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THE EQUATION OF STATE AT SUBNUCLEAR DENSITIES

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Abstract. - We first discuss the thermal properties of bulk hot nucleon matter with variable proton to total baryon ratio, Y. Of particular interest are the phase transitions and the coexistence curves, which have unusual features because of the additional degree of freedom associated with Y. Coulomb and surface effects, which modify these properties, are treated with a nuclear Thomas-Fermi model. The latter is shown to reproduce more than 80 % of the average level density (or specific heat) of known spherical nuclei.

1. Introduction. - The standard supernova model invokes an evolved star of a mass of 8 solar masses or higher, which has developed a dense iron core of about 1.5 solar masses, which, because of thermal dissociation of the nuclei becomes eventually dynamically unstable. The dynamic collapse continues to densities in excess of \(10^{14}\) g/cm\(^3\), when the equation of state stiffens sufficiently, mainly because of (non-degenerate) nucleon pressure and, later, because of nuclear repulsion, to cause a "bounce" and the formation of an outward moving shock wave. Electron captures during the collapse and bounce produce a copious amount of neutrini that tend to come into \(\beta\)-equilibrium with matter. The matter which one encounters is neutron rich at temperatures up to 20 MeV and at densities in excess of \(10^{11}\) g/cm\(^3\). It consists of an equilibrium (with respect to strong and electromagnetic interactions) mixture of nuclei, free nucleons, leptons and photons. Its equation of state and its opacity to neutrino radiation largely decide on the fate of the collapsing stellar core. In contrast, matter such as exists in neutron stars at subnuclear densities consists of nuclei in equilibrium with free nucleons, electrons and muons. Neutrini have had ample time to diffuse out and matter has established beta kinetic equilibrium, in addition to strong and electromagnetic equilibrium, except in the outermost crust of the star. The temperature although still of the order of \(10^{-10}\) K is essentially zero when compared to the chemical potentials of the constituent particles. The properties of such matter have been thoroughly studied and are believed to be well understood (Buchler et al. 1971, Negele and Vautherin 1973, Baym et al. 1971).

The finite temperature matter offers a much richer variety and it involves an intimate interplay between nuclear physics and statistical mechanics. In the following sections we are going to review first the nuclear properties of hot bulk matter, then discuss the excitation spectrum of the nuclei and the associated specific heat. In section IV we discuss the global problem of the equation of state and, finally, we analyze the effect of the trapping of neutrini on static neutron stars.

2. Bulk matter. - The properties of uniform nucleonic matter have recently received a great deal of attention. Of particular interest is the high degeneracy limit where the nuclear forces play an important role. Calculations have been performed with various nuclear interactions and with various approaches: A finite temperature Hartree-Fock model with a temperature independent
effective interaction has been used by Küpper et al. (1974), El Eid et al. (1977 and Lattimer et al. (1978); Buchler and Coon (1975, 1977) and Kiguchi (1977) have used the finite temperature many-body formalism of Bloch and De Dominicis with a particle-particle ladder summation which is necessary at the densities of interest here. A similar approach based on the Galitskii-Feynman formalism has been used by Bishop (1974). Some calculations have also been performed at the low degeneracy limit (Banerjee et al. 1977; Sahoo et al. 1979). Further work (Buchler and Datta, 1979) comparing the various approximations has shown that the restriction of phasespace in the intermediate states (Pauli principle) gives an improvement of a factor of 2 over a pure scattering approximation (Bethe-Uhlenbeck 1937). A high degeneracy approximation involving the use of the K matrices themselves, rather than of the associated phases, gives a rapidly increasing discrepancy as the degeneracy parameter η (chemical potential/kT) decreases below 2. This is an indication that the Hartree-Fock calculations, based on a high degeneracy assumption, break down at low η; fortunately this breakdown occurs when the relative contribution of the interaction energy to the total energy becomes small. Finally, the use of a temperature independent effective interaction leads to discrepancies of less than 5% and is therefore acceptable for the astrophysical problem at hand.

