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FORCED OSCILLATIONS OF NONSPHERICAL BUBBLES

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Résumé.- On examine différents aspects du mouvement d'une bulle qui oscille, en commençant par la formulation générale du problème et sa solution dans le cas d'une oscillation linéaire sphérique. On reprend alors ce problème au moyen d'une méthode variationnelle. On passe ensuite aux oscillations (spériques) sous-harmoniques et non-linéaires puis aux oscillations non-sphériques et à une discussion de l'équation (non-linéaire) de Mathieu. Ensuite, la méthode variationnelle est utilisée pour étudier le couplage par interaction non-linéaire entre les oscillations sphériques sous-harmoniques et les oscillations non-sphériques. Enfin, on évoque le rôle éventuel du couplage non-linéaire dans les études expérimentales.

Abstract.- Various aspects of an oscillating bubble are presented. The general formulation and the linear spherical oscillation are discussed first followed by a corresponding variational treatment. Nonlinear subharmonic spherical oscillations and nonspherical oscillation are presented next. The study of nonspherical oscillation leads to a discussion on nonlinear Mathieu equation. Finally, the variational method is employed for the study of the coupling between subharmonic spherical oscillation and nonspherical oscillation through nonlinear interaction. Experimental implication of the nonlinear coupling is briefly discussed.

1. INTRODUCTION.- A bubble in a liquid is a fascinating classical oscillator, all the more because it is an ordinary, visually observable physical object. It can execute simple harmonic, small amplitude, spherical oscillations with well-defined resonance. It has damping mechanisms due to viscosity and thermal processes. It is a nonlinear oscillator. Thus, among other things, subharmonic spherical oscillation can be generated under external oscillating pressure. /1,2,3,4/ Although the surface tension tends to keep a small bubble spherical, the spherical shape of bubble can often be distorted and induced by the spherical oscillation /5,6/. Indeed, the possible generation of nonspherical oscillation reveals clearly that the bubble system is a three dimensional nonlinear oscillator. Furthermore the bubble can split, and for vapor bubbles they can collapse or be created.

The purpose of this work is to explore one aspect of the oscillating bubble, i.e., the nonlinear interaction between the subharmonic spherical and nonspherical oscillations. In order to clarify the problems involved, a brief review on the general formulation, the subharmonic spherical oscillation and the linear theory of nonspherical oscillation will first be given. To facilitate the analysis of the complex problem, variational method of approximation is to be employed. Hence a variational formulation of the general problem will be presented. For the purpose of simplifying analysis, the dissipation processes are neglected at this stage.

The linear theory of nonspherical oscillation was shown to lead to Mathieu equations /6/. A study on nonlinear Mathieu equation /7/ will thus be given to enhance our understanding of the problem. Finally, the nonlinear coupling between the subharmonic spherical and nonspherical oscillation will be analyzed using the variational method.

2. GENERAL FORMULATION FOR A BUBBLE IN OSCILLATION. Let us consider a gas bubble in an inviscid, and incompressible liquid. Let us adopt the spherical coordinate system (r,θ,ξ) and let the bubble surface be defined as

$$r = R\,(\theta,\xi,t)\ . \tag{1}$$

Then, the dynamical problem can be formulated as follows /8/ :

$$\nabla^2\phi = 0, \quad \text{for} \quad r > R\ , \tag{2}$$

$$\frac{p}{\rho} + \frac{1}{2}\,(\nabla\phi)^2 + \frac{\partial\phi}{\partial t} = \frac{p_\infty(t)}{\rho}\ , \quad \text{for} \quad r > R\ , \tag{3}$$

where ϕ is the velocity potential, p is the pressure, and ρ is the density of the liquid. $p_\infty\,(t)$

is the externally applied pressure at infinity. Equations (2) and (3) are the continuity equation and the Bernoulli equation respectively. The kinematic and dynamic boundary conditions on the bubble surface are

