THE SUPERSELECTION STRUCTURE OF PARTICLE PHYSICS
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I want to report on some joint work with Sergio Doplicher and E. Roberts which extended over the past 5 years. It concerns the origin of charge quantum numbers and particle statistics in relativistic, local quantum theory with short range forces and the classification of the possibilities allowed by the basic principles. A bibliography of this and earlier work on the same lines is attached (ref. (1) to (7)). In the available time I can, of course, only give a very incomplete account but hope to be able to sketch the physical background, the theoretical framework used and the main results in an understandable way suppressing all detailed arguments and proofs.

1. BACKGROUND.

1.1. THE PROBLEM OF STATISTICS.

a) Wave mechanics. Let \( Z \) stand for the position and spin coordinates of an electron. Then a pure state of \( n \) electrons is described by a wave function \( \Psi(Z_1, \ldots, Z_n) \) which has to be totally antisymmetric under permutations of the arguments. Similarly, in the case of photons we have to restrict ourselves to totally symmetric wave functions. Generalization: to each type of particle there is attached a property called statistics. It determines (for arbitrary number \( n \) of such particles) the allowed permutation character of the multiparticle wave functions.

One has often asked whether only total antisymmetry (Fermi statistics) or total symmetry (Bose statistics) are allowed by general physical principles. Different answers to this question have been given by different people. Before stating our answer let me describe three other schemes.

i) Parafermions of order \( d \).

For arbitrary \( n \), precisely all those wave functions are allowed which cannot be antisymmetrized in more than \( d \) arguments.

ii) Parabosons of order \( d \).

For arbitrary \( n \), all wave functions are allowed which cannot be antisymmetrized in more than \( d \) arguments.

iii) Infinite statistics

All wave function irrespective of symmetry character are allowed.

Remarks.

For \( d = 1 \) (i) gives ordinary Fermi statistics and (ii) ordinary Bose statistics. Case (iii) may be regarded as a limiting case of either (i) or (ii) for \( d = \infty \).

We find that the schemes (i), (ii), or (iii) are the only ones allowed within the framework of local relativistic Quantum Theory, and the type of statistics depends only on the charge quantum numbers, two particles with the same charge also having the same type of statistics. Moreover the schemes (i) and (ii) with \( d \neq 1 \) should not be considered as strange situation because they can arise naturally in the following way from the case with \( d = 1 \). Suppose that \( Z \) corresponds to a maximal set of commuting observables for a single particle but that in addition the particle has a completely unobservable degree of freedom \( d \) which can take \( d \) possible values. Then, if the particle is treated as an ordinary fermion (boson) with respect to all degrees of freedom \( Z, \sigma \), it will be a parafermion (paraboson) of order \( d \) with respect to the observable degrees of freedom \( Z \). We should note that the case (iii) has some pathological features and should probably be excluded.

The converse question: "can parastatistics always be reduced to ordinary statistics by introducing some hidden degree of freedom" has not been answered by us in the general context. It can be answered affirmatively for the models of parastatistics suggested by H.S. Green (8) (see the discussion in ref. (8)). In the general case it is tied to the rather deep question as to whether every allowed structure of charge quantum numbers can be described in terms of a gauge group (see below).
b) Field theory. In field theoretical models the
statistics is determined by the commutation relations
at large distances of field operators which create
the particle in question from the vacuum. Now, the
principle of locality (or Einstein causality) demands
that all observables commute at space like distances.
In particular, they are not allowed to anticommute
or to have the more complicated commutation properties
suggested by H.S. Green (9) in the field theo-
dretic description of para-statistics. This means
that all particles which can be created from the
vacuum by operators belonging to the algebra genera-
ted by observables must be ordinary bosons. The oc-
currence of any other statistics, even ordinary Fermi
statistics, necessitates the existence of super-se-
lection rules (see below). We may conclude that par-
ticle statistics results from the principle of local-
ity and that the types of statistics which occur
are tied to the super-selection structure of the
theory.

1.2. CHARGE QUANTUM NUMBERS AND GAUGE GROUPS.--

In Lagrangian Field Theory conservation laws are
related to invariance groups via Noether's theorem.
In the particular case of conservation laws for
charges, the corresponding groups are the "gauge
groups". We use the term "charge" in a generalized
sense, including in it baryon number, lepton numbers,
but also quantities like the isospin if they are
strictly conserved in the theory. In contrast to
physical symmetry transformations, a gauge trans-
formation cannot be interpreted in operational terms.
It does not correspond to any changes in the labora-
atory but only a change in the description. If we
restrict attention to observables, the gauge trans-
formations do nothing. A Lagrangian theory of charge
carrying fields may be described as follows: the
field operators generate an algebra \( \mathcal{F} \), the "field
algebra", which contains the algebra \( \mathcal{A} \) generated
by observables as a subalgebra. Let \( \mathcal{H} \) be the Hilbert
space in which the field operators from \( \mathcal{F} \) act. Then
we can consider the group of all those unitary op-

