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To cite this version:

P. King. PARAMETRIC OSCILLATIONS IN A LIQUID FILLED ACOUSTIC RESONATOR. Journal de Physique Colloques, 1972, 33 (C6), pp.C6-281-C6-284. <10.1051/jphyscol:1972660>. <jpa-00215178>

HAL Id: jpa-00215178
https://hal.archives-ouvertes.fr/jpa-00215178

Submitted on 1 Jan 1972

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PARAMETRIC OSCILLATIONS IN A LIQUID FILLED ACOUSTIC RESONATOR

P. J. KING (*)

Department of Technical Physics, Helsinki University of Technology, SF-02150 Otaniemi, Finland

Résumé. — On a étudié les oscillations paramétriques d’un résonateur rempli de liquide constitué par deux transducteurs dont les faces internes sont à l’air libre. Les fréquences de résonance n’étant pas espacées régulièrement, les oscillations qui représentent $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc. de la fréquence appliquée peuvent être produites à l’aide de transducteurs appropriés. Une théorie basée sur les interactions paramétriques des modes du résonateur dues à la non-linéarité du liquide est ébauchée. Dans les circonstances limitées pour lesquelles elle a été testée, cette théorie prouve assez convenablement comment le seuil d’excitation dépend de la fréquence de pompage et de la longueur du résonateur ; elle prouve aussi l’importance du seuil.

Abstract. — A study has been made of the parametric oscillations of a liquid filled resonator formed between two air-backed transducer plates. Due to the unequal spacing of the resonant frequencies, oscillations close to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ etc. of the applied frequency may be produced by a suitable choice of transducers. A theory is outlined which is based on the parametric interactions between the resonator modes due to the nonlinearity of the liquid. In the restricted circumstances for which it has been tested this theory gives reasonable predictions for the dependence of the excitation threshold on the pump frequency and resonator length, and for the magnitude of the threshold.

1. Introduction. — Several acoustic systems are known where the application of a strong electric or acoustic excitation causes the appearance of lower frequencies than those applied. These lower frequencies only appear if the amplitude of the applied signal is raised above a certain « threshold » level. In some cases the frequencies generated are exact simple fractions, « fractional harmonics » of the applied frequency. In others the subfrequencies have no such exact property.

A variety of subfrequency effects have been reported in liquid filled resonators formed between two transducer plates [1], [2]. One of the transducers is used to excite the resonator and the other as a receiver for any subfrequency signals. Alternatively the scattering of laser light may be used as a detection method [3]. When the resonances are well defined the generation of two modes has been noted, the frequencies summing to the applied frequency. Under suitable conditions frequencies of close to $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ of the exciting frequency have been observed [4]. Adler and Breazeale have studied subfrequency generation in a variety of liquids and have attempted to explain their observations as due to the parametric variation of the length of their resonator [5].

In the present study an investigation has been made of the modes of a liquid filled resonator and of the threshold conditions for oscillation, with the object of obtaining a detailed understanding of the mechanisms involved.

(*) On leave until September 1972 from the Department of Physics, University of Nottingham, Nottingham, England.

2. The apparatus. — A block diagram of the apparatus is shown in figure 1. The acoustic resonator consists of two air backed plane X-cut quartz transducers mounted parallel to one another on a steel girder in such a way that the transducer separation can be changed and the parallelism adjusted. The transducers are immersed in a large tank of liquid which is stirred and temperature controlled.

A continuous sine wave is fed via a broad band amplifier and an impedance transformer to the exciting transducer (half wavelength resonant frequency $f_1$). This coupling circuit is inductively tuned to the frequency used, which is monitored on a digital frequency meter. The amplitude applied to the transducer is displayed on an oscilloscope.

The output of the second transducer (frequency $f_2$)
is fed via another impedance transforming circuit to an oscilloscope and frequency analyser so that the characteristics of any subfrequency signals generated may be determined. The circuits external to the resonator are designed so that the acoustic reflection losses at either transducer are small at all frequencies of interest.

Before making measurements the transducer plates must be made accurately parallel. At frequencies corresponding to acoustic resonances maxima occur in the output of the monitoring transducer. A swept frequency was applied to the exciting transducer and the parallelism adjusted by optimising the amplitude and shape of these maxima.

3. The observation of subfrequency oscillations.

The initial study was made on water at 24 °C. If a sufficiently strong «pump» excitation is applied at one of the resonant frequencies in the region of \( f_p \), one or more of the lower frequency modes are observed to break into oscillation. As the input level is raised no output is seen until a threshold is reached, when a large output is observed consisting usually of one or two frequencies. If a mode exists at close to one half of the pump mode frequency only this single mode oscillates, at a frequency exactly one half that applied, the phenomenon of the «fractional harmonic». If no such condition exists between the pump and lower frequency modes, a pair of modes whose frequencies sum to the applied frequency oscillate. For very large pump amplitudes more than two modes may be excited, but discussion will be restricted to the behaviour close to threshold.