$$\frac{\partial R}{\partial t} + \left(\frac{\partial R}{\partial \theta}\frac{\partial \phi}{\partial \theta}\Big/R^2\right) + \left(\frac{\partial R}{\partial \xi}\frac{\partial \phi}{\partial \xi}\Big/R^2\sin^2\theta\right) - \frac{\partial \phi}{\partial r} = 0, \text{ on } r = R , \quad (4)$$

and

$$p + \sigma\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right) = p_i \quad \text{on} \quad r = R , \quad (5)$$

where σ is the surface tension coefficient, and R_α and R_β *are the two principal radii of curvature* at the point of concern on the bubble surface. p_i is the internal pressure which will be assumed to be governed by the polytrophic relation :

$$p_i = p_e \left(\frac{V_e}{V_i}\right)^\gamma , \quad (6)$$

where p_e and V_e are the internal pressure and volume of the bubble at equilibrium and V_i is the present volume of the bubble.

Equation (2) can be solved in terms of the spherical harmonics $Y_n\,(\theta,\xi)$, and we obtain :

$$\phi = -\frac{\phi_o(t)}{r} + \sum_{n=1}^{\infty} \frac{\beta_n(t)}{r^{n+1}} Y_n\,(\theta,\xi) . \quad (7)$$

When R is a single-valued function of θ and ξ, we can also express R in terms of the spherical harmonics :

$$R = R_o\,(t) + \sum_{n=1}^{\infty} a_n(t)\, Y_n\,(\theta,\xi) . \quad (8)$$

It is readily seen that the equilibrium solution is given by :

$$\phi = 0,\ R = R_e,\ p_i = p_e, \text{ and } p_\infty = p_e - \frac{2\sigma}{R_e} \equiv p_a .$$

Let us denote

$$R' = R_o(t) - R_e$$

and

$$p_1 = p_\infty(t) - p_a .$$

When p_1, R', ϕ_o, β_n and a_n are small, and only linear terms of these quantities are retained in the equations, then due to the orthogonality of the spherical harmonics, the spherical harmonic modes are all decoupled, and we obtain

$$\frac{dR'}{dt} - \frac{\phi_o}{R_e^2} = 0 , \quad (9)$$

$$\frac{1}{R_e^2}\frac{d\phi_o}{dt} + \omega_o R' = -\frac{p_1}{\rho R_e} , \quad (10)$$

$$\frac{da_n}{dt} + \frac{n+1}{R_e^{n+2}}\beta_n = 0 , \quad n = 1,2,\ldots, \quad (11)$$

$$\frac{1}{R_e^{n-1}}\frac{d\beta_n}{dt} - \frac{\sigma}{\rho}(n-1)(n+2)\, a_n = 0, \quad n = 1,2,\ldots, \quad (12)$$

where

$$\omega_o^2 = \frac{3\gamma p_e}{\rho R_e^2}\left(1 - \frac{2\sigma}{3\gamma p_e R_e}\right) . \quad (13)$$

Or

$$\frac{d^2R'}{dt^2} + \omega_o^2 R' = -\frac{p_1(t)}{\rho R_e} , \quad (14)$$

and

$$\frac{d^2a_n}{dt^2} + (n-1)(n+1)(n+2)\frac{\sigma}{\rho R_e^3}\, a_n = 0 . \quad (15)$$

If we relax the restriction on the amplitude of the spherical mode yet maintain that the nonspherical amplitudes be small, then the varioux spherical harmonic modes are still decoupled, and we obtain :

$$R_o\frac{d^2R_o}{dt^2} + \frac{3}{2}\left(\frac{dR_o}{dt}\right)^2 = \left[\frac{1}{\rho}\, p_e\left(\frac{R_e}{R_o}\right)^{3\gamma} - \frac{2\sigma}{R_o} - p_\infty\,(t)\right], \quad (16)$$

and

$$\frac{d^2a_n}{dt^2} + \frac{3}{R_o}\frac{dR_o}{dt}\frac{da_n}{dt}$$

$$+ \left[(n-1)(n+1)(n+2)\frac{\sigma}{\rho R_o^3} - \frac{(n-1)}{R_o}\frac{d^2R_o}{dt^2}\right]a_n = 0 \ . \qquad (17)$$