Uniform nucleonic matter may be unstable and break up into two separate phases (Lattimer et al. 1978), indicative of the formation of clumps (nuclei) or holes. In order to get a feeling for the conditions under which this happens it is worthwhile to study the coexistence region of two bulk phases, i.e. to disregard surface and Coulomb effects.

Consider a mixture of N neutrons and Z protons, at temperature T, confined to a volume V. Let phase 1(2) have a proton to nucleon ratio, Y_{el} (Y_{e2}) and a density ρ_1 (ρ_2). Let ξ be the fraction of nucleons in phase 1, Y_e \equiv ξ/(N + Z) and ρ the overall density. Thus we have the relationship

\[ \frac{1}{ρ} = \frac{1}{ρ_1} ξ + \frac{1}{ρ_2} (1-ξ) \] and

\[ Y_e = Y_{el} ξ + Y_{e2} (1-ξ). \]

The densities and Y_e's are obtained through the solution of the phase-equilibrium conditions

\[ p(ρ_1 Y_{el}) = p(ρ_2 Y_{e2}) \]

\[ μ_n(ρ_1 Y_{el}) = μ_n(ρ_2 Y_{e2}) \]

\[ μ_p(ρ_1 Y_{el}) = μ_p(ρ_2 Y_{e2}), \]

where p is the pressure, μ_n and μ_p the neutron and proton chemical potentials, respectively. This phase equilibrium is more complicated than the one usually encountered, say in water condensation (liquid-vapour equilibrium) because the two phases not only have different densities, but also different proton to neutron ratios.

\[ \text{In figure 1 we exhibit the p versus } ρ \text{ isotherms for symmetric } (Y_e = 0.5) \text{ matter which exhibits a critical temperature of } T_{\text{crit}} = 15.5 \text{ MeV. This critical temperature is lowered as } Y_e \text{ decreases (fig. 3).} \]

These curves are calculated with the nuclear interaction energy described in Buchler and Epstein (1980). The critical temperature is very sensitive to the form of this interaction energy. Lattimer et al.
In figure 2 we exhibit the coexistence curves for bulk matter at an average $Y_e$ of 0.25. The subscript "1" refers to the phase whose mass fraction $\xi = 1$, whereas "2" denotes the phase which is just about to appear. Above the curve marked $\rho_1$ only one phase exists. Consider compressing nucleonic matter along an isotherm of $T = 8$ MeV. At point A a denser phase with a density of $\rho_1'$ then appears, with a $Y_e$ denoted by $\rho_1'$ on the curve for $Y_e$. Between points A and B we have the coexistence of two phases; during compression through this interval $\rho_1$, $\rho_2$, $Y_{e1}$ and $Y_{e2}$ all vary such that the average $Y_e$ equals 0.25. Finally at point B we witness the disappearance of phase 2 denoted by the point $B'$. It is interesting and in fact curious to note that the $\rho_1$ and $\rho_2$ curves and the two peaks do not coincide. If we were to decompress matter along an isotherm at 13.2 MeV at the intersection with the $\rho_1$ curve the incipient phase would now have a higher density because of the intersection of the $\rho_1$ and $\rho_2$ curves. In spite of this unusual behavior we have not found anything wrong or inconsistent. For example, when the coexistence points are plotted along isotherms in a $Y_e$ versus $\rho$ diagram no topological anomaly arises (Lattimer and Buchler, 1979). For $Y_e = 0.5$ no doubling of the curves occurs as shown in figure 3, whereas for $Y_e = 0.1$ the peaks split further apart. A similar situation arises for the nuclear potential used by Lattimer et al. (1978) although the split is not as dramatic. It should be noted that while the $\rho_1$ and $Y_{e1}$ curves can be quite different for the two interactions, the phase separation ($\rho_1$) curve, on the other hand, is very similar except toward the high temperatures.