If the threshold is measured for each mode it is often found that a smooth function of pump frequency results. In some regions however the thresholds of some pump modes are somewhat different from those of adjacent modes. Even in these regions there is usually a well defined lower limit to the data points and a limit curve may be found, most of the data lying close to this curve and a few points lying somewhat higher.

A graph of the threshold measurements for water are plotted against pump frequency in figure 2. The transducer frequencies were 5.949 and 2.989 MHz. A broken line indicates the use of a limit curve. It is seen that oscillations occur over a wide range of frequencies, but that it is difficult to excite oscillations both very close and very far from \( f_p \). The effects of lengthening the resonator are to raise the threshold levels and to narrow the frequency spread. Lower thresholds and also greater amplitudes of oscillation occur below rather than above \( f_p \). For pump frequencies below \( f_p \) the oscillation frequencies are close to one half of the applied frequency, but this is not generally the case at higher frequencies.

If the 2.989 MHz transducer is replaced by a 3.978 MHz unit it is found that a pair of modes can be made to oscillate one being in the region of 4 MHz and the other in the region of 2 MHz, the frequencies being closely \( \frac{1}{3} \) and \( \frac{2}{3} \) of the applied frequency. The measured threshold graph is shown in figure 3. The general form is as before, save that there is approximate symmetry about \( f_p \).

Although results have been given only for water several other liquids have been investigated, principally the lower alcohols. Although the threshold...
levels are somewhat different, no essentially new features were found and in this preliminary report only the results for water are discussed.

4. The subfrequency modes. — The subfrequency modes which oscillate when a particular pump mode is excited are found to be principally determined by the mode frequencies, being those which minimise the difference between their sum and the pump mode frequency. If several selections present much the same sum, mode switching occurs, the modes generated changing when fine input frequency adjustments are made.

Consideration of the equivalent circuits of the transducers [6] leads to the conclusion that since the boundary conditions presented by the transducers are frequency dependent the modes will not be equally spaced in frequency but bunched in the regions of \( f_\nu, f_\sigma, \) and their multiples. As a result of this bunching a clear choice of subfrequency modes can be made on the criterion given above, provided that the pump mode is in a bunched region. One of the subfrequency modes will then be in the region of \( f_\nu, \) and if \( f_\nu \) is close to a submultiple of \( f_\sigma, \) that mode should be closely the same submultiple of the applied frequency. Away from the bunched regions the choice of modes is not so clear.

Diffraction and effects due to the amplitude variations across the resonator due to the transducers produce modes whose spacings can be accurately predicted but whose actual frequencies cannot. Thus although the above predictions are usually true in practice, in some cases the measured mode frequencies are needed before the oscillatory modes can be predicted. This is the case for the circumstances of figure 1 when the pump frequency is above \( f_\nu. \)

5. Theory. — Although it seems likely that the oscillations just described are parametric in origin, a theory based on the periodic length variation of the type used by Adler and Breazeale [5] failed to give the correct order of magnitude for the threshold. Following a suggestion of Korpel and Adler [3] that the liquid nonlinearity may be responsible, a formulation based on the parametric interaction between well defined resonator modes was attempted.

The interaction between a strong pump displacement (labelled \( p \))

\[ u = p_u e^{3(\omega_p t - k_p x)} + p_u e^{3(\omega_p t + k_p x)} + \text{c.c.} \quad (1) \]

and two weak excitations (labelled 1 and 2) of the same form, may be described by a nonlinear wave equation:

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \left( 1 + S \frac{\partial u}{\partial x} \right). \quad (2) \]

Here \( c \) is the sound velocity of the liquid and \( S \) is a nonlinearity parameter equal to \( -(2 + B/A) \) in standard notation [7]. Assuming that \( \omega_p = \omega_1 + \omega_2 \)

and \( k_p = k_1 + k_2 \) the total displacement presented by the pump and the two subfrequency disturbances may be substituted into eq. 2 and the lowest order terms in the various frequency and wavevector dependences collected. Two pairs of coupled equations and their complex conjugates results, a typical pair being:

\[ \frac{\partial_1 u_+}{\partial t} = -c \frac{\partial_1 u_+}{\partial x} - \frac{Sc^2 k_p k_1 k_2}{2 \omega_2} \times p_+ e^{2 k_1 x} - \beta_1 u_+ \quad (3) \]

\[ \frac{\partial_2 u_+^*}{\partial t} = -c \frac{\partial_2 u_+^*}{\partial x} - \frac{Sc^2 k_p k_1 k_2}{2 \omega_1} \times p_+ e^{-2 k_1 x} - \beta_2 u_+^*. \quad (4) \]

The last term in each equation has been added empirically to describe the liquid losses, \( \beta \) being the reciprocal attenuation length.

