Anomalous diffusion in presence of boundary conditions
A. Giacometti, A. Maritan

To cite this version:

HAL Id: jpa-00212454
https://hal.archives-ouvertes.fr/jpa-00212454
Submitted on 1 Jan 1990

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Anomalous diffusion in presence of boundary conditions

A. Giacometti (1) and A. Maritan (2, *)

(1) Dipartimento di Scienze Ambientali, Università di Venezia, Calle Larga S. Marta, 30124 Venezia, Italy
(2) Dipartimento di Fisica dell’Università di Bari e I.N.F.N., Sezione di Bari, Italy

(Reçu le 10 juillet 1989, accepté sous forme définitive le 14 mars 1990)

Abstract. — Anomalous diffusion in presence of a (fractal) boundary is investigated. Asymptotic behaviour of the survival probability and current through an absorbing boundary are exactly calculated in specific examples. They agree with recent findings related with anomalous Warburg impedance experiments. The autocorrelation function for sites near the boundary is carefully discussed; in presence of reflecting boundary conditions it can be different from the (spatial) average of the autocorrelation at variance with the normal diffusion case. Interesting issues from a methodological point of view are discussed in the renormalization groups analysis. Scaling arguments are given relating various exponents.

1. Introduction.

Very recently the old problem of diffusion impedance (Warburg impedance) [1] has regained much attention in the framework of fractal geometry [2-5].

There exists both numerical and experimental [2] evidence that the long time behaviour of the Faradaic current, $J(t)$, across a fractal surface, scales like

$$J(t) \sim t^{-\alpha} \quad \text{with} \quad \alpha = \frac{d_s + 2 - d}{2}$$

(1)

where $d_s$ is the fractal dimension of the surface. Some heuristic considerations [2-4] and rather simple exactly solvable models [2, 3], have been suggested in order to justify equation (1).

In reference [4] it has been shown that the survival probability, $P_s(t)$, of a particle starting at random near the boundary has the same asymptotic behaviour of $J(t)$ and that the $\alpha$-exponent is related to the surface and bulk magnetic exponents of the Gaussian model associated to the diffusion equation.

One might ask what happens when a fractal bulk is substituted to the euclidean one.

Two differences are to be expected: an anomalous diffusion instead of the normal one and the presence of the fractal dimension $d_B$ of the bulk, instead euclidean dimension $d$.

(*) Mail Address: Dipartimento di Fisica dell’Università di Padova, via Marzolo 8, 35131 Padova, Italy
By means of general and exact arguments, the authors of reference 4 have been able to
generalize equation (1) to the case when besides fractal boundaries, a self-similar waiting time
distribution is present on an Euclidean d-dimensional bulk, thus leading to anomalous
diffusion.

In this case

$$\alpha = \frac{d_w + \bar{d}_s - d}{d_w}$$  \hspace{1cm} (2)

is found instead of equation (1). $d_w$ in equation (2) is the exponent governing the long time
behaviour of the average square distance, $r^2$, travelled by a random walker after a time
t, i.e. $r^2 \sim t^{2/d_w}$.

Concerning the behaviour of the survival probability $P_s(t)$, some heuristic arguments [4]
suggest that the power law decay is now different from the one of $J(t)$, and that is should be
characterized by the new exponent

$$\alpha_s = \frac{2 + \bar{d}_s - d}{d_w}.$$  \hspace{1cm} (3)

Our aim is three-fold: to present exactly solvable models (two- and three-dimensional
Sierpinski gaskets) where the scaling relations obtained in reference [4] are verified. We think
it is always welcome to have examples confirming scaling laws and indirectly the arguments
leading to them. Indeed as mentioned in reference [4] a straightforward generalization of
preexistig arguments leading to equation (1) [2] does not lead to equation (2).

If the waiting-time is an effect due to the presence of a self-similar distribution of symmetric
transition rates between neighbouring sites then the arguments of reference [4] would suggest
in this case $P_s(t) \sim J(t) \sim t^{-\alpha}$ with $\alpha$ given by equation (2). We will give an example that this
indeed occurs while other « bulk properties » like the average over sites of the autocorrelation
function and the mean end-to-end distance after a time $t$ behave like in the related waiting-
time problem.

Last but not least the renormalization group (R.G.) analysis of diffusion in presence of
boundary conditions is not a simple modification to be done in the already existing cases [7].
A first example of it was given some time ago in reference [6].

In section 2 the one dimensional case will be discussed in order to exemplify how the
implementation of a R.G. analysis must be done when boundary conditions are properly
processed. An application to the Huberman and Kerszberg [8] model of ultradiffusion is done
and some asymptotic behaviours are compared with those found for the related waiting-time
problem already discussed in reference [6]. In section 3 the two- and three-dimensional
Sierpinski gasket is studied where besides an absorbing boundary also a hierarchical
distribution of waiting time is present. In a sense our results could be considered
complementary to those of reference [7] where boundary conditions were not taken into
account. Section 4 contains scaling arguments and heuristic considerations supported by the
exact results of previous sections.

2. 1-D examples.

