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SOME MATHEMATICAL MANIPULATIONS OF INTERFEROGRAMS

A. S. FILLER

Laboratory for Research on the Structure of Matter,
and Department of Physics, University of Pennsylvania,
Philadelphia, Pennsylvania, U. S. A.

Abstract. — Brief discussions are given of the problems of zero-error corrections in interferograms, and of extrapolating interferograms in order to achieve increased resolution. It is pointed out that serious systematic errors may be introduced because of the use of finite sums, instead of the infinite sums required by the theory.

Résumé. — Nous discutons brièvement le problème de la correction du zéro dans les interférogrammes et de l'extrapolation des interférogrammes afin d'augmenter la résolution. Nous montrons que des erreurs systématiques sérieuses sont introduites par l'emploi de sommes finies au lieu des sommes infinies exigées par la théorie.

Last year, when papers were requested for this conference, I submitted the title "Phase errors in Fourier-transform spectroscopy". Early this year I learned that Forman, Steel and Vanasse had treated this problem, using the same approach that I had in mind. (Fortunately, I had not carried the problem to the computing stage, and I am grateful to them for having saved me much labor.) Their work has been published in the J. Opt. Soc. Amer., 1966, 56, 59. The first part of the present paper consists of a few remarks about this problem.

In a conversation with Dr. Vanasse, he mentioned the problem of extrapolating an interferogram, using the fact that it is an analytic function whose analytic continuation should be possible. Shortly thereafter a paper on this subject appeared, by Williams and Chang (J. Opt. Soc. Amer., 1966, 56, 167). The second part of this paper presents some work on this problem, since the subject is an intriguing one and may be of importance in practice. Also, since none of the papers announced for this conference included this subject, it was hoped that some useful discussion might be produced. Williams and Chang have not implemented their method in a practical way, and thus alternative procedures should be considered. However, there are doubts about the utility of the procedure given below, and its virtue may be that it is less formal than that of Williams and Chang, and it can serve at least to raise additional questions.

Phase Errors. — For simplicity we discuss the case of a symmetric interferogram $M(p)$ which is sampled asymmetrically, at $p = \delta + n\lambda$. Then, as Forman et al. point out, their formula reduces to the sampling theorem, which in truncated form is

$$M(m\lambda) = \sum_{s=\pm-p}^{m+p} M(\delta + n\lambda) \text{sinc}[(m - n) - \delta/\lambda]. \quad (1)$$

Since $p$ is not infinite in practice there is an error in this interpolation. (Of course this error decreases as $\delta$ decreases.) Forman et al. quote an error of 1 per cent for $p = 20$ (presumably for the worse value of $\delta$). There may be cases where such an error is not acceptable. It would seem to be desirable to have the error of interpolation small relative to the noise in the interferogram, so that effects of the error will not be apparent. Smaller interpolation error can be achieved by using larger values of $p$. Due to the slow decay of the function $\text{sinc}(z) = (\sin \pi z)/\pi z$ very large values of $p$ must be used to ensure very small errors of interpolation. Another procedure is to use an interpolation function which decays more rapidly, as discussed in J. Opt. Soc. Amer., 1964, 54, 762. For example, the function called $F$ in that paper reaches $10^{-4}$ for $p = 6$. A result of using such interpolation functions is that the calculated spectrum is weighted, going to zero as wavenumber $\sigma$ approaches $1/2 \lambda$.

If this weighting of the spectrum is annoying it can
be removed by dividing the spectrum by the weighting function. Since the interpolation reduces the noise in the same way that it does the spectrum, this division does not increase the signal-to-noise ratio. Near \( \sigma = 1/2 \lambda \) round-off errors will appear. If this is troublesome, then the interferogram can be sampled at smaller intervals, \( \Delta p = \lambda/n \), raising the limiting frequency to \( n/2 \lambda \). If the same total time is used to record the interferogram then each measurement will have more relative noise than for the minimum sampling case. However, only the minimum sampling points need be calculated by interpolation, and the noise in the interpolated values will be about the same as for the minimum sampling case. Further, the largest value of \( \delta \) now is \( \lambda/2 n \), smaller than \( \lambda/2 \), and in extreme cases this may serve to decrease the error of interpolation. Also, there will be some reduction of the effects of random errors in mirror position, since several mirror positions are of major importance in determining each interpolated value.

