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THE RIGID ROD MODEL OF DISLOCATION RESONANCE INCLUDING APPLICATIONS TO POINT DEFECT DRAG

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Abstract.—The rigid rod model of dislocation motion is used to unify, simplify and clarify various types of theories of dislocation damping effects. It leads to a simple Debye-type relaxation process for overdamped resonance. Deviations from the rigid rod phenomenological formalism are found in cases where restoring or viscous forces act at discrete points rather than continuously, and they are evaluated by comparisons with exact solutions. The deviations are found to lead generally to corrections which are smaller than other uncertainties, except for the case of point defect drag with very small pinning point densities. This establishes the range of validity of the rigid rod model.

1. Introduction.—The first comprehensive quantitative theory of internal friction phenomena due to dislocation motion was presented 25 years ago by Granato and Lücke /1/. Since then numerous papers have appeared in this field whose aim was to predict theoretically or to interpret dislocation induced modulus and damping effects. Some of these papers were kept within the framework of the Granato-Lücke theory, others appeared as completely new concepts. In many cases the interrelation of the different theories was not at all clear.

The principle purpose of the present paper is to clarify this situation. By choosing a rather simple theoretical concept, the rigid rod model, it has been possible to unify and simplify the theoretical description of dislocation resonance phenomena. In this model which has been stressed recently by Lenz and Lücke /2/, Granato /3/, and Lücke and Granato /4/, most of the different theories appear as different interpretations of constants in the same phenomenological equation. In particular, besides the old Granato-Lücke theory itself, the point defect drag case, recently treated thoroughly by Lücke and Granato /4/, could also be shown to be special cases of the same general formalism.

* After conference, authors proposed this paper as a contribution to the discussion following this session.
2. The Rigid Rod Model of Dislocation Resonance.— Under the influence of periodic external shear stresses, the dislocations contained in a crystal undergo forced vibration. This phenomenon is called dislocation resonance. The dislocation motion leads to an anelastic deformation and thus to an apparent decrease of the elastic modulus. Since, in general, frictional forces are associated with this motion, the vibrations of the dislocations, and thus also those of the crystal as a whole, are damped and the dislocation induced modulus defect becomes frequency dependent.

In most cases the exact geometry of the dislocation vibration and the nature of the acting forces are not known, so that it appears useful to approach the problem phenomenologically. Since the dislocations can be assumed to lie in equilibrium positions without an applied periodic stress, their motion will be opposed by a restoring force proportional to their displacement as long as the amplitude is small (Fig. 1a). The frictional forces acting on moving dislocations will in most cases be of a viscous nature, i.e. proportional to the velocity. Finally, due to their effective mass, inertial forces act on the dislocations.

The simplest geometry is an infinitely long straight dislocation with the forces being constant along its length so that the dislocation moves as a rigid rod (Fig. 1b). Then one has the following equation of motion:

\[ A\dddot{y} + B\dot{y} + Ky = b\sigma_0 e^{i\omega t} \]  

(1)

Here \( b \) is the Burgers vector, \( \sigma_0 \) the amplitude of the external shear stress, \( y \) the dislocation displacement, \( A \approx \rho b^2 \) the mass constant, \( \rho \) the density of the material, \( B \) the viscous drag constant and \( K \) the restoring force constant (all defined per unit length of the dislocation).

FIG 1 Schematic sketch of dislocation displacement \( y \) versus position \( x \). The dashed line represents the position under no external stress, and the solid line the displacement under stress. The solid points are fixed pinning points and the open points are movable pinners.

1a. General Case.
1b. Rigid Rod model for continuous restoring and viscous forces.
1c. Continuous restoring force and discrete viscous forces.
1d. String model. Discrete restoring force and continuous viscous force.
1e. Both restoring and viscous forces discrete.
dislocation). For the more general case where the dislocation is not straight and the forces are not constant (Fig. 1a), these quantities represent some kind of average along the whole dislocation depending upon the geometry.