2.1 Uniform case. — Let us start with the very simple and exactly solvable example of
diffusion on a semi-infinite one-dimensional lattice with an absorbing wall at $x = 0$. It will be
shown that of two possible choices for the sites to be decimated in order to implement a
rescaling of the system, only one is consistent with the boundary condition.
Let $P_{x_0,x}(t)$ be the probability of the particle to be at site $x$ after a $t$-step random walk beginning in $x = x_0$. $P$'s satisfy the discrete time master equation

$$P_{x_0,x}(t+1) = \frac{1}{2} P_{x_0,x+1}(t) + \frac{1}{2} P_{x_0,x-1}(t) \quad x = 1, 2, \ldots ,$$

(4)

where $P_{x_0,x}(t) = 0$, for any $t$, is the boundary condition and $P_{x_0,x}(0) = \delta_{x,x_0}$ is the initial condition.

We assume in the following that $x_0 = 1$, that is the particle starts its walk close to the absorbing wall.

By introducing the generating function (discrete Laplace transform)

$$\tilde{P}_{x_0,x}(\omega) = \sum_{t=0}^{\infty} P_{x_0,x}(t)(1 + \omega)^{-1-t},$$

(5)

equation (4) can be rewritten as :

$$\alpha_x(\omega) G_{x_0,x}(\omega) = G_{x_0,x-1}(\omega) + G_{x_0,x+1}(\omega) + \delta_{x,x_0}$$

(6)

where $G_{x_0,x}(\omega) = \frac{1}{2} \tilde{P}_{x_0,x}(\omega)$ and (1)

$$\alpha_x(\omega) = \begin{cases} \alpha(\omega) = 2(1 + \omega) & \text{if } x > 1 \\ \bar{\alpha}(\omega) & \text{if } x = 1. \end{cases}$$

(7)

In equation (7) $\alpha_{x=1}(\omega) = \bar{\alpha}(\omega)$ has been written in order to start with the minimal subset of parameters entering the master equation (6) which remains invariant under R.G. recursions. Pathological role played by $x = 1$ in fact, could make $\bar{\alpha}(\omega) \neq \alpha(\omega)$ under R.G. transformations even if actually the initial condition for the flow is $\bar{\alpha}(\omega) = \alpha(\omega)$. It is easy to see that the choice $\bar{\alpha}(\omega) = 1 + \omega$ corresponds to a reflecting barrier at $x = 0$. The Gaussian model related to this problem is described by the reduced Hamiltonian

$$H(\{\varphi_x\}; \bar{\alpha}, \alpha, h_1, h) = \frac{1}{2} \bar{\alpha} \varphi_x^2 + \frac{1}{2} \alpha \sum_{x=1}^{\infty} \varphi_x^2 - \sum_{x=0}^{\infty} \varphi_x \varphi_{x+1} - h_1 \varphi_1 - h \sum_{x=1}^{\infty} \varphi_x.$$

(8a)

Indeed using the identity

$$\prod_y \int_{-\infty}^{+\infty} d\varphi_y \frac{\partial}{\partial \varphi_x} (\varphi_{x_0} e^{-H}) = 0$$

(8b)

one sees that $\langle \varphi_{x_0} \varphi_x \rangle$ satisfies the same equation as $G_{x_0,x}(\omega)$ for $h = h_1 = 0$.

A R.G. transformation can be obtained by decimating the field variables in a subset $E$ of sites and recasting the effective Hamiltonian, for the surviving field variables $\varphi'$, in the form (8a) with renormalized parameters $\{ \bar{\alpha}', \alpha', h', h_{1}' \}$.

If the set of even sites $E_{\text{even}}$ is chosen as subset $E$, the following recursions for $\bar{\alpha}$ and $\alpha$ are found:

$$\bar{\alpha}' = \alpha \bar{\alpha} - 1,$$

(9a)

$$\alpha' = \alpha^2 - 2,$$

(9b)

(1) The coefficients $\alpha_x(\omega)$ should not be confused with the exponents in equations (2) and (3).
together the field variable renormalization $\varphi_{2x} = \sqrt{\alpha} \varphi_x$. R.G. flow in the \{\bar{\alpha}, \alpha\} plane is shown in figure 1a; the only fixed point $A = (\alpha^* = 2, \bar{\alpha}^* = 1)$ turns out to be repulsive and it cannot be reached by starting with $\bar{\alpha} = \alpha$. For $\omega \to 0$, the «fixed point» $(\alpha^* = 2, \bar{\alpha}^* = \infty)$ is reached which does not describe the problem of a walk starting at $x_0 = 1$ since $\bar{p}_{x_0 = 1,x} (\omega) = 2 \langle \varphi_1 \varphi_x \rangle = 0$ for $\bar{\alpha} = \infty$.

On the contrary if the set of odd sites $E_{\text{odd}}$ (which includes $x = 1$) is chosen, the same recursion (9b) is recovered, and

$$\bar{\alpha}' = \alpha^2 - \frac{\alpha}{\bar{\alpha}} - 1$$

(9c)

instead of equation (9a).

R.G. flow in the \{\bar{\alpha}, \alpha\} plane is shown in figure 1b; there are two fixed points $A = (\alpha^* = 2, \bar{\alpha}^* = 1), B = (\alpha^* = 2, \bar{\alpha}^* = 2)$ in this case, but $B$ is the only one that can be reached by starting with $\bar{\alpha} = \alpha$ i.e. in the case of absorbing boundary condition. The fixed point $A$ is related to the presence of a reflecting boundary at $x = 1$ since $(\alpha (\omega) = 2 + \omega, \bar{\alpha} (\omega) = 1 + \omega) \to (2, 1)$ as $\omega \to 0$.

