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ELASTIC ENERGIES AND DIRECTOR FIELDS
IN LIQUID CRYSTAL DROPLETS, I. CYLINDRICAL SYMMETRY

M. J. PRESS and A. S. ARROTT

Department of Physics, Simon Fraser University, Burnaby, B. C. V5A 1S6, Canada

Abstract. — The nature of point defects in nematic liquid crystals has been studied by placing droplets on a water substrate and viewing these between crossed polarizers. Three configurations are observed. These configurations are accounted for by minimizing the Frank elastic energies with suitable boundary conditions. The second order nonlinear simultaneous partial differential equations that describe systems with cylindrical symmetry are solved by relaxation methods using computer facilities. The importance of the cancellation of several contributions to the splay (divergence) is emphasized.

1. Droplets. — The purpose of the present research is to obtain descriptions of the minimum energy configurations of the molecular alignment in liquid crystal droplets. The energy of a drop depends upon surface tension, the angle the molecular alignment makes with the surfaces, and the Frank elastic energy in the volume [1]. The molecular alignment is described in terms of a director field \( \hat{n} \). A problem of some interest is to find \( \hat{n}(r) \) in a free falling drop when the director at the surface is prescribed. One can show from dimensional arguments that the drop will remain very close to spherical. If the director is perpendicular to the surface, a possible solution could be a purely radial director field. This will cost splay energy

\[
E_s = \frac{1}{2} S \int (\text{div} \, \hat{n})^2 \, dV = 8 \pi SR
\]

where \( S \) is the splay elastic constant (usually called \( K_{11} \)) and \( R \) is the radius of the sphere. But this is not necessarily the lowest energy configuration. The present paper has something to say about this, but has as its main purpose the treatment of a droplet which is experimentally somewhat more accessible than the free falling drop. Nematic liquid crystal droplets suspended from the surface of water take the shape of thin lenses. This geometry provides a preferred axis about which the droplets are cylindrically symmetric. The somewhat surprising result of microscopic observations of these droplets between crossed polarizers is that three different optical patterns occur as shown in figure 1. The configurations are named the right-handed, the left-handed and the normal solutions. In what follows it is shown that these configurations can be accounted for by considering the energy in a cylinder about the axis of the droplets. The problem of minimizing the energy of the entire droplet is not carried to completion.

FIG. 1. — Three droplets of MBBA on a water substrate viewed between crossed polarizers. The diameters are \( \sim 100 \mu m \).
left-handed solutions have slightly lower energy than
the normal solution for suitable choices of the elastic
constants. This difference is quite small compared to
the saving of energy which occurs due to a phenomena
which is here termed splay cancelling. If the boundary
conditions force a variation of the director in one
direction then a variation of the director in another
direction can lead to a cancellation of contributions
to a divergence or splay. This principle is clearly
illustrated in the example treated below. It is likely
that this is a general principle in the elasticity theory of
liquid crystals.

In Section 2 expressions are given for the Frank
elastic energies in cylindrical coordinates for the case of
complete rotational symmetry about the axis. From
these one obtains two torque equations to be satisfied
in the volume. In Section 3 the boundary conditions are
discussed and the particular problem to be solved is
described. In Section 4 solutions as determined by
relaxation methods using a computer are given. The
paper concludes with a discussion of the results and a
view of work in progress.

2. Energies and torque equations. — For cylin-
drical symmetry the director field is

\[ \hat{n}(\rho, z) = \sin \theta(\rho, z) \cos \phi(\rho, z) \hat{\rho} + \]
\[ + \sin \theta(\rho, z) \sin \phi(\rho, z) \hat{\Phi} + \cos \theta(\rho, z) \hat{z} \]  

(2)

where \( \hat{\rho}, \hat{\Phi} \) and \( \hat{z} \) are unit vectors in this cylindrical
coordinate system and \( \theta \) and \( \phi \) are the usual polar
angles with \( \phi \) referred to the \( \hat{\rho} \) axis.