By standard methods, the solution \( y(t) \) of Eq. (1), as well as the resulting modulus defect \( \phi = \Delta G/G \) and damping capacity \( \delta = \pi^{-1} \) (= \( \pi \) times logarithmic decrement), can be found:

\[
y = \frac{b \sigma_0}{K} \left( \frac{1 - \omega^2/\omega_0^2}{(1 - \omega^2/\omega_0^2)^2 + \omega^2 \tau^2} \right) - i \omega \tau \frac{b \sigma_0}{K} \left( \frac{1 - \omega^2/\omega_0^2}{(1 - \omega^2/\omega_0^2)^2 + \omega^2 \tau^2} \right)
\]

\[
\phi = \frac{\Delta}{2} \frac{1 - \omega^2/\omega_0^2}{(1 - \omega^2/\omega_0^2)^2 + \omega^2 \tau^2} ; \quad \delta = \frac{\Delta \omega \tau}{(1 - \omega^2/\omega_0^2)^2 + \omega^2 \tau^2}
\]

Here the resonance frequency \( \omega_0 \), the relaxation strength \( \Delta \), and the relaxation time \( \tau \), are given by

\[
\omega_0 = \sqrt{K/A} \quad ; \quad \Delta = \frac{A \sigma_0^2}{K} \quad ; \quad \tau = B/K
\]

where \( \Lambda \) is the dislocation density and \( G \) is the shear modulus.

One recognizes that these solutions are of resonance character, and hence two limit cases are of special interest:

(i) Underdamped resonance, where \( B \ll 2\sqrt{A/K} \). Then the term \( B^2y \) in Eq. (1) or the term \( \omega_0^2 \tau^2 \) in Eqs. (2), (3) can be neglected except for \( \omega \) in the near vicinity of the resonance frequency \( \omega_0 \) so that a sharp damping peak and strong modulus changes occur only near \( \omega_0 \). These conditions are fulfilled for sufficiently high restoring forces (e.g. for short loop length, c.f. Eq. (7), preferentially at low temperatures).

(ii) Overdamped resonance, where \( B \gg 2\sqrt{A/K} \). Then the term \( A^2y \) in Eq. (1) or the term \( \omega_0^2/\omega_0^2 \) in Eqs. (2), (3) can be neglected except for very large frequencies so that for a large frequency range the equation of motion, modulus defect and damping capacity are given by

\[
B^2y + Ky = b\sigma = b\sigma_0 \exp(i\omega t),
\]

\[
\phi = \frac{\Delta}{1 + \omega^2 \tau^2} \quad ; \quad \delta = \frac{\Delta \omega \tau}{1 + \omega^2 \tau^2}
\]

For numerical evaluations an orientation factor \( \Omega \) has still to be inserted which takes account of the orientation relationship between the geometries of the moving dislocations and the external periodic stress field.
The form of Eq.(6) means that overdamped resonance leads to a simple Debye-type relaxation process. It occurs preferentially at higher temperatures if the restoring force is sufficiently low (e.g. at long loop length). Since in metals most measurements have been made on very pure materials (i.e. for long loop lengths), only case (ii) will be considered further\(^+\) in the present paper.

3. Continuous and Discrete Restoring and Drag Forces.— Since Eq.(1) is purely phenomenological, it is not associated with any special type of restoring or dragging force, but applies to any kind of mechanism. Table I gives some examples which are of quite different nature, mathematically and physically. In most cases the drag constant \(B\) is independent of frequency. For the re-radiation loss, however, \(B\) is proportional to the frequency \(/5/\) so that the simple Debye formulas, Eq.(5), are not obtained. Of special interest for the present paper, however, is the fact that the restoring and dragging forces can act not only continuously at the dislocation (in the following denoted by \(K_c, B_c\)) but also discretely, i.e. pointwise (denoted by \(K_d, B_d\)). The latter occurs in the presence of firm or movable pinning points.