The recursions for the magnetic fields $h_1$ and $h$ are:

$$\begin{pmatrix} h'_1 \\ h' \end{pmatrix} = T_H \begin{pmatrix} h_1 \\ h \end{pmatrix} \quad \text{with} \quad T_H = \begin{pmatrix} \frac{\sqrt{\alpha}}{\bar{\alpha}} & \alpha + 1 \\ 0 & \frac{\sqrt{\alpha}}{2 + \alpha} \end{pmatrix}$$

(10)

whose eigenvalues $\lambda_H$ and $\lambda_{HS}$ are related to the bulk magnetic exponent $\gamma_H$ and surface magnetic exponent $\gamma_{HS}$ through the relation $2^\gamma = \lambda_i$ (2). We get $\gamma_H = \frac{3}{2}, \gamma_{HS} = \frac{1}{2}$ at $A$ and $\gamma_H = \frac{3}{2}, \gamma_{HS} = -\frac{1}{2}$ at $B$ which are the exact ones.

---

(2) The identification of $\lambda_H$ is obvious since $h' = \lambda_H h$.
The use of recursion equations like the ones obtained to derive the asymptotic scaling will be done in the next subsection in a more general model. It is well known that different R.G. schemes can lead to different fixed points but to equal exponents, that is the same physics must be described. However in the first of the two cases presented above, due to the R.G. flow, one is not allowed to derive exponents by standard scaling arguments. We must notice however that as far as the bulk properties are concerned only the recursion \(9b\) is important and it is independent of the R.G. schemes we use! It is worth noting that, in the analysis of reference [6], the right scheme has been chosen.

2.2 Hierarchical Transition Probabilities Distribution. — Let us apply what we have just learned to the non-trivial case of the Huberman and Kerszberg model [8]: the (discrete) Laplace transform \(\tilde{P}_{x_0,x}(\omega)\) satisfies the following equation

\[
\alpha_x \tilde{P}_{x_0,x}(\omega) = \delta_{x-x_0} + W_{x,x+1} \tilde{P}_{x_0,x+1}(\omega) + W_{x,x-1} \tilde{P}_{x_0,x-1}(\omega),
\]

with \(x \geq 1\) and

\[
\alpha_x = \omega + W_{x+1,x} + W_{x-1,x}.
\]

The transition probabilities \(W_{x,x+1}\) to jump from site \(x+1\) to site \(x = (2m + 1) 2^n \neq 1\) \(m, n = 0, 1, 2, \ldots\) are

\[
W_{x,x-1} = W_{x-1,x} = \begin{cases} 
\epsilon_1 R^{n-1} & n \geq 1, \\
\epsilon_0 & n = 0,
\end{cases}
\]

representing a particle hopping across barriers (low \(W_{x-1,x}\) means high barriers between \(x\) and \(x-1\)).

In order to take into account boundary conditions we choose

\[
W_{1,0} = 0, \quad W_{0,1} = W,
\]

indicating that \(x = 0\) is partially absorbing if \(W > 0\) or reflecting if \(W = 0\). By a simple redefinition of \(\tilde{P}\)'s one can fix \(\epsilon_0 = 1\). The model (11), without paying attention to boundary conditions, was studied by approximate renormalization group techniques in references [8, 9] and solved by an exact R.G. in reference [10]. It exhibits a « dynamical phase transition » at \(R = R_c = 1/2\) between an anomalous diffusion regime \((R < R_c)\) and a normal diffusion one \((R > R_c)\) [9, 10].

Let us see how the things go with boundary conditions. Also in this case it is more practical to work with the related Gaussian model whose Hamiltonian is

\[
H(\{\varphi_x\}; \{w, h\}, \omega) = \frac{1}{2} \sum_{x \neq 1} \alpha_x \varphi_x^2 - \sum_{x = 1}^\infty W_{x,x+1} \varphi_x \varphi_{x+1} - \sum_{x = 1}^\infty h_x \varphi_x.
\]

The minimal subset of « external field » parameters we need, which remain invariant under R.G. recursions, is \(h_x = h(1 - \delta_{x,1}) + h_1 \delta_{x,1}\).

The right subset of sites to be decimated is \(E = \{x = 4m + 1, 4m + 2 : m = 0, 1, 2, \ldots\}\) and apart from boundary effects was the same used in [10].

After the decimation the partition function can be rewritten as

\[
Z = e^G \int \prod_{x \notin E} d\varphi_x^* e^{-H} = e^G Z',
\]
where $H'$ has the same form as in equation (14) with the following renormalized fields and parameters

\[ \varphi'_{1/2} = \varphi \sqrt{\frac{\varepsilon_1}{(1 + \omega)(1 + 2\varepsilon_1 + \omega)}} \quad x \notin E, \]

\[ \varepsilon_1' = R(1 + \omega)(\omega + 1 + 2\varepsilon_1), \]

\[ \omega' = (\omega^2 + 2\omega) \frac{\omega + 1 + 2\varepsilon_1}{\varepsilon_1}, \]

\[ W' = (1 + 2\varepsilon_1 + \omega)(W + \omega)A, \]

\[ h' = h \frac{2 + \omega}{1 + \omega} \sqrt{(1 + \omega)(1 + 2\varepsilon_1 + \omega)} \]

and $A^{-1} = \omega^2 + \omega(1 + \omega + 2\varepsilon_1) + \varepsilon_1(1 + W) + W$. $G$ in equation (15) is constant and the relevant terms for our applications, are

\[ G = \left[ \frac{h_1^2}{2}(1 + \varepsilon_1 + \omega) + hh_1\varepsilon_1 \right] A + O(h^2) + \text{const}. \]

Now let us discuss what these recursions imply as far as the asymptotic behaviour $w \to 0$ (i.e. $t \to \infty$) is concerned.