3. Warm Nuclei.- The large nuclear specific heat associated with the level density of nuclear excited states plays an important role since the nuclei can act as an efficient heat reservoir during the compression phase in stellar collapse (Lamb et al. 1978). The problem of the internal partition function of nuclei has therefore received a great deal of attention with somewhat divergent results (Fowler et al. 1979; Mazurek et al. 1979; Lamb et al. 1978). To lowest order in temperature the internal free energy of a nucleus can be written as $F = E_0 - aT^2$ where the level density parameter, $a$, is related to the density of states near the Fermi surface, $g$, by $a = \gamma^2 g/6$. A Fermi gas model of a square well nucleus (Bohr and Mottelson, 1965) yields $a = A/16$ MeV$^{-1}$. However, a large body of experimental evidence suggests that the actual value of $a$ for spherical nuclei (after subtraction of shell effects) should be approximately $a = A/8$, about twice the simple Fermi gas model value. This discrepancy can be traced to several causes: (a) for actual nuclei the surface is diffuse which leads to a higher
level density at the Fermi surface. (b) Surface collective modes and nuclear deformation may contribute significantly to the level density at the relatively small excitation energies studied experimentally. (c) The effective mass of nucleons inside a nucleus is somewhat uncertain.

For the astrophysical conditions of interest to us here, the excitation temperatures are high, thus dwarfing shell effects as well as the effects of collective modes. In addition nuclei are expected to be spherical. A nuclear Thomas Fermi model should therefore be an ideal tool for the study of the nuclear level density at high temperatures.

We have generalized the nuclear Thomas-Fermi model of Brueckner et al. (1969) to finite temperatures. The problem consists of finding the neutron and proton densities, $\rho_n(r)$ and $\rho_p(r)$, which minimize the free energy $F_N = \int dr [f(\rho_n(r), \rho_p(r), T) + \xi(\mathcal{V}(\rho_n + \rho_p))^2 - \Theta(\mathcal{V}(\rho_n - \rho_p))^2 + \frac{\Theta}{2} \Phi_p(r) \Phi_c(r)]$ at constant given number of neutrons, $N$, and protons, $Z$. The free energy per unit volume, $f$, of the interacting neutron-proton fluid has been taken to be of the form

$$f(\rho_n, \rho_p, T) = f_{NI}(\rho_n, T) + f_{NI}(\rho_p, T) + \mathcal{V}(\rho_n, \rho_p),$$

where the non-interacting part $f_{NI}$ is well known and where we have chosen the interaction part $\mathcal{V}$ to be the same as at zero temperature (Brueckner et al. 1969, Lombard 1973). The only phenomenological parameters of the model, $\xi$ and $\Theta$ have been chosen to be independent of temperature and have been adjusted to yield a good fit to the binding energies, radii and surface thicknesses of a whole range of known nuclei. The variational problem is treated by solving the associated Euler-Lagrange equations. Further details can be found in Buchler and Epstein (1980).

The results of our calculation are well fitted by the expression

$$a = a_1 A + a_2 A^{3/3},$$

with $a_1 = 0.074$ and $a_2 = 0.154 \text{ MeV}^{-1}$, where in the spirit of the Weizsäcker mass formula we have introduced a bulk term and a surface term, (The symmetry term is absolutely negligible). As expected, the bulk term has a value very close to that of a uniform Fermi gas. When compared to experiment our results reproduce more than 80% of the general trends for the data and are expected to do even better at the higher astrophysical temperatures.

The nuclear Thomas-Fermi model thus seems to be able to account not only for the bulk properties of nuclei, namely energies, radii and surface thickness, but also for the average features of known nuclear excitations. It can thus be used with some confidence for the study of nuclei embedded in a hot sea of nucleons.