The Frank elastic energy is given by

\[ E = \pi \int d\rho \ dz \left\{ S(V, \hat{n})^2 + T(n \cdot \nabla \times \hat{n})^2 + \right. \]
\[ + B(n \times (\nabla \times \hat{n}))^2 \} . \]  

(3)

The notation \( S, T, \) and \( B \) refers to splay, twist, and bend
for which the notation \( K_{11}, K_{22}, \) and \( K_{33} \) is usual. The
strength of these constants will be of the order of
\( 10^{-6} \) erg. For the director as defined in (2), the inte-
grand of (3) becomes

\[ f = \frac{1}{2} \sin^2 \theta [S \cos^2 \phi + T \cos^2 \theta \sin^2 \phi + B \sin^2 \theta \sin^2 \phi] + \]
\[ + \rho \theta^2 \sin^2 \theta \cos^2 \phi + B \sin^2 \theta \sin^2 \phi] \]
\[ + \rho \sin^2 \phi \theta^2 [S \sin^2 \phi + T \cos^2 \theta \sin^2 \phi + B \sin^2 \theta \sin^2 \phi] \]
\[ + \rho \theta^2 \sin \phi \theta \theta [S \sin^2 \theta + B \sin^2 \theta] \]
\[ + \rho \sin^2 \phi \theta \theta [T \sin^2 \theta + B \cos^2 \theta] \]
\[ + 2 \sin \theta \cos \theta \theta \theta [S \cos^2 \phi + T \sin^2 \phi] \]
\[ + 2 \sin^2 \theta \sin \phi \cos \phi \theta \theta [- S \cos^2 \theta + B \sin^2 \theta] \]
\[ - 2 \sin^2 \theta \cos \phi \theta \theta \phi \theta \phi \sin \theta \cos \phi \theta \theta \phi \theta \phi \phi \]  

(4)

The subscripts denote partial derivatives. Also included is a term for the effect of a magnetic field \( H \) in the vertical
direction. \( \Delta \chi \) is the diamagnetic anisotropy of the molecules. The variation of the energy with respect to \( \theta \) and
\( \phi \) produces two torque equations

\[ 0 = \frac{1}{\rho^2} \sin \theta \cos \theta [S \cos^2 \phi + T(\cos^2 \theta - \sin^2 \theta) \sin^2 \phi + 2 B \sin^2 \theta \sin^2 \phi] + \]
\[ + \sin \theta \cos \theta \cos^2 \phi \theta^2 [S - B] \]
\[ + 2 \sin \phi \cos \phi \theta \theta \phi \theta \phi [S \cos^2 \theta - T + B \sin^2 \theta] \]
\[ - \left( \frac{1}{\rho} \theta^2 + \theta^2 \right) [S \cos^2 \theta \cos^2 \phi + T \sin^2 \phi + B \sin^2 \theta \cos^2 \phi] \]
+ \sin \theta \cos \theta \varphi_\rho^2 \left[ S \cos^2 \varphi + T(\sin^2 \varphi - 2 \sin^2 \theta \cos^2 \varphi) + 2 B \sin^2 \theta \cos^2 \varphi \right]
+ \sin \theta \cos \theta \varphi_\rho^2 \left[ B - S \right] 
- \theta_\rho \left[ S \sin^2 \theta + B \cos^2 \theta \right] 
+ \sin \theta \cos \theta \varphi_\rho^2 \left[ 2T \sin^2 \theta + B(\cos^2 \theta - \sin^2 \theta) \right] 
+ \frac{1}{\rho} \sin \theta \cos \theta \sin \varphi \cos \varphi \varphi_\rho \left[ S + T(4 \cos^2 \theta - 5) + 4 B \sin^2 \theta \right] 
+ \frac{1}{\rho} \sin^2 \theta \sin \varphi \varphi_\rho \left[ - S + T(2 - 4 \cos^2 \theta) + B(3 \cos^2 \theta - \sin^2 \theta) \right] 
+ \sin \theta \cos \theta \sin \varphi \cos \varphi_\rho \varphi_\rho \left[ S - T \right] 
+ \frac{1}{\rho} \sin \theta \cos \theta \cos \varphi \theta_\rho \left[ S - B \right] 
+ \sin \theta \cos \theta \sin \varphi \theta_\rho \varphi_\rho \left[ B - S \right] 
+ 2 \sin \theta \cos \theta \cos \varphi \theta_\rho \varphi_\rho \left[ S - B \right] 
+ \left[ \cos^2 \theta - \sin^2 \theta \right] \cos \varphi \theta_\rho \theta_\rho \left[ S - B \right] 
- \sin \theta \cos \theta \sin \varphi \theta_\rho \varphi_\rho \left[ S - B \right] 
+ \sin^2 \theta \cos \varphi \varphi_\rho \varphi_\rho \left[ - S + 2T(\sin^2 \theta - \cos^2 \theta) + B(3 \cos^2 \theta - \sin^2 \theta) \right] 
+ \sin^2 \theta \sin \varphi \varphi_\rho \varphi_\rho \left[ T - S \right] 
+ \sin \theta \cos \theta \Delta \chi H^2 \right),
(5)
These two simultaneous second order partial differential equations as well as the expression for the integrand of eq. (3) simplify considerably if $S = B = T = K$. This is called the one constant approximation with elastic constant $K$. Though the one constant approximation will not account for the right and left-handed solutions, it will give the normal solution and illustrate the principle of splay cancelling. In the one constant approximation