The fixed points of recursion equations (16b, c) are $O = (\omega^*, \varepsilon_1^*) = (0, 0)$ and $O' = (\omega^*, \varepsilon_1^{*-1}) = \left(0, \frac{1}{R} - 2\right)$ and the eigenvalues of the linearized recursions near them are $\lambda_1 = 4$, $\lambda_2 = 1/(2R)$ and $\lambda_1' = 2/R$, $\lambda_2' = 2R$ respectively. Thus for $R < 1/2 (> 1/2)$ $O'(O)$ is the stable fixed point implying an asymptotic behaviour for the average end-to-end distance after a large time $t$

\[ \langle x^2(t) \rangle \sim t^{2/d_w} \]

with [10]

\[ d_w = \begin{cases} \frac{\ln \lambda_1}{\ln 2} & R < 1/2 \\ \frac{\ln 2}{\ln \lambda_1} & R > 1/2 \end{cases} = \max \left(2, \frac{\ln 2/R}{\ln 2}\right) \]

leading to the above mentioned « dynamical phase transition » at $R = R_c = 1/2$ [9, 10].

From equation (15) the free energy density scales like

\[ f(\omega, \varepsilon, w, h, h_1) = \frac{1}{2} f(\omega', \varepsilon', w', h', h_1') + g \]

where $g$ is a non singular function of $\omega$, $\varepsilon_1$ etc. From equations (12) and (14) one has

\[ \frac{\partial f}{\partial \omega} = \lim_{N \to \infty} \frac{1}{N} \sum_{x_0=1}^{N} \left( \varphi_{x_0} \varphi_{x_0} \right)_t = \tilde{P}_{x_0, x_0}(\omega) \]
where the bar means average over sites. Using equation (19) at \( h = h_1 = 0 \) and at the stable fixed point, one has

\[
\bar{P}_{x_0, x_0}(\omega) \sim 2^{d_w - 1} \bar{P}_{x_0, x_0}(2^{d_w} \omega)
\]

implying

\[
\bar{P}_{x_0, x_0}(\omega) \sim \omega^{-1 + 1/d_w}
\]

i.e. \( \bar{P}_{x_0, x_0}(t) \sim t^{-1/d_w} \sim 1/\langle x^2(t) \rangle^{1/2} \), as one should expect from standard scaling arguments.

The fixed points of (16d) are \( W^* = 0 \) and \( W^* = 1 \) corresponding to the reflecting and absorbing wall at \( x = 0 \) respectively. It is easy to see that \( W^* = 1 \) is always attractive and thus as far as \( W \neq 0 \) the asymptotic behaviour is the one corresponding to the absorbing wall.

The Laplace transform of the probability to be at the boundary starting from the boundary, \( \bar{P}_{1, 1} = \langle \varphi_1 \varphi_1 \rangle \), can be obtained differentiating the partition function twice with respect to \( h_1 \). Recalling the recursion (15) and (16f) and \( G \) from equation (17) we get, at the fixed point,

\[
\bar{P}_{1, 1}(\omega) \sim \begin{cases} 
R^{-1} \bar{P}_{1, 1}(2 \omega / R) + (1 - R) / R & \text{at } W^* = 0 \\
R \bar{P}_{1, 1}(2 \omega / R) + 1 - R & \text{at } W^* = 1 
\end{cases}
\]

for \( R < 1/2 \). If \( R > 1/2 \) equation (23) still holds if \( R \) is substituted with \( 1/2 \). From (23) one derives the following asymptotic behaviour

\[
P_{1, 1}(t) \xrightarrow{t \to \infty} \begin{cases} 
t^{-1/d_w} & \text{Reflecting b.c. (} W^* = 0) \\
t^{-2 - 1/d_w} & \text{Absorbing b.c. (} W^* = 1) 
\end{cases}
\]

One can also derive the asymptotic expression for the probability, \( F_{1, 1}(t) \), to be for the first time again at the origin \( x = 1 \) at time \( t \). Since \( d_w \geq 2 \), using a simple result reported e.g. in reference [11] and equation (24a) or (24b) one has

\[
F_{1, 1}(t) \sim t^{-2 - 1/d_w}.
\]

From recursions (16e) and (16f) one gets (the procedure is the same as in the previous subsection)

\[
y_H = \frac{1 + d_w}{2},
\]

and

\[
y_{HS} = \begin{cases} 
\frac{d_w - 1}{2} & \text{Reflecting b.c.} \\
\frac{1 - d_w}{2} & \text{Absorbing b.c.}
\end{cases}
\]

It is easy to see that for large \( t \) \( P_{1, 1}(t) \sim t^{-(1 - 2y_{HS} / d_w)} \) with the \( y_{HS} \) appropriate to the chosen boundary condition.

It was shown in [4] that the current at an absorbing boundary of an initial uniform distribution of particles is

\[
J(t) = W \sum_x P_{x, 1}(t) = \frac{W}{Z} \frac{\delta^2 Z}{\delta h_1 \delta h}
\]
which is proportional to the survival probability \( P_s(t) = \sum_x P_{1, x}(t) \) of a particle starting near the boundary since in the present case (symmetric hopping probabilities) \( P_{x, 0} = P_{0, x} \). Using equations (26) and (27) it is immediate to show that

\[
J(t) \sim P_s(t) \sim t^{-\alpha}
\]

\[
\alpha = 1 - \frac{y_H + y_{HS}}{d_w} = 1 - \frac{1}{d_w}
\]

where \( y_{HS} \) is, of course, given by equation (26c). Equations (28a, b) confirm what we had already announced in the introduction concerning diffusion in presence of symmetric hopping probabilities.