4. The equation of state.- The zero temperature equation of state at subnuclear densities has been treated by several groups using different approaches: a liquid drop model (Baym et al., 1971), a self consistent Thomas-Fermi model (Buchler et al., 1971) and a Brueckner-Hartree-Fock model (Negele and Vautherin, 1973). The finite temperatures add several complications to the problem: (1) the thermal properties of bulk matter are more difficult to calculate and need further study; (2) a good theory for the excitation spectrum of an inhomogeneous system must be used, e.g. as described in the previous section. The specific heat of nuclei is large and plays an important role, because nuclei act as a thermostat, with an effective adiabatic index $\gamma = 1$, whereas the free, non-relativistic nucleons have a $\gamma = 5/3$. It turns out that, at low entropy, the two effects compensate (Pethick, private communication) and the global adiabatic index is very close to that of the leptons, i.e. $\gamma = 4/3$; (3) thermal structural changes in the nuclei,
or rather nuclear clusters, must be accounted for; (4) the statistical mechanics of these clusters, i.e. the motion and Coulomb interaction between these nuclear clusters is important. Fortunately in the region of interest the plasma is classical, $\lambda_{\text{thermal}} \ll \text{intercluster spacing}$; however, the plasma parameter, $\Gamma$, varies from about unity to several thousand, depending on the collapse adiabat, so that both the crystallized and the liquid phases (Hansen, 1973; Pollock et al. 1973) need to be considered. The problem is somewhat complicated by the fact that toward the higher densities the packing fraction gets large and it may actually be preferable to think of nucleon holes (or bubbles as suggested by Lamb et al., 1978). At the high temperatures, in the neighborhood of the critical temperature, say for $T > 15$ MeV one expects large fluctuations in the distribution of cluster sizes, which introduces enormous complications; among other things it no longer allows one to make use of the average, typical nucleus concept. The presence of alpha particles and their interaction with the other particles also introduces additional complications. Fortunately the ultra-relativistic electrons are very bad screeners and are essentially decoupled from the nucleons except for ensuring charge neutrality. The uncharged neutrino totally drops out from the equations. An elegant description of the properties of such matter has recently been given by Lamb et al. (1978) who have extended to finite temperatures a zero temperature liquid droplet model together with a planar Thomas-Fermi evaluation of the nuclear surface energy. While their approach constitutes a great step forward in our understanding of dense matter, we feel that it incorporates the various physical effects in a way which is not fully self consistent. We therefore go on to describe our own approach which is based on a self-consistent nuclear Thomas-Fermi model; the latter is a finite temperature extension of earlier work on neutron star matter (Buchler et al. 1971).

We assume (Barranco and Buchler, 1979 in preparation) that we can represent the statistical average of nuclear clusters (nuclei) by a single representative cluster. The advantage of the Thomas-Fermi model is that it allows the nuclear cluster to smoothly merge with the background nucleons as a matter of fact it does not distinguish between nucleons of high density (inside the cluster) and of low density (in the background sea). We shall refer to such a cluster together with the background as a cell. The size of the cells is determined by the overall charge neutrality with the background electrons and by the requirement that the radial derivative of the density vanishes at $r = R_{\text{cell}}$. Cells are assumed to be spherical.

Our problem consists of writing down an expression for the total free energy density and of finding the density profiles, $\rho_p(r)$ and $\rho_n(r)$ which minimize this free energy for a given overall baryon density $\rho_B$, proton concentration, $Y_e$, and overall charge neutrality. The nuclear free energy $F_{\text{nucl}}$ has already been described in a previous section. The treatment of the Coulomb energy presents a few complications: In the zero temperature calculations (Buchler et al. 1971) the Coulomb energy had the Wigner-Seitz form

$$F_{\text{Coul}}^{\text{WS}}[\rho_p] = \frac{e}{2} \int R_{\text{cell}} (\rho_p(r) - \rho_e)^\phi \text{Coul}_1(r) dr^3$$