$$f = \sin^2 \theta \left[ \frac{1}{\rho} + \rho \theta^2 + \rho \varphi_z^2 - 2 \cos \varphi \theta_z + 2 \rho \sin \varphi (\theta_z \varphi_\rho - \theta_\rho \varphi_z) + \rho \frac{\Delta z}{S} H^2 \right] +$$

$$+ \rho \theta^2 + \rho \varphi_z^2 + 2 \sin \theta \cos \theta \theta_\rho \quad (7)$$

and the torque equations become

$$\theta_{\rho \rho} + \frac{1}{\rho} \theta_\rho + \theta_{zz} = \sin \theta \cos \theta \left( \frac{1}{\rho^2} + \varphi_\rho^2 + \varphi_z^2 + \frac{\Delta z}{S} H^2 \right)$$

and

$$\varphi_{\rho \rho} + \frac{1}{\rho} \varphi_\rho + \varphi_{zz} = -2 \cot \theta (\theta_\rho \varphi_\rho + \theta_z \varphi_z). \quad (9)$$

Eq. (9) is satisfied by any constant value of $\varphi$. This is not so for eq. (6). If $\varphi$ is either 0 or $\pi$, eq. (6) is satisfied. For constant $\varphi$ and no magnetic field eq. (8) becomes

$$\theta_{\rho \rho} + \frac{1}{\rho} \theta_\rho + \theta_{zz} = \sin \theta \cos \theta \left( \frac{1}{\rho^2} + \varphi_\rho^2 + \varphi_z^2 + \frac{\Delta z}{S} H^2 \right), \quad (10)$$

which has among its many solutions two simple ones

$$\theta = \tan^{-1} \left( \frac{\rho}{z} \right) \quad (11)$$

and

$$\varphi = \tan^{-1} \left( -\frac{\rho}{z} \right). \quad (12)$$

Eq. (11) is a radial configuration and corresponds to the solution of the spherical drop considered in the introduction. Eq. (12) yields rectangular hyperbolas (see Fig. 2). These two solutions for a point singularity are well known [2]. But it seems that it is not generally appreciated that the energy of the point singularity of the second kind is 1/3 that of the first for equal spherical volumes. In the first case the energy is all splay. In the second case the energy is 3/5 splay and 2/5 bend. Thus the splay is reduced by a factor of 5 at a small cost in increased bend in going from the first case to the second.

In the treatment of the droplet to follow, the two point singularities occur at the surface. One of these is essentially one-half of the point singularity of the second kind.

(a) (b)

Fig. 2. — Two configurations for a point singularity with cylindrical symmetry about a vertical axis : a) The radial case, $\theta = \tan^{-1} (\rho/z)$. b) The hyperbolic case, $\theta = \tan^{-1} (-\rho/z).$
3. **Boundary conditions and the lens shaped drop.**

The droplets formed by the nematic liquid crystal Methoxybenzylidene-Butylaniline (MBBA) on the surface of the water take the form of thin lenses. At the water interface the director prefers to lie in the surface of a point singularity on the surface. The effects of interest here are on the scale of microns. Thus the surface deviations can be neglected.

In this case the preferred value of $\langle \hat{n}, \hat{m} \rangle$ may be rather weakly preferred, say with respect to $\hat{n} \cdot \hat{m} = 1$. Nevertheless it will be reasonable in the calculations given below to assume hard pinning at the air as well as at the water interface.

For the purpose of calculation at the water surface $\hat{n} \cdot \hat{m} = 0$ and at the air interface $\hat{n} \cdot \hat{m} = 0.93$ which is a pinning angle of 15 degrees from the normal to the surface.

The lens shape of the drop is determined by the surface tension. As the surface tension is comparable or larger than $K_p$, one can conclude that the surface will not deform significantly except within 1 nm of the center of a point singularity on the surface. The effects of interest here are on the scale of microns. Thus the surface deviations can be neglected.