These equations are essentially those first derived by Plesset. /5/

3. VARIATIONAL FORMULATION.- The previous formulation can be shown to be equivalent to the variational principle /9/ : the flow field of the system and the motion of bubble surface are such that an extremum is attained by the fuctional

$$I = \int_{t_1}^{t_2} J\,dt \ , \qquad (18)$$

where

$$J = \frac{p_e V_e^\gamma}{1-\gamma}\left(\int_{V_i} dV\right)^{1-\gamma}$$

$$+ \int_{V_o} \left\{ \rho\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right] + p_\infty\right\} dV$$

$$- \sigma\int_A dA \ , \qquad (19)$$

V_i is the volume inside the bubble ; V_o is the volume outside the bubble ; and A is the bubble surface area.

It may be readily shown that if we make a direct substitution of the bubble surface r = R and the velocity potential ϕ in terms of (7) and (8) in I, and then vary with respect to ϕ_o, R_o, β_n, and a_n, the equations obtained after linearization are exactly the same as those given in (16) and (17). /10/ The nonlinear equations are again coupled and extremely complicated. However, the variational formulation can offer a straightforward approximate scheme by simply taking only a finite term in the expansions (7) and (8). If these cut-off schemes could be in any way justified, then the amplitude of the nonspherical modes need not be restricted to be small. Therefore the variational formulation offers not only another perspective of the problem but also a possible approximate scheme for various theoretical explorations.

4. SUBHARMONIC SPHERICAL OSCILLATIONS.- The dynamics of a spherical bubble is governed by the equation (16). For small amplitude motion near equilibrium, the governing equation reduces to (14). Let us take

$$p_1(t) = p \sin 2\omega t \ . \qquad (20)$$

Then the solution of (14) is given by

$$R' = C_1 \sin \omega_o t + C_2 \cos \omega_o t + \frac{p \sin 2\omega t}{\rho R_e(\omega_o^2 - 4\omega^2)} \ , \qquad (21)$$

where the first two terms represent the free oscillations with the natural frequency ω_o.

When the dissipation processes are included in the problem, the free oscillation terms will die out for large t, and only the last term, the excited term, will persist. The excited oscillation, according to this linear theory, will oscillate with the excitation frequency (2ω).

Now if the excitation amplitude p is not small, we should go back to the equation (16) for correction to the linear theory. It is easy to see higher harmonic oscillations with multiples of the excitation frequency will be generated due to the nonlinear interactions. It is less obvious that oscillations with fractions of the excitation frequency would also be generated. These are called subharmonic oscillations. Theoretical studies on the subharmonic oscillations of bubbles have been carried out using the method of parametric reasonance /1/, the averaging method of Bogolyubov-Krylov /3/, as well as the variational method /4/. Some of the significant findings are :

i) When the dissipation process is included, the subharmonic oscillation can be generated only when the excitation amplitude exceeds certain threshold magnitude to overcome the damping.
ii) The subharmonic oscillations can be excited only in a definite frequency interval. The larger is the excitation amplitude, the wider would be the frequency interval. For the most important subharmonic oscillation with oscillation frequency ω, i.e., half the excitation

frequency, the permissible frequency interval is in the neighborhood of $\omega = \omega_o$, i.e. the excitation frequency is around twice the natural frequency.

When the dissipation process is not included in the formulation, there is no threshold for the excitation amplitude, but there is again a permissible excitable frequency range. These qualitative and approximate behavior can be most simply demonstrated by the variational method. /4/

5. SMALL AMPLITUDE NONSPHERICAL OSCILLATIONS.- When the amplitudes of the nonspherical modes are small, the equations of motion are given by (16) and (17). Let us introduce b_n by

$$b_n = (\frac{R_o}{R_e})^{3/2} a_n .$$

then the equation (17) becomes

$$\frac{d^2b_n}{dt^2} + B_n b_n = 0 , \qquad (22)$$

where

$$B_n = \omega_n^2 - \frac{(n+\frac{1}{2})}{R_o}\frac{d^2R_o}{dt^2} - \frac{3}{4}\frac{1}{R_o^2}(\frac{dR_o}{dt})^2 , \qquad (23)$$