2.3 Hierarchical Waiting Time Distribution. — Let us now compare the exponents obtained above with the ones of a related model where the bulk transition probabilities are

\[
W_{x \pm 1, x} \propto R^n \quad x = (2m + 1) 2^n.
\]

This means that at a site \( x = (2m + 1) 2^n \) there is an average waiting time \( \sim R^{-n} \) and that this distribution is hierarchical in the same sense as the Huberman and Kerszberg model previously studied. At variance with the model of the previous subsection (Eq. (13a)) now sites represent wells (low \( W_{x \pm 1, x} \) means deep well at \( x \)) and the transition rates are not symmetric. This model with boundary conditions taken into account was studied in full detail in reference [6] (\( R \) in Eq. (29) is \( R^{-1} \) in Ref. [6]) so that we report here only the results.

Concerning « bulk properties » like \( \langle x^2(t) \rangle \) and \( P_{x, 0}(t) \) they behave like the previous case with the same exponent \( d_w \) given by (18).

The « magnetic exponent » \( y_H \) and \( y_{HS} \) are the same as in the uniform case of subsection 1 and thus do not coincide with those given in equation (26) in the anomalous diffusion regime \( R < 1/2 \). Concerning the asymptotic behaviour of the correlations introduced above they are

\[
P_{1, 1}(t) \sim \begin{cases} t^{-\left(1 - 1/d_w\right)} & \text{Reflecting b.c.} \\ t^{-\left(1 + 1/d_w\right)} & \text{Absorbing b.c.} \end{cases}
\]

\[
J(t) \sim t^{-\left(1 - 1/d_w\right)}
\]

\[
P_s(t) \sim t^{-1/d_w}.
\]

Equations (32a) and (32b) lead to the exponents (2) and (3) respectively for the 1-D case.

We believe these two last examples are instructive at least for two reasons: i) in spite of having the same « bulk properties » that could also be obtainable by heuristic arguments of references [7] and [6] they have completely different « surface properties ». We would like to remark that such a difference was already partially guessed on the basis of numerical simulations by Havlin and Ben-Avraham [12] but never explained. ii) Renormalization group techniques we have applied, explicitly exhibits the care that must be payed to the boundary conditions. This is uncommon especially because most of the subtleties related to an exact R.G. are lost in approximate approaches.
3. Sierpinski gaskets with boundary conditions.

Let us consider now an example of diffusion on a fractal structure. With reference to figure 2, consider a 2-dimensional Sierpinski gasket with a side $\partial A$ acting like a perfecting absorbing surface. We assume that the particle could start the walk in each of the site belonging to $\partial A = \{x_0: |y - x_0| = 1, y \in \partial A\}$ with equal probability.

If $A$ is the bulk, define $E_m$ as the set of sites $x$ belonging to $A$, such that $x$ is a vertex of a $m$-th order triangle ($m = 0, 1, 2, \ldots$); a hierarchical distribution of waiting times is obtained by assigning a probability $q_x = \tau_m^{-1}$ to jump when the particle is one a site $x$ belonging to $E_m$ [7]. Thus $\tau_m$ is the waiting time on sites of $E_m$. It can be further assumed $\tau_0 = 1$ by a proper rescaling of $\omega$.

The fractal dimension of the bulk is $d_B = \ln 3 / \ln 2$ and the master equation is such a case is:

$$P_x(t+1) = (1-q_x)P_x(t) + \sum_{y(x)} \frac{q_y}{z_y} P_y(t)$$

where, as usual, $\sum_{y(x)}$ means sum over all nearest-neighbour of $x$ whose number is $z_x$.

In our model:

$$q_x = \begin{cases} 
\tau_m^{-1} & \text{if } x \in E_m \subset A \\
\tau^{-1} & \text{if } x \in \partial A \\
0 & \text{if } x \in \partial A 
\end{cases}$$

(34a)

Fig. 2. — Subset of an infinite Sierpinski gasket. The waiting times related to some representative site are shown together the boundaries $\partial A$ and $\partial A$ defined in the text.
where \( z = 4, 2 \) before the first decimation is performed, corresponding to absorbing and reflecting boundary conditions respectively.

The probability of being trapped in \( \delta \Lambda \), after a \( t \)-step walk, \( Q(t) \), satisfies:

\[
Q(t + 1) = Q(t) + \frac{\bar{q}(z - 2)}{z} \sum_{x \in \delta \Lambda} P_x(t), \quad \bar{q} = \bar{\tau}^{-1}
\]

(35)

as it can be inferred by using the constraint of conservation of the total probability. By means of (5) and our definition of the initial conditions, we get from (33):

\[
\alpha_x(\omega) G_x(\omega) = \sum_{y(\omega)} G_y(\omega) + \frac{\delta_{x, \delta \Lambda}}{|\delta \Lambda|}
\]

(36)

where

\[
G_x(\omega) = \frac{q_x}{z_x} \bar{P}_x(\omega), \quad (37a)
\]

\[
\alpha_x(\omega) = z_x \left( 1 + \frac{\omega}{q_x} \right), \quad (37b)
\]

\(|\delta \Lambda|\) is the number of sites on the surface and \( \delta_{x, \delta \Lambda} = 1 \) if \( x \in \delta \Lambda \) and zero otherwise.

