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SQUINT-A NONLINEAR FEATURE OF FLOW PATTERNS IN NEMATICS

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Abstract. — The curvature $\psi$ of an electronemodynamic flow pattern obeys, in the steady state, the nonlinear equation $\psi'' + a^2 \psi - \psi^3 + b \psi' = 0$. The term $\psi' \psi''$, called squint, is studied. Squint originates in the distortion torque and the distortion stress. Squint brakes the axial inversion symmetry of a simple flow roll but preserves the reflection symmetry of a roll pair. Flow patterns are predicted to show an observable pairing of domain stripes. Terms unjustifiably discarded in the one-constant approximation are explicitly introduced. The mean angle approximation which replaces the variable coefficients $a^2$ and $b$ by constant mean values is also discussed and justified.

Applied electric, magnetic and thermal fields induce in nematic liquid crystals a variety of interesting flow patterns [1-4]. The well known Williams domains may be taken as a typical representative. Recently, the importance of nonlinear phenomena in these flow patterns has been increasingly recognized [5-11].

Moritz and Franklin [7] presented a concise but reasonably self-contained derivation of the governing equations for the charge density $q$ and the pattern curvature $\psi$. A comprehensive and detailed treatment was given by Moritz [12]. Their result is:

$$\frac{dq}{dt} = a_{11} q + a_{12} \psi,$$
$$\frac{d\psi}{dt} = a_{21} q + a_{22} \psi + b_1 \frac{d^2 \psi}{dx^2} + b_2 \frac{d \psi}{dx} + b_3 \psi^3. \tag{1}$$

This paper focuses on the remarkable nonlinear term with $\psi \psi'/dx$ that has been neglected heretofore. For reasons that will become clear later, let us call it squint.

The presentation will be concise. A full report will be published elsewhere.

We deal with a nematic in a standard sandwich geometry, independent of the $z$-coordinate. An electric field is applied across the sample in the $y$-direction. The angle between the director and the $x$-axis is $\phi$.

The squint terms arise from two sources: the distortion torque density $m_z$ and the distortion stress density $\sigma$.

Moritz, in his analysis [7,12], follows de Gennes [1] and chooses the pattern curvature $\psi = \partial \phi / \partial x$, rather than the director angle $\phi$, as the relevant variable. Accordingly, we are interested in the derivatives of the distortion torque density:

$$m_{xx} = \frac{\partial m_z}{\partial x} = k(\phi_{xxx} + \phi_{xyy}) + 2 \delta \left\{ C[1/2(\phi_{xxx} - \phi_{xyy}) + \phi_{xx} \phi_y + 3 \phi_{xy} \phi_x - \phi_z^3 + \phi_x \phi_y^2] + 
S(\phi_{xx}^2 - \phi_{xy}^2 + \phi_{xy} - \phi_{yy}^2 + \phi_x^2 + \phi_y^2 - \phi_z^2) \right\},$$
$$m_{xy} = \frac{\partial m_z}{\partial y} = k(\phi_{yy} + \phi_{xxy}) + 2 \delta \left\{ C[1/2(-\phi_{yy} + \phi_{xxy}) + \phi_{yy} \phi_x + 3 \phi_{xy} \phi_y + \phi_z^3 - \phi_x \phi_y^2 - \phi_x^2 \phi_y^2 \phi_z^2] + 
S(\phi_{xx}^2 + 2 \phi_{yy} \phi_y - \phi_{xy} \phi_x - \phi_{xx}^2 - \phi_{yy}^2 - 2 \phi_x^2 \phi_y^2 \phi_z^2) \right\}. \tag{2}$$

Here we have introduced the notations:

$$C = \cos 2 \phi, \quad S = \sin 2 \phi. \tag{3}$$

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and
\[ k = \frac{1}{2}(k_{33} + k_{11}), \quad \delta = \frac{1}{2}(k_{33} - k_{11}), \]

where \( k_{33} \) and \( k_{11} \) are the bend and splay modulus, respectively. The subscripts \( x, y \) denote partial derivatives, of course.

The body force density components \( f_x, f_y \) due to \( \tau \) are given by
\[
- f_x = k(2 \varphi_{xx} \varphi_x + \varphi_{xy} \varphi_y + \varphi_{yy} \varphi_x) + \delta[C(2 \varphi_{xx} \varphi_x - \varphi_{xy} \varphi_y - \varphi_{yy} \varphi_x + 4 \varphi_y^2 \varphi_y) + \nonumber \\
+ S(\varphi_{xx} \varphi_y + 3 \varphi_{xy} \varphi_x - 2 \varphi_x^2 + 2 \varphi_x \varphi_y^2)], \nonumber \\
- f_y = k(2 \varphi_{yy} \varphi_y + \varphi_{xy} \varphi_x + \varphi_{xx} \varphi_y) + \delta[C(-2 \varphi_{yy} \varphi_y + \varphi_{xy} \varphi_x + \varphi_{xx} \varphi_y + 4 \varphi_x^2 \varphi_x) + \nonumber \\
+ S(\varphi_{yy} \varphi_x + 3 \varphi_{xy} \varphi_y - 2 \varphi_y^2 - 2 \varphi_y \varphi_x^2)].
\]

In eqs. (2) and (5) we have conveniently separated the terms kept in the one-constant approximation from the «correction» terms which by no means small.