erators on \( \mathcal{H} \) which commute with all observables. This
group will be called the gauge group, \( \mathcal{G} \). We may note
incidentally that \( \mathcal{G} \) will be a connect group (with
respect to the strong topology of operators on \( \mathcal{H} \)),
if no particle with zero mass, nor any infinite mul-
tiplet of stable particle types with the same mass
occurs, and if the theory has a complete particle inter-
pretation (\( \mathcal{F} \) is irreducible). We may then consider the
abelian algebra \( \mathcal{Z} \) consisting of all operators which
commute with all elements of \( \mathcal{G} \) as well as with all
elements of \( \mathcal{A} \). (\( \mathcal{Z} \) is the center of the group algebra
of \( \mathcal{G} \)) and we may simultaneously diagonalize all op-

erators in \( \mathcal{Z} \). Due to the compactness of \( \mathcal{G} \) the spectrum
of \( \mathcal{Z} \) is discrete and we obtain a decomposition of \( \mathcal{G} \)
to a direct sum of subspaces \( X_\xi \) such that each \( X_\xi \)
is invariant both under the action of \( \mathcal{A} \) and
under the action of \( \mathcal{G} \). The set of \( \xi 's \) (spectrum of
\( \mathcal{Z} \)) are the charge quantum numbers.

1.3. SUPERSELECTION RULES.--
The situation described under (7) exemplifies a
phenomenon which was first pointed out in (9) and
called "superselection rules". It is a partial break-
down of the superposition principle. Namely, if we take
two state vectors \( |\psi_1\rangle \), \( |\psi_2\rangle \), belonging to different
subspaces \( X_\xi_1 \), \( X_\xi_2 \), then a linear superposition
\( |\psi\rangle = \alpha |\psi_1\rangle + \beta |\psi_2\rangle \) will not give us just a pure state
but will just amount to a statistical mixture of the
states 1 and 2 with weights \( |\alpha|^2, |\beta|^2 \). In other words,
the relative phase between \( \alpha \) and \( \beta \) can have no signi-
ficance because it can be changed by a gauge trans-
formation and this should have no observable effect.

One may ask: what is the origin of superselection
rules if we do not start from a Lagrangian theory?
The most natural answer is the following. We should
not consider the algebra of observable \( \mathcal{A} \) a priori
as an operator algebra in a Hilbert space, but as an
abstract algebra defined (like an abstract group) by
its algebraic structure (the relevant operations in
\( \mathcal{A} \), being multiplication of elements, linear combi-
ation of elements with complex coefficients and in-
volution \( A \leftrightarrow A^* \) corresponding to the adjoint
formation) to be mathematically precise we may consider \( \mathcal{A} \) as
an abstract \( C^\ast \)-algebra, i.e., a state must be de-
defined as a positive linear functional on the algebra.
It assigns to each \( A \in \mathcal{A} \) a (complex) number \( \rho(A) \)
such that the dependence on \( A \) is linear and the value
of \( \rho(B) \) is real and non-negative if \( B \) is a positive ele-
ment of \( \mathcal{A} \), i.e., if \( B = A_\ast A \). I.E. Segal (10) has
shown that in this way one obtains indeed an adequa-
te mathematical framework for Quantum Mechanics. The
way how this frame may be used in Quantum Field The-
ory has been described in (11) and there it was also
pointed out that in this frame superselection rules
appear very naturally: the abstract algebra \( \mathcal{A} \) may
have several inequivalent representations by opera-
tors algebras on Hilbert spaces. Each such represen-
tation carries with it a specific family of states
namely all those expectation functionals on \( \mathcal{A} \) which
arise from the vectors in the representation space.
Two disjoint representations* give rise to disjoint families of states. If the abstract algebra \( \mathcal{A} \) is the primary object of the theory then the superselection structure corresponds to the set of equivalence classes of irreducible representations of \( \mathcal{A} \).

2.- STRUCTURE ANALYSIS FOR RELATIVISTIC LOCAL QUANTUM THEORY.

2.1.- FRAME.- Two points have to be added to the previous remarks. First we have to consider not just the algebra \( \mathcal{A} \) generated by all observables. The abstract structure of this algebra does not contain much physical information. Rather we have to associate with each finitely extended region \( \mathcal{O} \) in space-time an abstract algebra \( \mathcal{A}(\mathcal{O}) \), interpreted as the algebra of observables in the region \( \mathcal{O} \). I shall not describe in detail here the general properties which this "net of algebras \( \mathcal{A}(\mathcal{O}) \)" should have but only indicate them by a few catch words: Poincare's covariance, existence of a vacuum state \( \omega_0 \) (Poincare-invariant state with minimal energy in its representation space), additivity**, causality. It is then claimed that a net with these properties constitutes a physical theory in the sense that it uniquely determines which types of particles occur and what their collision cross-sections are.