As we learnt from the one dimensional case the decimation procedure consists of the elimination of the whole set \( E_0 \), including \( \delta \Lambda \), in favour of \( E_m \) \((m > 0)\); the whole system is thereby scaled by a factor 2. This leads to new expression \( G'_x, \alpha'_x, \bar{\alpha}' \) in terms of \( G_x, \alpha_x, \bar{\alpha}(x \in E_n, x' \in E_{n-1}) \).

After some manipulations the recursion equations, in the limit \( \omega \to 0 \), are:

\[
G_x'(\{\alpha'\}) = \frac{3}{5} (z - 1) G_x(\{\alpha\}) \quad (38a)
\]

\[
\omega' = \frac{5}{2} (2 + \tau_1) \omega \quad (38b)
\]

\[
R'_n = R_{n+1} \quad (38c)
\]

\[
\tau'_1 = 1 + \frac{\tau_1 - 1}{\tau_1 + 2} R_1 \quad (38d)
\]

\[
z' = \frac{16}{3} - \frac{10}{3(z - 1)} \quad (38e)
\]

\[
\bar{\tau}' = [a(z, z') \bar{\tau} + b(z, z') \tau_1 + C(z, z')]/(2 + \tau_1) \quad (38f)
\]

where

\[
a(z, z') = \frac{2z}{z'(z - 1)^2}; \quad b(z, z') = \frac{4}{z'}; \quad c(z, z') = \frac{124}{15} z' + \frac{64}{15} \quad (39a)
\]

and \( R_n \) is defined as follows:

\[
R_n = (\tau_{n+1} - \tau_n)/(\tau_n - \tau_{n-1}) \quad n = 1, 2, \ldots \quad (39b)
\]
The recursions for $\omega$ and $\tau_n$'s are obtained by imposing $\tau_0' = 1$ while the ones for $\bar{r}$ and $z$ are derived from the recursion for $\bar{a}$ by requiring $\bar{a}' = z'(1 + \omega' \bar{r}')$.

It is important to stress that the $G$'s renormalization is determined by the inhomogeneous term in equation (36) which, for the renormalized problem, must be $\frac{\delta x, \delta A'}{[\delta A']}$ ($G'_x$ in Eq. (38a) has the same functional dependence on $\{\tau'\}$ and $\omega'$ as $G_x$ has in terms of the unrenormalized counterparts).

The fixed points of the recursions (38b-e) are:

\begin{align*}
\omega^* &= 0, \quad R^* = R \\
\tau_1^* &= 1, \quad R = 2 \\
\tau_n^* &= (\tau_1^* - 1) \frac{R^n - 1}{(R - 1)} + 1 \\
z^* &= \frac{13}{3}, \quad 2.
\end{align*}

The fixed point reached by $\bar{r}$ under recursions is

$$\bar{r}^* = \frac{A}{1 - B},$$

where

$$A = \begin{cases} 
824/45 & \text{if } R < 3 \\
300 R - 76/45 & \text{if } R > 3
\end{cases}, \quad B = \begin{cases} 
3/50 & \text{if } R < 3' \\
9/50 R & \text{if } R > 3.
\end{cases}$$

We have used that, since $(\partial z'/\partial z)_{z^* = 2} = 10/3$ and $(\partial z'/\partial z)_{z^* = 13/3} = 3/10$, $z^* = 13/3$ is the stable fixed point and it is reached starting with the initial condition $z = 4$. $z^* = 2$ corresponds to reflecting boundary conditions and it is unstable as usual.

If $R_n = R$ recursion equations (38b-d) imply that the stable fixed point describing the asymptotic behaviour $\omega \to 0$ is $\tau_1^* = \text{Max}(1, R - 2)$, where the maximum eigenvalue of the matrix associated to the linearized recursion gives the diffusion exponent defined through the scaling law $r^2(t) \sim t^{2/d_w}$ of the average square distance travelled by the particle after a long time $t$.

As expected a « dynamical phase transition » at $R_c = 3$ [9, 10, 7] arises, separating the normal and anomalous regime.

In the following we will call $d_w^{(0)}$ the diffusion exponent in equation (42) when $R < R_c$.

The result (42) has already been given by Robillard and Tremblay [7]. We point out, however that neither boundary conditions nor magnetic exponents have been discussed in their work.

In view of the fixed points quoted above, it is straightforward to derive the long time behaviour of survival probability, which is found out to be strongly connected with the value $z^* = 13/3$. Indeed equation (35) at the fixed point yields:

$$\omega \tilde{Q}(\omega) = (z^* - 2) \tilde{G}(\omega)$$
where $\tilde{Q}$ is the discrete Laplace transform of $Q$ and

$$\tilde{G}(\omega) = \lim_{|A| \to +\infty} \sum_{x \in A} G_x(\omega)$$

(44)

which satisfies, by using the recursion equations (38a, b) at the fixed point:

$$\tilde{G}(\omega) = \frac{3}{10} [\tilde{G}(\omega') + 1].$$

(45)

Equations (43), (45), (35) and the recursion (38b) at $\tau_1 = \tau_1^*$ lead to the following asymptotic behaviour of the survival probability and autocorrelation function

$$P_x(t) = 1 - Q(t) \sim t^{-\alpha_s}$$

(46a)

$$P_{\delta A}(t) = \sum_{x \in \delta A} P_x(t) \sim \frac{dQ}{dt} \sim t^{-(1 + \alpha_s)}$$

(46b)

where

$$\alpha_s = \frac{d^{(0)}_w + 1 - \bar{d}_B}{d_w(R)}$$

(47)

which agrees with the prediction (3), since in this case $\bar{d}_s = 1$ and the normal diffusion exponent in the present case is $d^{(0)}_w$ and not 2 as was the case equation (3) refers to (in order to obtain Eq. (46b), Eq. (35) has been used).