The computer technique uses difference equations on a rectangular grid. For this reason it was decided to try the assumption of flat surfaces. Only the central cylindrical region of the droplet is treated. This is mathematically convenient, and it is justified by the observations that disturbances of the outer regions of the droplets do not affect the optical patterns in the central regions. The artificial termination of the central region raises the problem of how to handle the boundary conditions there. The integration by parts used to produce the Euler-Lagrange torque equations leads to two surface equations to be satisfied at the limits of the integration in the $\rho$ direction. These are

$$
0 = \rho \theta_p [S \cos^2 \theta \cos^2 \varphi + T \sin^2 \varphi + B \sin^2 \theta \cos^2 \varphi] + \sin \theta \cos \theta [S \cos^2 \varphi + T \sin^2 \varphi] + \rho \sin \theta \cos \theta \sin \varphi \cos \varphi \rho [T - S] + \rho \sin \theta \cos \theta \sin \varphi \cos \varphi \rho [B - S] - \rho \sin^2 \theta \sin \varphi \rho T
$$

Eq. (16) is approximately

$$
\theta_p \approx - \frac{\sin \theta \cos \theta}{\rho} \approx 0
$$

while eq. (17) is approximately

$$
\sin \varphi \theta_s \approx - \varphi_p
$$

at sufficiently large radius. Inasmuch as $\theta_s$ cannot be zero in view of the different pinning angles on the upper

(1) This can be argued on the basis of electric dipole interactions with a highly polarizable substrate.
This is approximately
\[ \varphi_z \approx \sin \varphi \rho, \]  
(21)
but, as \( \theta \) is fixed on the surface, \( \varphi_z \) is then approximately zero. For the water surface where \( \theta = \pi/2 \), \( \varphi_z = 0 \) exactly.

The problem to be solved numerically consists of the difference equation equivalents to the torque eq. (5) and (6) on a rectangular grid with boundary conditions \( \theta = \pi/2 \) on the upper surface, \( \theta = 0.26 \) on the lower surface, \( \theta = 0 \) at \( \rho = 0 \), and \( \varphi \) given by eq. (16) at \( \rho = \rho_0 \), while \( \varphi \) satisfies the differential equations given by eq. (20) for the upper and lower surfaces and by eq. (17) for \( \rho = \rho_0 \). \( \varphi \) is undefined at \( \rho = 0 \), inasmuch as \( \theta = 0 \) there.

4. Numerical solutions. — The relaxation method can be illustrated using the difference equation corresponding to eq. (8),
\[ \frac{\theta_k + \theta_L - 2 \theta}{4(\Delta \rho)^2} + \frac{\theta_k - \theta_L}{2 \rho \Delta \rho} + \frac{\theta_A + \theta_B - 2 \theta}{4(\Delta z)^2} = f(\theta, \varphi_\rho, \varphi_z) \]
(22)
where \( f(\theta, \varphi_\rho, \varphi_z) \) stands for the right side of eq. (8). \( \theta_k, \theta_L, \theta_A, \text{and} \theta_B \) are the values of \( \theta \) at grid points to the right, left, above, and below the point of interest.

Eq. (22) is solved for \( \theta \) as it appears on the left, but \( \theta \) as it appears in \( f(\theta, \varphi_\rho, \varphi_z) \) is left explicitly in \( f(\theta, \varphi_\rho, \varphi_z) \). A similar procedure is carried out for eq. (9) and that is solved for \( \varphi \) at the point of interest. One assumes starting values of \( \theta \) and \( \varphi \) over the entire grid and then proceeds to run through the grid recalculating \( \theta \) and \( \varphi \). If the procedure converges, it is then self-consistent.

The procedure was found to work for eq. (8) and (9) and then extended to eq. (5) and (6). The grid used most often had 39 points in the \( \rho \) direction and 30 in the \( z \) direction. These were spaced closer together near the boundaries and the axis. The convergence takes from 500 to 2000 passes depending upon the choices of elastic constants.