The restoring force between two partial dislocations (Table I) is definitely of continuous nature (e.g./6/). The same is true for the restoring force supplied on a dislocation by the stress fields of other dislocations (e.g./7/). In the case usually considered, however, namely the case where the dislocation is anchored at firm pinning points (string model, Fig.1d), the restoring force acts only at the dislocation pieces situated directly at the pinning point. Such a distinction also applies for the drag forces (Table I). E.g. dislocation-phonon and dislocation-electron interactions lead to continuously acting drag forces, but point defects firmly connected to but able to migrate together with the moving dislocation also lead to drag forces on the dislocation which, however, act only pointwise (Fig.1c).

As indicated in Fig.1, the dislocation line is no longer straight for the case of pointwise acting forces, and thus the rigid rod Eq.(1) is not strictly applicable. But an approximation is obtained also here by forming average displacements and forces along the dislocation. For this purpose an infinitely long straight dislocation under constant stress \(\sigma\) is considered. For the case of discrete restoring force (Fig. 1d), one may set the drag term in Eq.(5) to zero \((B = 0)\), and calculate

\(^+\)E.g. for copper at 300K (4.2K), one obtains \(B = 4 \cdot 10^{-4} (4 \cdot 10^{-5})\) (cgs units), so that loop lengths \(L > 2 \cdot 10^{-5}\) cm \((2 \cdot 10^{-4}\) cm\) are necessary to obtain the overdamped state.
the mean displacement of a dislocation loop of length $L_N$. One then has
\[ \bar{y} = b_0 L_N^2 / 12 C \]
and obtains with $y = \bar{y}$
\[ K_d = b_0 / \bar{y} = 12 C / L_N^2 \quad . \tag{7} \]
For the case of discrete dragging forces with mobile pinning points separated by a distance $L_d$ (Fig. 1c), one may set the restoring term in Eq. (5) to zero and obtain
\[ B_d = b_0 / \dot{y} = 1 / mL_d \quad , \tag{8} \]
Here $1/L_d$ is the number of point defects per unit length, $m = D / kT$ is the mobility of the point defect, where $D$ is its diffusion constant.

It should be mentioned that there exists also the possibility of pointwise action of the forces without participation of point defects or anchor points: If the dislocation motion takes place by the motion of kinks, the inertial, drag and restoring forces act on the kinks, which for high Peierls energies have a length of about $b$. Although this case of kink induced dislocation motion can be included in the above scheme, it will not be treated in the present paper.

4. Deviations from Rigid Rod Behaviour for Discrete Forces.- In the case of discrete restoring and pinning forces the dislocation in reality is not straight so that deviations from the rigid rod case solutions are expected. They can be found only by treating the exact problem for each case. To derive the exact solutions, a restoring force $-Cy''$ due to the curvature of the dislocations, as well as the exact displacement of the pinning points ($y_N$ for firm and $y_v$ for movable pinning points) must be adequately considered. Thus for the general case one obtains instead of Eq. (5)
\[ B \ddot{y} + K c y - Cy'' = b_0 \quad \exp (i \omega t) \quad . \tag{9} \]
\[ B \ddot{y}_{\nu} = (C / L_d) \left( y'_{\nu+} - y'_{\nu-} \right) ; \quad y_0 = y_N = 0 \quad . \tag{10} \]
Here $1/B_d L_d = m$ is the mobility of a movable pinning point, $C(y'_{\nu+} - y'_{\nu-})$ is the force on such a point, and $y'_{\nu+}$ and $y'_{\nu-}$ are the slopes of the dislocation line to the right or left, respectively, of such a pinning point.