which contains both the protons' self-energy $\frac{e^2}{2} \int \rho_p^\phi \text{Coul}_1(r) dr^3$, usually included in the nuclear energy ($Z^2/A^{1/3}$ term in the Weizsäcker mass formula), as well as the (remaining) lattice energy. For an extension to finite temperatures we rely on statistical mechanics results. The latter have been obtained for point particles or for hard spheres. The finite size of the clusters enters both into the Coulomb energy and into the energy of thermal-motion. It is customary to include all finite size effects into the so-called Coulomb excess energy and then to simply add the free
energy of thermal motion for point particles:
\[ F_{\text{TH}} = -T \ln \left| \psi_{\text{cell}}^* \left( \frac{\text{AM}}{2\pi \hbar} \right)^{1/2} \right|, \]
e.g. in the liquid phase. The Coulomb excess energy for a gas of hard spheres of radii \( a \) is of the form
\[ F_{\text{coul}} = \Gamma \times \text{funct.}(\Gamma, \sigma), \]
and \( \sigma = (a/R_{\text{cell}})^{1/2} \). The available expressions for the excess Coulomb energy have three deficiencies: (a) they do not include the protons' self-energy (b) they do not take into account the diffuseness of the clusters' surface and (c) the replacement of the compressible cluster by a hard sphere is an approximation. In order to include the major surface diffuseness effects into the Coulomb energy, on the one hand, and to avoid double counting, on the other, we approximate the total Coulomb energy by the following expression:
\[ \frac{1}{V_{\text{cell}}^2} \left[ F_{\text{coul}} + F_{\text{coul}} - F_{\text{TH}} - \mu_1 N - \mu_2 Z \right] + f_e(\rho_p) + f_\nu(\rho_\nu) \]
where the Lagrange multipliers \( \mu_1 \) and \( \mu_2 \) are to be chosen such that the average baryon density is \( \rho_B \) and the average proton concentration is \( Y_e \). The electron density is fixed by the overall charge neutrality
\[ \rho_e = \frac{1}{V_{\text{cell}}} \int_{R_{\text{cell}}}^{} \rho_p(\tau) d\tau. \]

The neutrino density can be specified separately. When beta kinetic equilibrium obtains, we have the additional constraint that
\[ \rho_e + \rho_\nu = \rho_{\text{lepton}} \text{ (given)}, \]
which can easily be handled with an additional Lagrange multiplier and which yields the Gibbs relation:
\[ \mu_\nu + \mu_\nu = \mu_p + \mu_e. \]
The minimization proceeds through the solution of the Euler-Lagrange equations associated with the variational problem. The solution of these very nonlinear differential equations, which involves a delicate eigenvalue search, is now in progress (Barranco and Buchler).

5. Neutrino trapped neutron stars.- It has only recently been recognized (Mazurek, 1977; Yush and Buchler, 1977; Arnett, 1977) that the neutrinos diffuse out of the core on a timescale longer than dynamical and thus that they have time to build up their degeneracy to essentially beta kinetic equilibrium values at densities in excess of \( 10^{12} \text{ g/cm}^3 \). An important implication for the equation of state is that electron captures are thus inhibited and that the value of \( Y_e \) and \( Y_L \) (lepton to baryon ratio) stays much larger than previously believed (Sato, 1975).

The pre-collapse specific entropy \( s \) of matter composed predominantly of Fe is very low and electron captures during the initial phase of the collapse \( (\rho \leq 10^9 \text{ g/cm}^3) \) do not add significantly to \( s \) (Bethe et al., 1979); during the later stages \( (\rho \geq 10^{12} \text{ g/cm}^3) \), since the neutrinos are strongly trapped, there again is only a small energy and entropy leakage, so that the global collapse should be close to adiabatic. This physics is of course implicitly contained in the complex numerical hydrodynamic calculations, but it has only recently been explicitly pointed out by Bethe et al. (1979) who have paraphrased the core-collapse calculations in terms of entropy considerations. Bethe et al. argue that a collapse adiabat of low entropy should prevent nuclei from boiling off free nucleons (neutron drip), since the latter, non-degenerate at first, have a high entropy. As a result the dominant contribution to the pressure should come from the degenerate, relativistic leptons which have a low adiabatic index \( (\gamma = \frac{5\text{MeV}}{\text{MeV}} \approx 4/3) \), therefore giving rise to a soft equation of state until about nuclear densities when the nuclei are dissolved (see also Lamb et al., 1978).