In order to compute magnetic exponents, we write down the Gaussian model associated to the equation (36), i.e.

$$H(\{\phi_x\}; \{\alpha_x\}, \bar{\alpha}, h, h_1) = \frac{1}{2} \sum_{x \in A} \alpha_x \phi_x^2 + \frac{1}{2} \sum_{x \in \delta A} \bar{\alpha} \phi_x^2 - \sum_{(x,y)} \phi_x \phi_y - h \sum_{x \in A - \delta A} \phi_x - h_1 \sum_{x \in \delta A} \phi_x$$

(48)

A decimation of $E_0$ similar to the one performed in the one-dimensional case, yields the previous recursions and a new couple of recursion equations involving $h$ and $h_1$. In the limit $\omega \to 0$ and at the fixed point, these latter turn out to be:

$$h' = 5 \left( \frac{3}{5} \right)^{\frac{1}{2}} h$$

(49a)

$$h_1' = 2 \left( \frac{5}{3} \right)^{\frac{1}{2}} h + \left( \frac{3}{5} \right)^{\frac{1}{2}} h_1$$

(49b)

independent of $\tau^*$! Thus bulk and surface magnetic exponents are

$$y_H = (\bar{d}_B + d^{(0)}_w)/2$$

(50a)

$$y_{HS} = (\bar{d}_B - d^{(0)}_w)/2$$

(50b)

which can be thought as generalization of the standard Euclidean case [4].

As shown in reference [4] the current arriving at the absorbing wall for initially uniform distribution of particles through the fractal, scales like $J(t) \sim t^{-\sigma}$ with
and using equation (50a, b)
\[ \alpha = \frac{d_w(R) + \bar{d}_s - \bar{d}_B}{d_w(R)} \]  
which generalizes equation (1) to the present case.

In the reflecting boundary condition case one finds
\[ \gamma_{HS} = 1 + \frac{(d_w^{(0)} - \bar{d}_B)}{2}, \]  
confirming a naive scaling argument according to which the sum of \( \gamma_{HS} \)'s corresponding to the reflecting and absorbing boundary conditions is equal to the fractal dimension of the boundary (see next section).

At least we want to check our result in the three dimensional Sierpinski gasket. This time the surface is fractal and it is the two dimensional Sierpinski gasket of the previous example, i.e. \( \bar{d}_s = \frac{\ln 3}{\ln 2} \). Rather tedious calculations follow the same procedure as the two dimensional case.

Instead of (42) we have (see also Ref. [7]).

\[ d_w(R) = \text{Max} \left( \ln \left( \frac{3}{2} R \right) / \ln 2, \ln 6 / \ln 2 \equiv d_w^{(0)} \right) \]  
and the « dynamical phase transition » occurs at \( R_c = 4 \). Equation (47) is substituted by
\[ \alpha_s = \frac{d_w^{(0)} + \bar{d}_s - \bar{d}_B}{d_w(R)} \]
while equations (50) and (51) retain their form.

In this case it is found \( z^* = 13/2 \).

4. Scaling arguments and conclusions.

The exact asymptotic behaviours obtained in the previous sections together the results contained in references [4, 6], help us to formulate naive scaling arguments which might hold in general.

The singular part of surface free energy density of the Gaussian models, associated to the diffusion problems studied above, scales like
\[ f_s^{\text{sing}}(\omega, h, h_1) = \ell^{-\bar{d}_s} f_s^{\text{sing}}(\omega \ell^{d_w}, h \ell^{\gamma_H}, h_1 \ell^{\gamma_{HS}}) \]  
under a spatial rescaling \( \ell \). In all cases we have studied \( \langle \phi_x \phi_{x_0} \rangle = A_x \tilde{P}_{x_0, x}(\omega) \) where \( A_x \) does not depend on \( x \) for the barrier case (see, e.g., Sect. 2.2); for the wells case, \( A_x \) depends on \( x \) except when \( x \in \partial A \). From equation (53) we thus have that the Laplace transform of the average autocorrelation function behaves like
\[ \tilde{P}_{\partial A}(\omega) = \frac{1}{|\partial A|} \sum_{x, x_0 \in \partial A} \tilde{P}_{x_0, x}(\omega) \sim \frac{\partial^2 f_s^{\text{sing}}}{\partial h_1^2} \bigg|_{h = h_1 = 0} \sim \omega^{(d_s - 2 \gamma_{HS})/d_w}, \]
for small \( \omega \), implying the following large \( t \) behaviour

\[
P_{\delta A}(t) \sim t^{-x} \quad x = \frac{d_w + \bar{d}_s - 2 \gamma_{HS}}{d_w}. \tag{54b}
\]

\( \gamma_{HS} \) at variance with respect to \( \gamma_H \), will depend on the boundary conditions.