Exact solutions for the most important cases are compiled and discussed in /4/. They are found to have either the form of simple
Debye-type relaxation (continuous restoring forces) or of a super-
position of such processes (for discrete restoring forces). The latter
case is difficult to handle analytically. However, the deviations from
simple Debye-type expressions are small, so that an approximation by
such expressions is practical. As discussed in /4/ there are several
such approximations possible which lead to different values of the
relaxation strength $\Delta$ and of the relaxation time $\tau$ for different point
defect densities. Very frequent use is made of the low frequency approx-
imation where $\Delta$ and $\tau$ are chosen in such a way that the damping and
modulus are described correctly by the approximation at low frequen-
cies, and the first term Fourier approximation, where the leading term
of a Fourier expansion of the dislocation displacement is used.

A third approximation appears to have special value since it
recovers the two zero-order terms, the low frequency modulus defect
$\phi_L = \Delta$ and the high frequency damping $\phi_H = \Delta/\omega \tau$ (c.f. Eq.(6)). Moreover
this "zero-order" approximation $\phi_o(x,t)$ can be derived rather easily
without solving the complete differential equations (9),(10) but by
separately solving the equation for low frequencies (by omitting the
drag terms with $\dot{y}$) and for high frequencies (by omitting the restoring
force terms with $y$ and $y'$).

Then the modulus defect calculated from the first solution gives
the zero-order relaxation strength $\Delta_o$, and the damping calculated
from the second solution gives the corresponding quantity $\Delta_o/\omega \tau_o$ and
thus $\tau_o$. For the rigid rod case, the expressions given in Eq.(4) for
$\Delta_o$ and $\tau_o$ (here referred to as $\Delta_o$, $\tau_o$) would be obtained. In the case
of point forces, however, deviations from rigid rod behaviour are
found which can be characterized by the factors

$$
\kappa_o = \Delta_o /\Delta_c \quad , \quad \gamma_o = \tau_o /\tau_c
$$

The values for these factors can be taken from the exact solu-
tions /4/. Some are listed in Table II. This table also gives the
factors $\kappa_L$, $\gamma_L$ and $\kappa_F$, $\gamma_F$ which are obtained if in Eq.(11) instead of
$\kappa_o$, $\gamma_o$ the values $\kappa_L$, $\gamma_L$ for the low frequency approximation and $\kappa_F$,
$\gamma_F$ for the first term Fourier approximation are inserted.

They show that the deviations from values near one, i.e. from
the rigid rod solution, are obtained only for discrete drag at suffi-
ciently low dragging point densities. In all other cases the internal friction effects can be described in a good approximation by a simple
Debye-type relaxation process with relaxation time and strength given
by the rigid rod model. Four fundamental cases will be discussed
briefly:
(i) Both restoring and drag forces are of continuous nature. (K = K_C, B = B_C, Fig.1b.) This case is the pure rigid rod case and leads exactly to a single Debye-type process. Here \( \kappa = \gamma = 1 \). Since there is only a single relaxation, all the approximations of Table II yield the same result.

(ii) The restoring force is continuous (K = K_C) and the drag force is discrete (B = B_d, Fig.1c). In this case equidistant movable point defects with spacing L = L_d are assumed. According to \(/4c/\), there is a single Debye relaxation with \( \kappa = \gamma^{-1} = (L_d/L_0)/\tanh (L_d/L_0) \), where \( L_0 = 2\sqrt{C/K_C} \). For a high density of dragging points (\( L_d/L_0 \ll 1 \)), \( \kappa = \gamma^{-1} \to 1 \), while for only one dragging point per length \( L_0 \) (\( L_d/L_0 = 1 \)), \( \kappa \) and \( \gamma \) deviate from one by 24%. For \( L_d/L_0 \gg 1 \) one gets values \( \kappa = \gamma^{-1} = L_d/L_0 \) which may become \( \gg 1 \). In these cases \( \tau \) is independent of \( L_d \) and \( \Delta \) increases linearly with \( 1/L_d \). Then the restoring forces correspond to the case where there is less than one movable pinning point per segment \( L_N \).