The threshold condition for oscillation corresponds to the state when the losses to the disturbances 1 and 2 are just due to the liquid damping and due to the reflection losses at the transducers, are just balanced by the parametric gain. This condition can be obtained by solving the set of equations represented by (3) and (4), with \( \partial \partial t = 0, \) and under the restrictions imposed by the boundary conditions at the transducers, which are positioned at \( x = 0 \) and \( x = L. \) These boundary conditions can be described in terms of \( \theta^p \) and \( \theta^d \) the phase angles upon reflection and corresponding amplitude losses \( Q^p \) and \( Q^d. \) The condition that the equations have solutions gives the threshold, and one other condition on the variables.

Although the derivation of a general expression for the threshold is straightforward if lengthy the result is difficult to handle except by numerical methods. However if the approximation is made that the liquid and reflection losses are identical for the frequencies 1 and 2 \( (\beta_1 = \beta_2, \; \theta^p_1 = \theta^p_2, \; Q^p_1 = Q^p_2 \text{ etc.}) \) the following expression may be obtained:

\[ V = \left\{ \frac{(2 \beta_1 L + Q_1^2 + Q_0^2 + \delta_1^2)^{15}}{S | k_1 k_p L G(\omega_p) H(\omega_p) \cos \{ (\pi - \gamma)/2 \}} \right\}. \quad (5) \]

Here \( \delta_1 = \theta^p_1 + \theta^d - 2 k_1, \) \( L, \) \( \delta_p = \theta^p_0 + \theta^d - 2 k_p L \) and \( \gamma = \theta^p_0 - \delta_0. \)

\( \delta_1 = 0 = \delta_0 \) formally defines the pump and subfrequency mode frequencies. \( G(\omega_p) H(\delta_p) \) is the bandwidth function relating the mean pump-displacement amplitude in the resonator to the applied voltage \( V, \) \( G(\omega_p) \) being that relation on resonance and \( H(\delta_p) \) being the pump mode lineshape. In addition to this expression for the threshold, an exact condition \( \delta_1 = \delta_2 \) is obtained.

6. Experimental results and discussion. — In order to test the above theory measurements were made on the 5.949 and 2.898 MHz transducer combination to
which the approximations made apply. It was found that provided modes selected for their good lineshape were chosen the relation $\delta_1 = \delta_2$ was true to the accuracy of the experiment. If only one subfrequency mode was involved oscillation at exactly half the applied frequency was found as predicted. It is convenient to present the threshold results for $\delta_1 = 0 = \delta_p$. These may be obtained by either measuring $\delta_1$ and $\delta_p$ or by finding the lower limit of the threshold data for each length, since for some pump modes this relation is in any case closely true. It is often useful to take several sets of data with the resonator lengths varied slightly about the length of interest. The « on tune » threshold results for a range of lengths are shown in figure 4.

Theoretical predications for comparison were made using eq. 5 and the equivalent circuits of the transducers [6] including mounting losses, from which $\theta, Q, \gamma$ and $G(\omega_p)$ were obtained. The nonlinearity parameter $B/A$ was taken as 5.0 [7]. The liquid attenuation lengths were measured as $95 \pm 10$ cm at 6 MHz and $110 \pm 10$ cm at 3 MHz. The latter figure includes diffraction losses which were found to behave for the purposes of this experiment in the same way as viscous losses in the liquid, at least for short resonators.

The experimental curves are seen to compare well with the theoretical ones, which depend mainly on the factor $(G(\omega_p) k_p k_1)^{-1}$ modified by the cosine factor, which raises the threshold close to $f_t$. The minima in the threshold for each length are reasonable well predicted. The dependence on length is good for short resonators but the experimental thresholds increase more rapidly than predicted for the greater lengths possibly due to diffraction becoming more dominant.

Although in this preliminary study the theory has only been tested in a limited range of circumstances the agreement is sufficiently striking to establish that a parametric theory based on the liquid nonlinearity is essentially capable of explaining the results.

Fig. 4. — The dependence of the « on tune » threshold voltage on pump frequency for a water filled resonator with

$$f_t = 5.949 \text{ MHz}$$

and $f_c = 2.989 \text{ MHz}$. The light curves are experimental and the heavy curves are calculated from theory. The resonator lengths are marked in centimetres.

Further experiments are needed on a range of liquids and transducer combinations to verify the detailed predications, taking care in future design to eliminate the severe diffraction problems encountered.

Acknowledgments. — The author is grateful to the Royal Society of London and to the Finnish Academy of Science for a maintenance grant and to Professor M. Luukkala for the hospitality of his laboratory.

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