Given the asymptotic law (54b) a simple theorem (see e.g. [11]) states that the asymptotic behaviour of the first return probability, \( F_{\delta A} \), will be

\[
F_{\delta A}(t) \sim \begin{cases} 
  t^{-(2-x)} & x < 1 \\
  t^{-x} & x > 1.
\end{cases} \tag{55}
\]

It is crucial to observe that, by definition, \( F_{\delta A} \) cannot depend on the boundary conditions and thus the scaling laws in equation (55) must coincide for the absorbing and reflecting cases (we will distinguish the two cases with the superscripts \( a \) and \( r \), respectively).

This implies that

\[
P_{\delta A}^{(r)}(t) \sim t^{-x^{(r)}}, \tag{56a}
\]

\[
P_{\delta A}^{(a)} \sim t^{-x^{(a)}} = t^{-2-x^{(r)}}, \tag{56b}
\]

\[
\gamma^{(r)}_{HS} + \gamma^{(a)}_{HS} = \bar{d}_s. \tag{56c}
\]

Since \( x^{(r)} < x^{(a)} \), we have \( x^{(r)} < 1 \), \( x^{(a)} > 1 \) and \( P_{\delta A}^{(a)} \sim F_{\delta A} \).

For hierarchical distributions of symmetric transition rates (see e.g. subsect. 2.2) with a simple redefinition of the time scale one has that the particle moves at each time step. If after a \( t \)-step walk the particle visits (almost) uniformly a region of linear size \( \xi(t) \sim t^{1/d_w} \) then we should have

\[
P_{\delta A}^{(r)} \sim \xi(t)^{\bar{d}_B - \bar{d}_B} \tag{57a}
\]

implying

\[
x^{(r)} = \frac{\bar{d}_B - \bar{d}_s}{d_w}, \quad \gamma^{(r)}_{HS} = \bar{d}_s + \frac{d_w - \bar{d}_B}{2}, \tag{57b}
\]

\[
x^{(a)} = 2 - x^{(r)}, \quad \gamma^{(a)}_{HS} = \frac{\bar{d}_B - d_w}{2}, \tag{57c}
\]

i.e. the same behaviour we would get if \( \delta A \) were a (fractal) subset in the bulk.

The exponent \( \gamma_H \) can be determined by observing that the zero field bulk susceptibility must behave like

\[
\chi = \frac{\partial^2 f_B}{\partial h^2} \bigg|_{h=0} = A \sum_x P_{x_0, x}(\omega) \sim \omega^{-1} \tag{58}
\]

due to probability conservation (\( f_B \) is the bulk free energy). Proceeding as we did for equation (54a) we get

\[
\gamma_H = \frac{d_w + \bar{d}_B}{2}. \tag{59}
\]
Now it is easy to see that equation (28a) holds with

\[ \alpha = \alpha_s = 1 - \frac{y_H + y_{HS}^{(a)} - \bar{d}_s}{d_w} = \frac{d_w + \bar{d}_s - \bar{d}_B}{d_w} = 1 - x^{(r)} \]  

(60)
to be compared with equation (2).

The case of hierarchical distribution of waiting times (equal probability to jump from a site to any of its neighbours like in Subsect. 2c and Sect. 3) was already considered in reference [4] for diffusion on a regular d-dimensional lattice. The arguments given there are easily generalized to a fractal bulk. Indeed if renormalization transformations are analytic in a neighbour of \( \omega = 0 \) then the exponents \( y_H \) and \( y_{HS} \) are the same as in the case without waiting time. In the example of section 3 if \( \omega = 0 \) the Hamiltonian (48) becomes independent of \( \{q_x\} \) (see Eq. (37b)) and thus the recursions for the bulk and surface field \( h \) and \( h_1 \) are the same as in absence of waiting times!

Thus using the results of the previous case we get

\[ y_{HS}^{(a)} = \bar{d}_s + \frac{d_w^{(0)} - \bar{d}_B}{2} \]  

(61a)

\[ y_{HS} = \frac{\bar{d}_B - d_w^{(0)}}{2} \]  

(61b)

\[ y_H = \frac{d_w^{(0)} + \bar{d}_B}{2} \]  

(61c)

where \( d_w^{(0)} \) is the diffusion exponent in absence of waiting times. Proceeding as in reference [4] we easily get the decay exponents

\[ \alpha = \frac{d_w + \bar{d}_s - \bar{d}_B}{d_w} \]  

(62a)

\[ \alpha_s = \frac{d_w^{(0)} + \bar{d}_s - \bar{d}_B}{d_w} \]  

(62b)

for current and survival probability, respectively.

In the examples considered here the waiting times on the boundary do not depend on the site and equation (54b) holds implying

\[ x^{(r)} = 1 - \alpha_s, \quad x^{(a)} = 1 + \alpha_s. \]  

(63)

All the exponents found in the previous sections by an exact R.G. analysis agree with the corresponding ones heuristically derived here. In particular we wish to stress the independence of \( y_{HS}^{(a)} \) from the surface fractal dimension as predicted already in reference [4] and the relations (60) and (63) relating the asymptotic behaviours of the average autocorrelation function and of the survival probability.

We should mention that recently adsorption of self-avoiding walks have been studied in reference [13] on the same Sierpinski gasket used here.

It would be extremely interesting to study the case of the ideal chain adsorption, i.e. the problem of reference [13] without the self-avoidance constraint, on self-similar structures with non-uniform coordination. This problem is identical to the one studied here when the
coordination is uniform, and in general can present rather unexpected surprises, at least on deterministic self-similar structures [14].

Acknowledgment.

We wish to thank Attilio Stella for discussions and for a critical reading of the manuscript. We thank also the referee whose criticism stimulated the scaling arguments of section 4.

References