(iii) The restoring force is discrete (K = K_d) and the drag force continuous (B = B_C, Fig.1d). This is the case treated originally by Granato and Lücke where equidistant discrete firm pinning points with spacing L = L_N are assumed. It leads to a superposition of an infinite number of Debye processes for which, however, the one with the lowest relaxation time alone represents a very good approximation, and the others lead mainly to a slight broadening of the peak.

(iv) Both restoring and drag forces are discrete (K = K_d, B = B_d', Fig.1e). Here \( n-1 \) movable pinning points with spacing \( L_d = L_N/n \) are located in each dislocation segment \( L_N \) determined by firm pinning points. The exact solution is given by Lücke and Granato in \(/4/\). It consists for \( n=2 \) and \( n=3 \) in a single Debye-type relaxation process but with \( \Delta \) and \( \tau \) deviating from Eq.(4). For \( n \geq 4 \) a superposition of several such processes \(^+\) is obtained, but this results again only in a slight broadening of the peak determined by the first term. The largest deviation from one is 33%, occurring for \( n=2 \). The differences between the approximations are less than 18% in the worst case. For the case of less than one pinning point per loop length \( L_N \), the same behaviour as in case (ii) for large \( L_d \) is obtained: the decrement decreases proportionally to \( 1/L_d \) without change of the relaxation time. (One then has only double loops.) This means that also here \( \kappa \) can reach values \( \gg 1 \) if \( L_d/L_N \gg 1 \).

\(^+\) Since only the odd modes contribute, one has \( n/2 \) processes for even \( n \) and \((n-1)\) for odd \( n \).
5. Summary and Discussion.- In the literature dislocation resonance and dislocation relaxation are often distinguished in terms of the drag mechanism. E.g., the term resonance is often used for the case of drag due to dislocation-phonon and dislocation-electron interaction, and the term relaxation for the case of point defect drag or Bordoni drag. The reason for this kind of classification is not clear to the authors, but might be connected with the fact that in the latter case the relaxation time usually depends exponentially upon temperature. In the former cases it depends only linearly or not at all on the temperature. In contrast to this, the present authors prefer a distinction based upon the frequency dependence, the principal parameter in the phenomenological description of the response of a solid to external forces. It is proposed that all dislocation internal friction phenomena be collected under the name of dislocation resonance. The case of overdamped resonance, however, might be denoted as dislocation relaxation, since it leads to a simple Debye-type relaxation process (or a superposition of such processes). It then also becomes clear that dislocation resonance phenomena can also occur at very low frequencies (for large B).

The rigid rod model, which leads to a pure Debye-type relaxation process, has been chosen for a general description of dislocation resonance and relaxation. If properly averaged values for K and B are used, it is found to give a good approximation also for the case of pointwise acting restoring and dragging forces for dragging forces of sufficient density of pinning points, i.e. at least one per network length $L_N$ in case (iv), and one per length $L_o$ in case (ii). The deviations from the rigid rod results for $\Delta$ and $\tau$ are then usually small and still only 33% for the worst case. They also show up as a slight broadening of the damping peak corresponding to the same differences in $\tau$. These deviations are smaller than the uncertainties in knowing the constants B and K and than the changes due to the distributions of loop lengths and point defects. On the other hand, for dragging forces of sufficiently low density ($L_d/L_o > 1$ in case (ii), or $L_d/L_N > 1$ in case (iv)), strong deviations from rigid rod behaviour are obtained. This defines the range of validity of the rigid rod model.

The exact treatment of the problem of discrete restoring forces with continuous drag was already given in the original publication of Granato and Lücke /1/ (string model). In contrast, the problem of discrete drag forces (point defect drag) was mainly treated by using the above continuous approximation (e.g. in /8/) by which the strong deviations from rigid rod behaviour would not be detected. After
investigation of certain aspects of the problem of truly pointwise acting drag forces by Lücke and Schlipf /9/, Blistanov and Shaskolskaya /10/, and Simpson and Sosin /11/, the complete solution of this problem was given by Lücke and Granato /4/. They also clarified the question of superposition of continuous and discrete drag /4b/ and also gave results concerning the loop length and point defect distributions /4c/. Ogurtani /12/ also claims to have solved the problem of discrete drag completely but in fact his solution is only that for the continuous approximation †).