**Extrapolation.** — Analytic continuation is based on the fact that any continuous portion of an analytic function can be used to evaluate all the derivatives of the function, permitting the full Taylor series expansion of the function to be constructed and allowing the function to be evaluated everywhere. In practice, if an interferogram is known at a large number of points (say \( 10^8 \)) only a finite (although similarly large) number of derivatives can be found. In general it can be expected that all these derivatives will be of appreciable magnitude, i.e., the interferogram cannot be represented satisfactorily by a polynomial of lower order. The best that can be done by this approach is to find the polynomial which passes through all the measured points. Such a polynomial will be very good for interpolation, but not for extrapolation. The reason for this is that for extended extrapolation the omitted higher order terms in the series become important. Williams and Chang do not use a Taylor series expansion, but one based on prolate spheroidal functions. I must confess to complete ignorance of these, and thus can hope that they may be more suitable for this purpose. However, because of a distrust for the technique of analytic continuation I have used a different approach, based on interpolation techniques.

The problem of extrapolation of experimental data must be of interest in many fields. This discussion will be in terms of step-by-step recorded interferograms of band-limited spectra, which are assumed to be symmetric. Let the largest wavenumber in the spectrum be less than \( \sigma_0 \), and call the minimum sampling path differences, \( p = n/2 \sigma_0 = n \Delta \), the «primary» sampling points. In order to make an extrapolation additional information is required, and this is in the form of «secondary» sampling points, one or more of which lie between each pair of primary points. For simplicity, we take one secondary point in each primary interval. The recorded half of the symmetric interferogram contains \( N + 1 \) primary sampling points, from \( p = 0 \) to \( p = N \Delta \). If the true interferogram were zero for \( p > N \Delta \) then it would be possible to calculate exactly the values of the interferogram at the secondary points,

\[
M_s \left( \left[ j - \frac{1}{2} \right] \lambda \right) = \sum_{k=-N}^{N} M(k \Delta) \text{sinc} \left( \left[ k - j + \frac{1}{2} \right] \lambda \right),
\]

\( j = 1, 2, ..., N \) \hspace{1cm} (2)

Since the interferogram is not zero beyond the sampled range, the measured values at the secondary points are given by a similar sum, but with infinite limits,

\[
M \left( \left[ j - \frac{1}{2} \right] \lambda \right) = \sum_{k=-\infty}^{\infty} M(k \Delta) \text{sinc} \left( \left[ k - j + \frac{1}{2} \right] \lambda \right),
\]

\hspace{1cm} (3)

The differences between measured values and those calculated from the truncated interferogram are

\[
\delta_j = \sum_{k=N+1}^{\infty} M(k \Delta) \left( \text{sinc} \left( \left[ k - j + \frac{1}{2} \right] \lambda \right) \right) + \text{sinc} \left( \left[ k - j + \frac{1}{2} \right] \lambda \right)
\]

\[
= \sum_{k=N+1}^{\infty} A_{jk} M(k \Delta).
\]

If the \( \delta_j \) are very small, so that there is a good probability that they may be the result of the noise in the interferogram, then it is futile to continue. In order to make an extrapolation the noise level must be low enough to give significance to the \( \delta_j \). If this is the case, then there is extrapolation information in the \( \delta_j \) and we face the problem of extracting this information. Since there are \( N \delta \)'s and an infinite number of \( M \)'s, there cannot be a unique extrapolation. The obvious procedure is to terminate the sum after \( N \) terms, and write

\[
\delta_j = \sum_{k=N+1}^{2N} A_{jk} M(k \Delta).
\]

(5)

The error introduced by the termination certainly will result in large errors for those \( M \)'s near the terminus. How far toward \( k = N \) the errors are important requires further investigation. It seems likely that the
range over which the errors are large will be relatively independent of $N$ (since the neglected terms are infinite in number), and thus the procedure may work best, if it works at all, for large $N$.

Eqn. (5) can be solved for the $M'$ s, using $B_{jk}$, the matrix which is inverse to $A_{jk}$

$$M(k \Delta) = \sum_{j=1}^{N} B_{jk} \delta_j.$$  

The $A_{jk}$ are defined in eqn. (4) and are simple functions of $N, j$ and $k$. Therefore the $B_{jk}$ are also functions of these parameters. If their explicit form can be found, then the use of eqn. (6) becomes practical. If the form of the $B_{jk}$ is not known, then it is still possible to extract the $M'$ s from eqn. (5), although the computation time may be excessive. The $N$ equations given by eqn. (5) can be handled two at a time, thus requiring $2(N+1)$ words in memory. The $A_{jk}$ can be generated as needed.

We now consider, briefly, some effects of the noise in the interferogram. $M_6$ in eqn. (2) receives its largest contributions from the terms with $k = j - 1$ and $j$, and has an uncertainty about equal to the noise at the primary points. $M$ in eqn. (3) has the noise at the secondary sampling points. $\delta$ in eqn. (4) is a combination of these and will have the minimum noise, for a given total time of measurement, if the noise is the same for the primary and secondary points. Thus the measurement time should be the same for all the points. For a normal spectra the $\delta'$ s will be small relative to the $M'$ s, and very good signal to noise ratios will be required in the interferograms.