$$\omega_n^2 = (n-1)(n+1)(n+2)\frac{\sigma}{\rho R_o^3} . \qquad (24)$$

If the excited spherical mode is oscillating according to the linear theory, then we can write

$$R_o = R_e (1 + \delta \sin 2\omega t) .$$

Thus, we have

$$B_n = \left[\omega_n^2 + 0(\delta^2)\right]$$

$$+ \{\left[4(n+\tfrac{1}{2})\omega^2 - 3\omega_n^2\right]\delta + 0(\delta^2)\} \sin 2\omega t . \qquad (25)$$

If we ignore $0(\delta^2)$ terms, then equation (22), which is a Hill's equation, becomes a Mathieu equation. /16/ Thus when δ is large enough, the amplitude of some of the nonspherical modes will grow according to the stability theory of the Mathieu equation /11/. The critical value is given by /12/

$$(\omega^2-\omega_n^2) = 2\delta\left[(n+\tfrac{1}{2})\omega^2 - \tfrac{3}{4}\omega_n^2\right] . \qquad (26)$$

It is significant that at the critical value the frequency of oscillation of the nonspherical mode is ω, i.e. half the excited frequency. Therefore, it is also a subharmonic oscillation. There is also a limited range of permissible frequency interval, which is related now to ω_n rather than the natural frequency ω_o. It is likewise worth noting that the parametric resonance approach to the study of spherical subharmonic oscillations also leads to a type of Hill's equation.

6. NONLINEAR MATHIEU EQUATION.- When the nonspherical mode, according to the equation (22), is unstable, what can we say about the final state of the motion ? One possible outcome is that an asymptotic state of steady finite amplitude oscillation will be reached due to the nonlinear interactions so far neglected. A study of a model nonlinear Mathieu equation could shed lights on this question.

A nonlinear Mathieu equation of the following type :

$$\frac{d^2x}{dt^2} + (\alpha + \beta \cos 2t)x + rx^3 = 0 , \qquad (27)$$

has been chosen for the study, and the variational method has been employed for the analysis /7/. When the constant parameter $r = 0$, equation (27) is the ordinary linear Mathieu equation.

We may recall that for the linear Mathieu equation, the (α,β)-plane is divided into stable and unstable regions by the so-called characteristic curves. In the stable region, the two linearly independent solutions both tend to zero asymptotically, whereas in the unstable region, both solutions tend to grow indefinitely. Only on the characteristic curves, a solution periodic in time can exist, which is called a Mathieu function. The other linearly independent solution on the characteristic curve is unstable /11/. Therefore, except for the exceptional values of (α,β) on the characteristic curves, asymptotic periodic solutions are generally not to be expected to exist for linear Mathieu equations.

For nonlinear Mathieu equation, however, it may be shown by the variational method /7/, that asymptotic periodic solution of finite amplitude can exist for a finite region in the (α,β)-plane, where the linear Mathieu system is unstable. The possible existence of such asymptotic periodic solution suggests that the linearly unstable nonspherical modes of oscillation of bubble may also settle down to finite amplitude steady oscillation.

7. THE COUPLING OF SUBHARMONIC SPHERICAL AND NONSPHERICAL OSCILLATIONS.- As we have described in section IV, when the excitation amplitude is large enough, subharmonic spherical oscillations are to be generated. The most easily excited mode is the subharmonic oscillation with frequency half the excitation frequency. At the same time, as described in section V, nonspherical oscillations with half the excitation frequency are also induced by the external spherical excitation. It is conceivable that these finite amplitude oscillations will interact with each other.

We do not attempt to solve this extremely difficult problem in its general form. We shall try to establish qualitatively and approximately that finite amplitude asymptotic subharmonic spherical and nonspherical oscillations can both be generated by the externally applied spherical excitation. Variational method of analysis is to be employed, since this method is most straightforward to apply and simplest to single out the relevant modes of oscillation.