It is hoped that the present paper not only gives exact and approximate solutions for different types of restoring and drag forces, but mainly clarifies the interrelations between the different concepts and thus demonstrates the unity and also the simplicity of the whole field of dislocation resonance.

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/1/ A.V. Granato and K. Lücke, J. Appl. Phys. 27, 582 (1956)
/4b/ Part II: "Superposition of Continuous and Pinning Point Drag Effects"
/4c/ Part III: "The Effect of Distributions of Pinning Points on Dislocation Damping", to be published

†) This can be seen most clearly in Eq. (8) of /12/, or in Eq. (12) of /13/, where the solution for the displacement of the dislocation is given.
/6/ A.V. Granato, R.B. Schwarz and G. Kneezel, (see paper in this conference)

/7/ J. Weertmann, J. Appl. Phys. 28, 193 (1957)


/12/ T.O. Ogurtani (See paper this conference)

Table I  Examples /2,3/ of Sources of Continuous and Discrete Restoring and Drag Forces

<table>
<thead>
<tr>
<th></th>
<th>Restoring Force</th>
<th>Drag Force</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>Interaction between partials Dislocation interactions</td>
<td>Dislocation-phonon interaction Dislocation-electron interaction Reradiation Loss</td>
</tr>
<tr>
<td>Either</td>
<td>Snoek Atmosphere</td>
<td>Dipole reorientation (Snoek atmosphere)</td>
</tr>
<tr>
<td>Discrete</td>
<td>Line tension (String $\kappa = C/12L^2$)</td>
<td>Cottrell atmosphere point defect drag Breakaway drag (immobile point defects)</td>
</tr>
</tbody>
</table>

Table II  $\kappa$ and $\gamma$ Factors for Different Uniform Densities of Restoring and Drag Forces

<table>
<thead>
<tr>
<th>Approximation</th>
<th>Case</th>
<th>Restor. Force</th>
<th>Drag Force</th>
<th>i</th>
<th>ii</th>
<th>iii</th>
<th>iv</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rigid</td>
<td>Contin.</td>
<td>Contin.</td>
<td></td>
<td></td>
<td>Discrete</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Rod</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(n = $\infty$)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Contin.</td>
<td>Discrete</td>
<td></td>
<td></td>
<td>Discrete</td>
<td></td>
</tr>
<tr>
<td>Zero Order</td>
<td>$\kappa_0$</td>
<td>1</td>
<td>$\nu \tanh \nu$</td>
<td>$1$</td>
<td>$1/(1-1/n^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\gamma_0$</td>
<td>1</td>
<td>$1/\gamma_0$</td>
<td>$1$</td>
<td>$1/(1+1/n)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low Frequency</td>
<td>$\kappa_L$</td>
<td>$\kappa_0$</td>
<td>$\kappa_0$</td>
<td>1</td>
<td>$1/(1-1/n^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\gamma_L$</td>
<td>$\gamma_0$</td>
<td>$\gamma_0$</td>
<td>$5/6$</td>
<td>$5/6/(1+1/n^2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First-Term Fourier</td>
<td>$\kappa_F$</td>
<td>$\kappa_0$</td>
<td>$\kappa_0$</td>
<td>1.0147</td>
<td>$n^3 \sin^2(\pi/2n)/b_1^n \cot(\pi/2n)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\gamma_F$</td>
<td>$\gamma_0$</td>
<td>$\gamma_0$</td>
<td>0.8225</td>
<td>$(n^2/3)\sin^2(\pi/2n)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* $n-1$ is the number of dragging points in a network length $L_N$.  
* $\nu = L_d/L_0$, where $L_0 = 2\sqrt{c/K_c}$.  
