric, derived from the potential U, lead to the conclusion that R is always positive when γL^3 > 1/8, which disagrees with the observed stochastic transition at γ = L = 1. That should not surprise us however, since the "negative curvature criterion" is known to fail in some cases [45, 47, 48, 28].

The application of the Chirikov criterion of overlapping resonances [32] as well as of the renormalization approach (to the case of two resonances and the destruction of the "last invariant torus" separating them) of Escande and Doveil [49] hinges upon the fact that it is not easy to express the diamagnetic perturbation term by the Keplerian action-angle variables in an explicit way. Hence, the classical perturbation problem in the limit γL^3 → 0 is difficult to treat. Similarly, but for other reasons, one faces difficulties in the opposite extreme γL^3 → ∞. Here, the problem is that the motions decouple.

The only case where the action-angle formalism is easy to handle is the neighbourhood of the critical point, i.e. near the bottom of the potential well. To the lowest approximation the Hamiltonian is harmonic (being nonresonant except at \( \lambda_0 = \lambda(\gamma L^3/2) = 1.4/7, \ldots \)), while the higher order terms in the power expansion of the potential can be treated as perturbations, and can be easily expressed in action-angle variables. A criterion ala Chirikov, saying that the stochasticity will set in when the third-order and the fourth-order resonant terms have comparable amplitudes, results in

\[
E_{\text{crit}} - E_{\text{min}} = \left(1 - \frac{3}{4} \frac{\lambda_0}{\sqrt{\gamma} L^2 \lambda_0^2}\right),
\]

which yields \( E_{\text{crit}} = -0.05 \) when \( \gamma = L = 1 \). The agreement with the observations described in the foregoing section is excellent. However, such an expansion is efficient only when a few low-order terms represent a significant departure from an integrable system. This is not true in the limiting cases \( \gamma L^3 → 0,∞ \), which renders the validity of eq. (15) to the neighbourhood of \( \gamma L^3 \approx 1 \). (As can be seen, the expression (15) does not have the correct asymptotic behaviour in these limits.) It is not quite clear whether the excellent agreement in this region is accidental, since the criterion is largely qualitative in nature. Nevertheless, it shows that around \( \gamma L^3 \approx 1 \) the expansion is meaningful and could be made rigorous by a detailed study of the truncated Hamiltonian. If we succeed to construct a few stable periodic orbits and some of their period doubling bifurcations, then this would enable us to use the formula (13).

A final remark of this section concerns again the important case of the weak diamagnetic term, i.e. the case of the slightly perturbed Kepler-system. The KAM Theorem admits the destruction of all tori, since all tori of the Kepler problem are resonant, but we can rely on the structural stability of the real physical systems, and believe, that only high energy tori will be destroyed. To be more precise but still qualitative, we expect this to happen when the Coulomb potential and the diamagnetic potential are comparable, 1/r = γL^3/8, whence \( r = 2 \sqrt{\gamma L^3}/3 \). This yields the correct scaling \( E_{\text{crit}} \approx -1/r = -0.5 \sqrt{\gamma L^3}/3 \), and (incidentally) also the correct numerical factor. The importance is not in rederiving a result of the previous section, but in showing that in the limit of weak magnetic field, interesting for experimental investigations, the classical irregular region is coincident with the inter-mixing regime.

5. An approximate integral of the motion below the critical energy

The third integral of motion exists almost everywhere on the energy surface below the critical energy, except in thin, macroscopically invisible, gaps. It can be exact there, as the KAM Theorem suggests, but is certainly not analytic, and therefore difficult to construct. It would be useful to find an approximate formal integral of motion, using the computer algebra to overcome the difficulties in formal expansions [50]. If three integrals E, L and F of the Hamiltonian (7) are known, and are in involution, we can use the tori quantization (2).

In a recent work Solov'ev [51] considers the time averages of the Poisson brackets \( \{H, L\} \) and \( \{H, A\} \), where \( L = r \bar{p}, \ A = pxL - \bar{r}/r \) are the angular momentum and the Runge-Lenz vector. He finds two approximate integrals of motion Q and Λ,

\[
Q = \bar{L}^2/(1-\bar{A}^2), \quad \Lambda = 4\bar{A}^2 - 5A_z^2.
\]

(16)
Their mean variation in time is of the order $\gamma^4$. (The Poisson brackets \{H,Q\} and
\{H,λ\} are still of the order $\gamma^2$.) When we introduce the splitting $H=H_0 + H_D$, $H_0=γ<\frac{e^2}{8}$, and use the relation $A^2=2L^2H_0 + 1$, we see that $O=1/2H_0$.