The study of zero temperature matter in
the same density range has shown that free neutrons abound (e.g., Buchler and Barkat, 1971, Baym et al., 1971). Zero temperature matter is, however, also a very low entropy (in fact zero entropy) matter, so that a low entropy cannot be the dominant reason for preventing nucleon boil-off, at least when the temperature is low.

It is possible to show Gudmundsson et al. (1980) from very simple analytical considerations that it is this large $Y_L$, resulting from neutrino trapping, which is the dominant factor in preventing nucleon drip at low temperatures. They show that the dominant nuclear species $(N,Z)$ is essentially determined by the neutron chemical potential, $\mu_n$, independently of the concentration of neutrini and that, at fixed $\mu_n$, an increase of $Y_L$ increases the relative proportion of bound nucleons, as well as the total baryon density itself; neutrino trapping forces nuclei to absorb the free nucleons. Conversely, with neutrini trapped, therefore, at a given total baryon density, the nuclear species is the same as would be for matter at lower baryon density with fewer trapped neutrini. Gudmundsson et al. (1980) have exploited this result to extrapolate the zero temperature equation of state of Baym et al. (1971), computed for $Y_L = Y_e$, to a whole range of $Y_L > Y_e$. This equation of state has been used in conjunction with the Oppenheimer-Volkov equations to construct neutrino-trapped neutron stars in hydrostatic equilibrium. The presence of trapped neutrini causes a dramatic increase of the minimum neutron star mass. Because of the additional lepton pressure the equation of state now has an adiabatic index $\gamma = 4/3$ between $10^{11.5}$ to $10^{13}$ g/cm$^3$, thus increasing the gravitational mass it can support. The lepton rich neutron stars are smaller by about a factor of 2 in radius, as compared to a standard neutron star, with a concomitant reduction in the central density. It is somewhat frustrating to note that even so the central density of a typical 0.8 $M_\odot$ neutron star lies around $7 \times 10^{14}$ g/cm$^3$ where our understanding of the equation of state starts to break down. For the collapsing core the situation is even worse because of an overshoot about the hydrostatic equilibrium state. Of special interest is also the gravitational binding energy: For a 0.8 $M_\odot$ core (which is the size of the homologously collapsing supernova core) a binding energy of up to $8 \times 10^{52}$ erg can be released during the final contraction. A sizeable part of this energy can be converted into kinetic energy during the expected Rayleigh-Taylor overturn of the core and give rise to a powerful ejection of the envelope.

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Appendix.-

The behavior observed in figures 2 and 3 suggests the following qualitative evolution of the phases in the coexistence region as we go from below the crossing point (fig 4a) of the $\rho_1$ and $\rho_2$ curves (refer to figs. 2 and 3), through the point (fig 4b) and finally above it. The dashed line denotes the average values of the density and of $Y_e$, respectively. On the left-hand side on the coexistence curve ($\epsilon=1$) the incipient phase (A') is always different from the main phase (A), both in density and in $Y_e$. On the other hand, on the right-hand side on the coexistence curve ($\epsilon=0$), the two phases are usually different as well, $\rho_A > \rho_{A'}$ below the crossing point and with $\rho_B < \rho_{B'}$ above it. At the crossing point itself, however, the two phases are exactly the same (B=B') so that we have what may be called a second-order phase separation. Above this crossing point a qualitatively different behaviour occurs. In an isothermal compression a denser phase first appears, reaches a maximum massfraction and then disappears again. We are in the process of checking these conclusions numerically.
References:


4/ Beth, E. and Uhlerbeck, G.E., Physica 4, 915 (1937)


6/ Bishop, R. F., P. R. A10, 2423 (1974)


15/ Gudmundsson, E. H. and Buchler, J. R., Astrophys. J. (June 1980)


20/ Lattimer, J. and Buchler, J. R. (unpublished)


22/ Lombard, R. J., Ann Phys. 77, 380 (1973)


27/ Sahoo, N. and Tripathi, R. K., preprint, 1979