In its simplest form, we shall look for solutions such that

$$p_\infty = p_a + p \sin 2\omega t \ , \tag{28}$$

$$R = R_i + R_1 \sin 2\omega t + R_2 \cos \omega t + a \cos \omega t \; Y_n\,(\theta,\phi) \ , \tag{29}$$

$$\phi = -\frac{1}{r}\,(\psi_1 \cos 2\omega t + \psi_2 \sin \omega t) + \frac{\beta}{r^{n+1}} \sin \omega t \; Y_n\,(\theta,\phi) \ , \tag{30}$$

where p, R_1, R_2, a, ψ_1, ψ_2, and β are all taken to be constants. These are the asymptotic trial solutions to be substituted into the expression (18) to be varied. They are certainly not the most general form of the solutions. Guided by our experience with the subharmonic spherical oscillations and the Mathieu equations, they are the simplest relevant solutions to be expected. The relation (26) is to be used to determine the value n, since this mode is expected to be most unstable according to the linear theory.

Even though (29) and (30) are the simplest form of trial solutions which encompass the coupling between subharmonic spherical and nonspherical modes, the results are already quite involved. The averaged functional computed up to $O(\varepsilon^4)$ is given elswhere /15/.

For small amplitude motions, the linear equations are readily found to be :

$$\left[(2\omega)^2 - \omega_o^2\right] R_1 = \frac{p}{\rho R_e} \ , \tag{31}$$

$$\left[\omega^2 - \omega_o^2\right] R_2 = 0 \ , \tag{32}$$

and

$$\left[\omega^2 - \omega_n^2\right] a = 0 \ . \tag{33}$$

If we carry to next order, without going into detail, the equations take the following form :

$$\left[(2\omega)^2 - \omega_o^2\right] R_1 = \frac{p}{\rho R_e} + R_1 C_1 \tag{32}$$

$$\left[\omega^2 - \omega_o^2\right] R_2 = R_2 C_2 + c a^3 \ , \tag{33}$$

and

$$\left[\omega^2 - \omega_n^2\right] a = a C_3 \ , \tag{34}$$

where C_1, C_2 and C_3 are quadratic in R_1, R_2 and a, and c is some constant coefficient. Therefore we can see qualitatively that the subharmonic spherical oscillation is generated when the excitation frequency 2ω is about twice the value of ω_o, while the nonspherical oscillation is generated when the excitation frequency is about twice the value of ω_n. However, as seen from (33), the nonspherical oscillation will induce the subharmonic spherical oscillation through the nonlinear interaction. To this order of calculation the coupling is essentially unidirectional. The excitation of finite amplitude nonspherical oscillation would lead to the subharmonic spherical oscillation but not necessarily vice versa.

8. DISCUSSION.- The problem of nonlinear coupling between spherical and nonspherical oscillation is extremely complex. In order to obtain some concrete understanding of the problem we have made some drastically simplifying assumptions. The dissipation processes are neglected, and we are thus unable to calculate the threshold amplitude for these nonlinear oscillations. We have singled out and chosen some particular modes of oscillation. Although the choices are plausible from physical reasoning and variational argument, quantitative accuracy is expected to suffer. However despite these defects and incompleteness, a qualitative demonstration of the coupling of the subharmonic spherical and nonspherical oscillations through nonlinear interaction was established, and a framework for more accurate quantitative calculation was set up for this complex problem.

From the complexity of analysis even for our simplified approach, it seems to be not very productive to attempt for more refined theoretical study at this stage before some experimental study is made to establish the basic validity of the nonlinear coupling between subharmonic spherical and nonspherical oscillations. One important implication of the coupling is worth discussing. As we have mentioned before, the subharmonic spherical oscillation will oscillate in the neighborhood of the natural frequency ω_0, when the excitation frequency 2ω is about $2\omega_0$. On the other hand, the most critical situation for the nonspherical mode to be excited is for $2\omega \approx \omega_0$ and $\omega_n \approx \frac{\omega_0}{2}$, i.e., when ω_n is half the excitation frequency, wile the normal resonance is approached to yield the largest value of R_1. Thus through the nonlinear coupling, subharmonic spherical oscillation can be generated to oscillate in the neighborhood of $\omega_0/2$.

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