Following Penz and Ford [4, 7, 12], we now eliminate the dependence on \( y \) by introducing an arbitrary dispersion parameter \( p \) such that
\[
\frac{\partial}{\partial y} = p \frac{\partial}{\partial x}.
\]

All derivatives in eqs. (2) and (5) can now be expressed in terms of
\[
\psi = d\varphi/dx, \quad \psi' = d^2\varphi/dx^2, \quad \psi'' = d^3\varphi/dx^3.
\]

Then, eqs. (2) and (5) yield, respectively, the following squint terms:
\[
S(q(m_x)) = S(q(m_y))/p \quad = \quad 4 \delta[2 C - (1 - p^2) S] \psi \psi',
\]
\[
S(q(f_x)) = S(q(f_y))/p = -2 \{ k[1 + p^2] + \delta[(1 - p^2) C + 2 pS] \} \psi \psi' .
\]

Obviously, there is no restriction for the sign of squint, although in most cases we would have \( \delta > 0 \). It is certain, however, that zero squint would be something of a miracle. Eqs. (8) and (9) clearly imply that a change of boundary conditions from parallel to homeotropic reverses the sign of the torque squint (8) and at least the tendency of the force squint (9). This ought to be interesting from the experimental point of view.

We confine ourselves to seeking steady state solutions. Then the time derivatives on the left-hand side of eqs. (1) vanish. The charge density \( q \) can now be eliminated.

After some tedious but straightforward algebra, for which the reader is referred to [12], and after trivial rescaling one arrives at the equation
\[
\psi'' + q^2 \psi - \psi^3 + b \psi \psi' = 0, \tag{10}
\]

where \( q \) is the characteristic wave vector of the flow pattern (obviously, \( q \approx \pi/d \), where \( d \) is the sample thickness) and \( b \) is the coefficient of squint. Clearly, the coefficients, as well as an implied natural unit of length, are functions of the material constants and the applied fields.

At first sight it might seem more natural to normalize eq. (10) so that \( q^2 \) is unity. Since, however, due to the nonlinear terms, \( q \) is eventually renormalized, this would lead to spurious divergencies, which are avoided in the present normalization.

The simplicity of eq. (10) is deceptive. Apart from being nonlinear in \( \psi \), its coefficients depend on \( \varphi \). To overcome this difficulty, Moritz and Franklin [7], rather arbitrarily, choose to deal with fluctuations about a fixed mean director angle \( \langle \varphi \rangle \). A priori, this mean angle approximation appears to be rather unwarranted and certainly calls for some discussion.

First, we notice that \( \varphi \) enters the coefficients only through \( S = \sin 2 \varphi \) and \( C = \cos 2 \varphi \). Curiously enough, Moritz and Franklin [7] overlooked the obvious fact that one may set \( \langle \varphi \rangle = 0 \) without any further loss of generality. A very significant simplification results since we drop all terms with \( S \). For the mean value of \( C \) we obviously have \( 0 < \langle C \rangle < 1 \). A more detailed analysis shows that for an amplitude of \( \varphi \) as high as 30° the error is less than 20 % within 80 % of the domain of \( \varphi \).

From a physical point of view, we are dealing with a situation where the director angle is still far from saturation but sufficiently above threshold so that nonlinearities become important.

An a posteriori scrutiny of the results shows that the mean angle approximation is actually much better than expected. The solutions, in terms of either \( \varphi \) or \( \psi \), are elliptic functions. Thus, whenever \( | \varphi | \) is large, \( \psi \) and its derivatives almost vanish and vice versa. This has been corroborated independently by Ben-Abraham [6] and Akahoshi and Miyakawa [11].
We now return to eq. (10). First, we notice that it has the trivial solutions $\psi = 0$, and $\psi = \pm q$. They are of no interest to us. The null solution, as an approximation, leads back to linearization. The nonnull constant solutions are nonphysical. They imply a director angle $\varphi$ proportional to $x$, in contradiction to assumptions of the mean angle approximation.

The squintless cubic case of eq. (10), i.e. $b = 0$, is well known. It is exactly solvable in terms of Jacobi's elliptical functions (cf. [17]).

The extreme special case $b = 1$, has an exact solution [13]:

$$\psi = \frac{\wp'(k(x - x_0), 12, C)}{\wp'(k(x - x_0), 12, C - 1)},$$

where $k = q/(2 \sqrt{3})$, and $x_0$ and $C$ are integration constants; $\wp'$ and $\wp''$ are the Weierstrass elliptic function and its derivative (cf. [14]). Although this solution displays the characteristic features of squint in a rather extreme way, it is not very illuminating and seems to be only of academic interest.

However, the effect of squint can be understood by means of the following simple argument. Suppose, we tried to solve eq. (10) by perturbation theory, starting with its linearized version. We might choose, say,

$$\hat{\psi} = \cos qx$$

as a trial function. This can be always achieved by shifting the origin to a point of zero director angle $\varphi = 0$. Then $\hat{\psi}$ is an even function with period $\lambda = 2 \pi/q$. Apart from that, however, $\hat{\psi}$ also has the symmetry

$$\hat{\psi}(x - 1/4 \lambda) = - \hat{\psi}(1/4 \lambda - x).$$

This corresponds to the inversion symmetry of a domain pattern with respect to the axis of a single roll. The cubic term will repeatedly triple the wave vector and thus will leave this symmetry intact. That of course, can be ascertained from the exact solution of the squintless cubic equation.

On the other hand, the squint term doubles the wave vector. While possibly conserving the parity of $\psi$, it immediately destroys the symmetry specified by eq. (13). The roll axis is displaced by a distance $\varepsilon$ and ceases to be a symmetry axis. The cross section of a pair of adjacent rolls will be distorted into the form of squinting eyes (Fig. 1). Hence the term.

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Carefully planned experiments to detect and measure squint are needed. I wish to acknowledge the hospitality of Kent State University that made this research possible. I am grateful to Will Franklin whose sudden and premature death put an end to our fruitful and promising interaction. I thank Elan Moritz for helpful comments and E. Gelerinter for making available his pictures.
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