The trick behind this idea of averaging is connected with the degeneracy of the Kepler Hamiltonian. Because of the degeneracy the invariant tori of $H_0$ are not unique \[34\]. In other words, Kepler orbits can be embedded into several different systems of nested tori. The point is to find those tori on which the secular changes can be neglected. Indeed, $L_2H_0=-1/2Q$ and $λ$ are three independent integrals of the Kepler problem, they are in involution and thus define the invariant tori. The result of the averaging is that these tori are preserved to the order $γ^4$ under the diamagnetic perturbation $H_D$.

The semiclassical topological quantization leads to the usual result for the Coulomb spectrum, $E_0 = -1/2n^2$, where $n=1,2,...$ is the main quantum number, while the diamagnetic contribution is obtained by averaging $H_D$ over the torus labelled by $L_z$, $H_0,λ$, and then expressed by the corresponding quantum numbers. Solov’ev’s result reads,

$$E = E_0 + \langle H_D \rangle = -\frac{1}{2n^2} + \frac{γn^2}{16} (n^2 + m^2 + n^2λ_k)$$

where $λ_k$ is the quantized value of the integral $λ$, determined by

$$\oint dθ/n^2(1 + \frac{λ}{2}) - \frac{m^2}{1-5\sin^2θ/\sin^2θ} = 2π(k + 1/2).$$

Here $n=L$ is the quantized $z$-component of the angular momentum, equal to an integer. The variable $θ$ in the integrand is defined as the angle between the Runge-Lenz vector $λ$ and the z-axis. The integrand is then the momentum canonically conjugate to $θ$.

Solov’ev points out the importance of the pole in eq. (18). It implies that tori belong to two distinct classes, according to the sign of $λ$. When projected down onto the configuration space, those with $λ$ negative lie inside the cone $0 < θ < θ_0$, while others lie in the complementary cone. Hence, tori differing in the sign of $λ$ have nonoverlapping projections in configuration space, thus the semiclassical wavefunctions associated with them are also nonoverlapping. Two levels corresponding to such states are therefore allowed to cross in the semiclassical approximation. However, since the exact wavefunctions do not vanish exactly beyond the caustics, but decay exponentially instead, the corresponding matrix elements $\langle ψ_+|H_D|ψ_-\rangle$ will be exponentially small. These matrix elements for states within an $n$-manifold with constant $m$ are of the order $\exp(-2n+m)$, a result obtained by Solov’ev which is in beautiful agreement with the numerical calculations of the splittings at the avoided crossings from the work of Delande and Gay \[26\]. (The splittings are proportional to the matrix elements \[52\].) The result is a nice example in favour of the conjecture by Berry \[29\] that the exact spectra of nonseparable integrable systems will generically show exponentially small separations at avoided crossings.

6. The correspondence between the classical and the quantum system

Above the critical energy, i.e. in the inter-$n$-mixing regime, the tori spanned by the integrals $L_z, H_0$ and $λ$ will begin to break up, and when the last large-scale torus is destructed, the system will become completely stochastic. As a consequence, there will be no caustics and no localized states, so that the corresponding levels will no longer be exponentially small separated, but will display avoided crossings typical of the ergodic systems. The spectrum will be largely irregular, which has been observed by Clark and Taylor \[16\] to appear at the energy of the order $γ^{2/3}$ below the zero-field-ionization limit. The only "large scale torus" remaining in this irregular region is the boundary of the Poincaré SOS (see Fig. 4). It corresponds to plane orbits in $z=0$, i.e. along the gradient line $z=0$ of Fig. 3, and its quantization may explain the precursors of quasi-Landau resonances reported by Delande and Gay \[24\]. Obviously above the escape energy such plane orbits become even more important due to the adiabatic invariants. The correspondence between the classical regular and irregular motion and the structure of the quantum mechanical spectrum is shown in Figure 8 for the case of laboratory field strengths.
At very strong magnetic fields, $\gamma >> 1$, the spectrum of the hydrogen atom is again Coulomb-like for states with the same $m$, except for the ground state, which scales as $\ln^2 \gamma$. Apart from the ground state all levels are within $\approx 1\text{Ry}$ below the ionization energy. Since the ionization energy is higher than the escape energy by $\sqrt{\gamma}$, this implies that all levels are above the irregular region, and that the chaotic features of the spectrum will eventually disappear as $\gamma \to \infty$. However, in white dwarfs $\gamma$ is not higher than 0.1, so that there remains the possibility of explaining some continuum-like features [1,2] of their spectra by stochasticity involved in the problem.

Acknowledgements

The author thanks the Sonderforschungsbereich Radioastronomie for financial support. The valuable and stimulating communications by J.C. Gay and C.W. Clark are gratefully acknowledged. Sincere thanks are due to William P. Reinhardt for his comments on the incompleteness of the hydrogenic basis (see the introduction). The work has not been supported by any military agency.

References

11. GARSTAND, R.H., contribution to this proceedings (1982).
12. MCDOWELL, R.M.C., ibid (1982).
52. VON NEUMANN, J. and WIGNER, E., Phys. Z. 30 (1929) 467.