For the one constant approximation \( S = B = T \) the only solutions found were for constant \( \varphi \), and in view of the boundary condition at \( \rho = \rho_0 \) for which eq. (19) is exact that constant is either 0 or \( \pi \). These two solutions differ remarkably in energy. The configuration for \( \varphi = \pi \) is obtained from figure 3 by reflecting each column about a vertical axis through that column. Note that the configuration in the center of the upper surface (upper left-hand corner in Fig. 3) for \( \varphi = 0 \)

This corresponds to the point singularity shown in figure 2b. The configuration for \( \varphi = \pi \) corresponds to the point singularity shown in figure 2a. As can be anticipated from the discussion of splay cancellation the \( \varphi = 0 \) solution has lower energy in the region of the center of the upper surface. But the major saving of energy between these two solutions occurs in the region where the splay from the enforced variation of \( \theta \) from the top to the bottom surface is cancelled by a radial splay coming from the cylindrical symmetry.

Specifically for \( \cos \varphi = \pm 1 \) the splay is given by
\[ \text{div} \, \mathbf{n} = \sin \theta \left( \frac{1}{\rho} - \theta_z \right) \pm \cos \theta \varphi_\rho. \]  
(23)
As \( \theta_z \approx \pi/2 \), \( z_0 \), where \( z_0 \) is the separation of the upper and lower surfaces, the coefficient of \( \sin \theta \) cancels for \( \cos \varphi = 1 \) near \( \rho = 2 \, z_0/\pi \). Near the top surface where \( \cos \theta \) and \( \theta_z \) both vanish the splay is completely cancelled for this radius.

Because the singularity at the center of the bottom surface (lower left corner in Fig. 3) looks more like the radial singularity of figure 2a, one might anticipate that a lower energy solution would be possible if this singularity also were to be more like the hyperbolic singularity of figure 2b. This would require a switch in \( \varphi \) along the bottom surface inasmuch as \( \varphi = 0 \) will be a lower energy at large radius. It turns out that this is not the case for the bottom surface pinned at \( \theta = 15 \) deg. The cost of the twist more than offsets the splay saving at the bottom surface. But for a large magnetic field in the vertical direction the two surfaces are decoupled and the energy is lower for \( \varphi = 0 \) on the top surface and \( \varphi = \pi \) on the bottom surface. The details of the phase change in vertical magnetic fields will be discussed elsewhere.

After failure to produce the right-handed and left-handed solutions for the one constant case, the calculations were extended to the three constant cases. Values of the ratios of the elastic constants appropriate for MBBA [4] were taken as \( B/S = 5/4 \) and \( T/S = 5/8 \). With these values the twisted configurations were found. Figure 4 shows the vertical cross section of the

![Figure 4](image-url)

**Fig. 4.** Solution to the problem of the cylinder with \( \varphi \neq 0 \). The unit vectors of the director field are shown as cylinders in perspective. \( B/S = 5/4, T/S = 5/8 \).
Elastic Energies and Director Fields

The top. An enlargement of the central region is shown in figure 6. The darkened cylinders are those that are within 0.1 radian of the edges of the page. Inasmuch as the variation of \( \phi \) with \( z \) is small, this diagram approximates the result of a more sophisticated optical calculation using Maxwell’s equations [5] and corresponds to what would be observed between crossed polarizers aligned with the edges of the page. The difference in energy between the solutions with handedness and the solution with \( \phi = 0 \) for this choice of elastic constants is less than 1 percent of the total energy of the central region. A finite value of \( \phi \) is unfavorable in the region of the singularity at the center of the upper surface. It is favorable for small values of \( \theta \) as long as \( T < S \) and also for the region of the singularity at the center of the lower surface. As the \( \phi = 0 \) solution has produced a great deal of splay cancelling near \( \rho = \pi \rho_0/2 \) one can appreciate that having a finite \( \sin \phi \rho_0 \) in this region can only be unfavorable. All these differences are small in terms of the total energy.

The calculations have been carried out primarily for a grid with the ratio of 80/19 for \( \rho_0/z_0 \). The \( \theta \) variation is essentially that of \( \rho = \infty \) for \( \rho/z_0 > 2 \) and is the same in this region for \( \phi = 0 \) and \( \phi \neq 0 \). That the behavior in this region is determined by the enforced variation of \( \theta \) in the \( z \) direction is consistent with the idea that the behavior in the central region is not influenced by the outer regions of the droplet. And this is part of the justification for neglecting the curvature of the surfaces. From calculations with several choices of the ratio of the elastic constants and for several choices of the pinning angles it is possible to conclude that it is necessary to have \( T \) somewhat less than \( S \) to obtain a solution with handedness. The critical value, for the chosen boundary conditions, is likely near \( T = 0.93 \) \( S \). In this region the solutions are extremely slow in convergence if \( \phi \) is not set equal to zero. The ratio of \( S/B \) is not particularly important for ratios near unity. One might anticipate that for very large \( B \) the singularity at the top surface might change from hyperbolic to radial, but this has not yet been checked. As \( T/S \) approaches 0.93, the angles \( \phi(\rho, z) \) approach zero continuously in what looks like a second-order phase change. This is in contrast to the apparently first-order phase change which occurs with increasing magnetic field in the vertical direction.