The following example may be an interesting way to test any extrapolation technique. Consider the band-limited spectrum $S(\sigma) = 1 - \cos 2 \pi p_0 \sigma, | \sigma | < \sigma_0$. Its interferogram consists of sinc functions centered at $p = 0, \pm p_0$. Suppose that the recorded interferogram does not include the side peaks. Then the test is to see how well the extrapolation gives them. In one sense this is a difficult test, since it is desired to extrapolate to a large value without too much error. But from the point of view of noise it is an easy test. In eqn. (6) the noise on the rhs is approximately constant, so that when the lhs is large the signal-to-noise ratio of the extrapolated value is good.

**Conclusion.** — Mathematical manipulations can become fascinating, and there is the danger that they may be used to an unnecessary extent. This is to be avoided, not only for simplicity, but also because such manipulations are a source of systematic errors. This is a serious danger which Fourier spectroscopy does not share with conventional spectroscopy. It is one thing to have a mathematical theorem. It is something different to implement it numerically. The best example of this is the use of a truncated interferogram, where the production of structure in the apparatus function is well known, and the use of apodization to correct this is common usage. The present paper presents other examples of truncation. The danger here is not just that errors are introduced into the interferogram, but that these errors are systematic, not random, and their effect on the calculated spectrum may be of an unexpected nature. This is a problem which probably merits further study, and not merely by the calculation of a few particular examples, a procedure which can be misleading.

In spite of this warning a good argument can be made for the routine use of a procedure for symmetrizing interferograms (at least for those cases where a full double-sided interferogram is not produced). It requires extreme care in adjustment to produce an interferogram in which no asymmetry can be detected. If the original interferogram is reasonably symmetrical, both in form and in sampling points, then the amount of interpolation will be small, so that the errors of interpolation will be small.

On the other hand, to try to increase resolution by extrapolation of an interferogram clearly is an act of desperation. There is no good substitute, in terms of time or effort, for an original interferogram. Still, there do occur cases where it is not possible to record the necessary span of interferogram. One instance is the study of transient phenomena, which do not last long enough to permit a long scan. And surely every spectroscopist has found at least one spectrum which needed just a little more resolution. What does one do when the mirror reaches the end of its track? The production of a good solution for the extrapolation problem is not merely an intellectual exercise, and it is hoped that one will be forthcoming.

**INTERVENTIONS**

L. MERTZ. — The method described by Williams and Chang (J. Opt. Soc. Amer., 1966, 56, 167) of separation into prolate spheroidal wave functions is all very well to increase a spectrum from perhaps 2 to 5 resolution elements. This is very small resolution for Fourier spectrometry. For Fourier spectrometry an increase from 100 to 102 resolution elements might be possible, but the $\gamma$ coefficients drop off incredibly fast, so that one disagree with their assertion that signal-to-noise can be evenly traded off for resolution.
P. Fellgett. — I agree with what Larry Mertz has just said. Also, I prefer to consider the phase as an essential part of the transform, rather than to «correct» the zero of path difference by interpolation; as A. S. Filler said, one’s concepts should not be confined to symmetrical interferograms.

There is a general method for obtaining wrong theoretical conclusions about the solution of convolutions. One invents for oneself a method of undoing a convolution; the proposal to extrapolate the interferogram is equivalent to this. One then works this method up in sufficiently complicated analysis to ensure that it is difficult to notice when the convoluting function is of the kind that cannot be undone, e.g. one having a Fourier transform which is either 0 or 1 everywhere.

For some curious historical reason communication engineers often understand these matters better than mathematicians.

I will stick my neck out and say that, in the approximation that the interferogram can be made both band-limited and of finite range in its own argument, no extrapolation of the interferogram is necessary.

On a minor point, the signal-to-noise relations are exactly the same whether one makes 1, 2 or $\infty$ measures per sampling interval (a special case is that continuous and step-by-step methods are equivalent) provided that the total time actually used is the same. This follows from the theorem that each set of sampling values is equivalent to the whole band-limited function and therefore to every other set of sampling values.

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THE INFLUENCE OF PATH-DIFFERENCE IN A DISPERSIVE MEDIUM ON TWO-BEAM INTERFERENCE

H. H. Hopkins
Imperial College of Science and Technology
University of London, England

The temporal coherence between two light beams having constant spatial coherence is given by Fourier transform of the spectral distribution of energy in the source when the path-difference arises in air. If there is also a path-difference in a dispersive medium, this is no longer the case. The temporal coherence as a function of the total path-difference for the mean wavelength is found for a gaussian profile; and this is used to derive a tolerance on the equality of glass thickness in the two arms of an interferometer.