The second point to recognize is that the physical states in which we are interested for the purpose of elementary particle physics are only a tiny subset of all states (positive linear functionals over the net). We are not interested, for instance, in states describing a distribution of matter filling the universe with finite density, but restrict our attention to those states which differ from the vacuum only locally. Correspondingly we are only interested in a tiny subset of all possible Hilbert space representations of the net. In the case where there are only short range forces, no zero mass particles, no infrared problems, (which is the only case we have analyzed), the criterion which selects the representations of interest can be formulated in the following way.

\( \mathcal{O} \) denotes a finitely extended region in space-time, \( \mathcal{O}' \) its causal complement i.e. the set of points which lie space-like to all points of \( \mathcal{O} \). Let us denote the vacuum representation by \( \pi_0 \) (i.e. the irreducible representation of \( \mathcal{A} \) in which the vacuum state appears) and let \( \pi \) be another representation. Then \( \pi \) is of interest for elementary particle physics if the restrictions of \( \pi \) and \( \pi_0 \) to the subalgebra \( \mathcal{A}(\mathcal{O}') \) are unitarily equivalent for all \( \mathcal{O} \). Thus all representations of interest are very closely related. They may be inequivalent only for the total algebra and are equivalent whenever we restrict attention to any space-time region which has a non-void causal complement.

One finds then that these representations are obtainable from the vacuum representation by applying a "localized morphism" to the algebra \( \mathcal{A} \). This means that any such representation can be written as

\[ \pi(A) = \pi_0(\rho(A)), \quad A \in \mathcal{A} \]

where \( \rho \) maps \( \mathcal{A} \) into \( \mathcal{A} \) conserving the algebraic structure, and \( \rho \) has a "localization region" such that

\[ \rho(A) = A \quad \text{for} \quad A \in \mathcal{A}(\mathcal{O}') \]

i.e. \( \rho \) acts trivially on the algebra of the causal complement of \( \mathcal{O} \). The simplest case is when the image of \( \mathcal{A} \) under \( \rho \) is the whole of \( \mathcal{A} \) ( \( \rho \) is an automorphism). But the case where the image is not the whole algebra ( \( \rho \) is an endomorphism) and yet is given by (1) is irreducible is also possible. This is in fact the case in which nonstatistics and non abelian gauge groups appear. A morphism is called pure if it leads via (1) to an irreducible representation. Two morphisms \( \rho_1 \) and \( \rho_2 \) are called equivalent if they lead to equivalent representations and this is again synonymous to saying that they differ by an inner automorphism i.e.

\[ \rho_1(A) = \rho_2(U A U^{-1}) \]

with \( U \) a unitary element from \( \mathcal{A} \).

The equivalence classes of pure morphisms correspond to the charge quantum numbers (superselection sectors). We denote such a class by the symbol \( \xi \).

2.2.- RESULTS.-

a) There is a composition law for charges, which arises from the obvious composition of morphisms. This composition of charges turns out to be commutative:

\[ \xi_1 \xi_2 = \xi_2 \xi_1 \]

However, in general, the product of the two pure
morphisms will no longer be pure and thus the composition of two pure charges will in general lead to mixture. We can express this symbolically by:

\[(2) \quad \xi_1 \cdot \xi_2 = \sum \xi_i\]

An example for this is the case where the "charge" \(\xi\) is the magnitude of the isospin, the composition corresponds to the vector addition of two isospins and the right hand side of (2) is the Clebsch-Gordan series.

b) There is a charge conjugation. To each pure \(\xi\) there corresponds a unique (pure) conjugate class \(\xi\), determined by the requirement that in the composition \(\xi \cdot \xi\), the right hand side of (2) contains one term with zero charge (the vacuum representation).

c) For each charge, one has a numerical parameter \(\lambda\), the statistics parameter. It may take one of the values \(0, \pm \frac{1}{2d}\) (d, a positive integer).

All particles of charge \(\xi\) are parabosons (parafermions) or order d when \(\lambda = \pm \frac{1}{2d}\) (respectively \(\lambda = \frac{1}{2d}\)). They have "infinite statistics" if \(\lambda = 0\).

d) Conjugate charges have equal statistic parameters.

e) All states of charge \(\xi\) have integer angular momentum if \(\lambda\) is positive, half integer angular momentum if \(\lambda\) is negative. (connection of spin and statistics).

f) If \(\xi = \bar{\xi}\) (self-conjugate sector) there is an additional sign associated with \(\xi\) (which is not related to the sign of \(\lambda\)). If this sign is negative a particle with this charge must have an antiparticle different from it; if this sign is positive the particle and antiparticle are identical. (Carruthers theorem).

\(\eta\) The energy spectrum is positive in all sectors.

h) Scattering theory can be worked out in terms of observables and morphisms without reconstructing the charge-carrying fields. The construction of fields and of the gauge group has not been done in general. In the simple case where all pure morphisms are automorphisms the construction of the field algebra and the gauge group (then abelian) has been given.

REFERENCES