5. Conclusions. — The present paper gives the Frank elastic energy in systems with cylindrical symmetry and derives the Euler-Lagrange torque equations for these systems. These equations are solved numerically for the rather simple geometry of a cylinder with the polar angle pinned at one angle on the top surface and at a different angle on the bottom surface and with free boundary conditions on the cylindrical surface. The two solutions with the elastic constants equated to one another both have the director confined to the \( \rho-z \) plane. One of these is similar to the hyperbolic point singularity and the other is similar to the radial point singularity. These two differ in energy by a factor of three with the hyperbolic-like solution benefiting from what is here termed splay cancelling. The saving is such that the energy per unit area \((2)\) of the drop is less than that to be found in a layer with translational invariance for which the upper and lower surfaces are pinned at the angles used in the calculation with cylindrical symmetry. The energy per unit area goes to

\[ \text{(2)} \text{ This means the energy of a cylindrical shell between } \rho \text{ and } \rho + d\rho \text{ divided by } 2 \pi \rho \, d\rho. \]
infinity as $1/R$ as $R$ goes to zero. It goes to a constant for large $R$ and in between goes through a minimum (see Fig. 7). What is more interesting is that the total energy per unit area also goes through a minimum. This is seen in figure 7 where

$$E_A = \int_0^{s_0} \int_0^R f \, d\rho \, dz + \int_0^{s_0} \int_0^R f_\infty \, d\rho \, dz \quad (24)$$

is plotted against $R$ for the one constant case. In eq. (24) $f$ is given by eq. (7) and

$$f_\infty = \rho \theta^2 \left( \rho = \infty, z \right). \quad (25)$$

For comparison, eq. (24) evaluated for the solution with $\varphi = \pi$ is also shown in figure 7. The minimum observed in figure 7 may be of some importance in the stability of point singularities in liquid crystal films.

The three constant equations have been shown to yield the right-handed and left-handed solutions as observed in liquid crystal droplets on liquid (and solid) substrates. The equations also yield the normal solution and show it to have slightly higher energy. One might account for the appearance of all three configurations under very similar conditions as arising from metastability. As yet this metastability is not proven by calculation.

Line singularities have been studied by Williams et al. [6] using capillaries with the director pinned perpendicular to the tube at the cylindrical surface. They have shown that the director changes from perpendicular at the surface, $\rho = \rho_0$, to parallel to the singular axis at the axis. They treat this problem in the one constant approximation for which the solution has $\theta = 2 \tan^{-1} \rho/\rho_0$ and $\varphi = 0$. For the three constant case with $T < S$ the energy density (eq. (4)) for $\rho < \rho_0$ is readily shown to have a minimum for $\sin \varphi = 1$. Thus it seems likely that for $T$ sufficiently less than $S$, $\varphi$ should change from 0 at $\rho = \rho_0$ to a finite value at $\rho = 0$. This should be accomplished with little change in the $\theta$ behavior and with little change in the total energy. The variation of $\varphi$ with $\rho$ should increase the energy while the finite value of $\varphi$ lowers the energy near the axis. If the end of the line singularity terminates at a free surface one would then expect a point singularity of the hyperbolic type, but twisted.

Observations of line singularities in MBBA films on the surface of water do show twisted configurations.

The successful treatment of a cylindrical portion of the droplet by computer techniques opens the way to consideration of more complex geometries. One of those under study is the complete thin lens droplet. Another is the sphere. In these cases it is necessary to make a suitable first guess for the relaxation method to proceed. From what has been learned in the treatment of the central cylinder it seems reasonable to guess that solutions with $\varphi = 0$ may give good approximation to the minimum energy for a particular geometry, but twisted configurations may lower the energy slightly more. The $\varphi = 0$ solution may well be studied with the one constant approximation; while the full equations may be necessary to find minimum energy twisted configurations for the actual elastic constants.

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