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Modulated spiral phases in doped quantum antiferromagnets

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Résumé. — Nous étudions la phase spirale découverte par Shraiman et Siggia, dans le régime où sa constante de raideur devient négative. Nous montrons que cette instabilité ne signale pas une séparation de phase mais plutôt la formation de murs suivant les directions (1, 1) ou (1, T) sur un réseau carré, où les trous se localisent. Nous discutons brièvement l'effet d'une répulsion coulombienne à longue portée dans une telle structure.

Abstract. — We study the spiral phase discovered by Shraiman and Siggia, in the regime of coupling constant where the effective spin wave stiffness is negative. We show that this instability does not signal a phase separation but rather the formation of domain walls in the (1, 1) or (1, T) directions on a square lattice, where holes get trapped. We discuss briefly the interplay between dipolar and Coulomb interactions in the resulting structure.

1. Introduction.

Recently, Shraiman and Siggia [1] have written within the so-called t - J model a semiphenomenological Hamiltonian describing the motion of holes in a locally Néel ordered antiferromagnet. Their major finding for a single hole was that for wave vectors around the energy minimum (at $k = (\pm \pi/2, \pm \pi/2)$, at least for not too large values of $t/J$), the vacancy is dressed by a twist of the staggered magnetization taking a dipolar configuration at large scales. This effect comes physically from the coupling through hopping of the hole momentum to the spin current carried by the magnetic medium around. The approach followed in [1] (hereafter referred as I) was essentially of a semiclassical nature and applies a priori to the limit $t/JS \ll 1$. Although the picture found in I is less obvious in the large $t$ limit, where we know that the hole disorders very strongly the spins within a core region [2], it seems to be confirmed qualitatively in recent numerical studies [3]. One may therefore think that the Hamiltonian derived on I is still of physical relevance in this limit although with some renormalized coupling constants.

In a subsequent work [4] (hereafter referred as II), Shraiman and Siggia considered the situation of a small but finite density of holes. Thanks to the semiclassical analysis of I, one...
can very easily understand the nature of the interaction between holes at large scales. Indeed the long range tails of the static spin clouds around each hole mediate a dipolar-like interaction between them. It appears that beyond a critical value \( g_c \) of the coupling of the holes to spin waves, holes prefer to have their dipole moments aligned. This results in the formation of a « spiral » phase, characterized by a uniform planar rotation of the staggered magnetization with an incommensurability wave number scaling like the density of holes.

However, it was pointed out in II that if \( g > \sqrt{2} g_c \), the spin wave stiffness as renormalized by particle hole fluctuations becomes negative in the same way as it does in an homogeneous Néel phase for \( g > g_c \). It is the main purpose of this paper to elucidate the nature of the phase which should appear above this second threshold. We find that this second instability corresponds to the formation of domain walls in the spin orientation, along the \((1, 1)\) or \((1, \bar{1})\) directions on a square lattice, where holes get self-consistently trapped. Once they are formed, it is energetically favourable for these domain walls to accomodate as many holes as they can. A proper description of the final structure consisting of sharply localized soliton lines would require more knowledge on the short range physics in a doped antiferromagnet. This matter is clearly beyond the scope of our long wave length approach which nevertheless is able to prove the formation of a « modulated » spiral phase right above \( g = \sqrt{2} g_c \). Since there is an accumulation of charges on these domain walls, it is worth investigating how their formation is affected by a long range Coulomb repulsion. If the associated Bohr radius \( a_0 \) is of the order of the lattice spacing \( a \), then the spiral phase is restabilized by Coulomb effects which are however likely to promote in this case the formation of a Wigner crystal at low density. If on the contrary \( a_0 \) is much greater than \( a \) (a somewhat unrealistic hypothesis) we find that domain walls could still appear provided that the density of holes is high enough.

The paper is organized as follows. In section 2 we recall some basic features of the Shraiman-Siggia theory and introduce a simple version of their Hamiltonian which contains all the instabilities we are interested in. In section 3, we calculate the energy of a single domain wall and show that it becomes negative at \( g = \sqrt{2} g_c \), independently of the density of holes within the wall. We argue that the interaction between domain walls is repulsive. Finally Coulomb long range effects are discussed in section 4.

2. Simple considerations on the spiral phase.

We first recall some basic features of the Shraiman-Siggia theory. Holes are represented by spinless fermions, spin degrees of freedom by Schwinger bosons or in the large \( S \) limit a unit vector \( \hat{\Omega} \) whose dynamics is controlled by a non-linear \( \sigma \) model. Local Néel order halves the Brillouin zone and it is convenient to introduce two components for the hole annihilation operator, \( \psi_A \) and \( \psi_B \) corresponding to the two sublattices. These two components form a spinor, hereafter noted \( \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \). One complementary important assumption of this work is that the energy minimum for a single hole is at \( k_1 = (+\pi/2, +\pi/2) \) or \( k_2 = (+\pi/2, -\pi/2) \). Although this fact is confirmed by various numerical works at values of \( t/J \) smaller than 4, finite size effects make it more difficult to check for larger \( t/J \) [3, 5]. As will be clear in the following, the physics to be discussed in this paper is very sensitive to the position of the energy minimum.

At low hole density \( (n < 1) \) the hole Fermi surface consists of two small pockets around \( k_1 \) and \( k_2 \), which define two valleys 1 and 2. Inside each valley it is convenient to separate rapid and slow components of the spinors \( \psi_i \) by writing \( \psi_i = e^{i k_i \cdot r} \chi_i(r) \) where \( \chi_i \) embodies
the long wave length components of $\psi_i$. It is straightforward to take the continuum limit of the Shraiman-Siggia Hamiltonian for the fields $\chi_i$. It reads

$$\mathcal{K} = + \sum_{i=1,2} \left\{ - \chi_i^+ \frac{1}{2} \mu_{ij} \partial^2 \chi_i + g a \chi_i^+ p_i \cdot (\sin \theta \nabla \varphi \sigma_x - \nabla \theta \sigma_y) \chi_i \right\} +$$

$$+ \frac{1}{2} \left\{ \rho \left[ \sin \theta \nabla \varphi \right]^2 + (\nabla \theta)^2 \right\} + \chi^{-1} m^2 \right\}.$$ (1)

In this equation the last term is purely magnetic: it is the non-linear $\sigma$ model governing the spin fluctuations at large scales. $\theta$ and $\varphi$ are the Euler angles parameterising the local direction of the staggered order parameter $\hat{\Omega}$. $m$ is the local magnetization operator which is conjugate to $\theta$. $\chi \sim J^{-1}$ is the magnetic susceptibility and $\rho \sim JS^2$ the spin wave stiffness.

The first two terms take their origin from the hopping processes. The first one gives the band structure of the holes within each valley. It builds up predominantly from the hopping of holes to high energy virtual spin fluctuations. The band structure is known to be very anisotropic and we have to specify here our choice of spatial coordinates. We take the $(1,1)$ and $(1,\bar{1})$ directions as principal axes i.e. we define $x = \frac{x + y}{\sqrt{2}}$ and $z = \frac{-x + y}{\sqrt{2}}$. In such coordinates the effective inverse mass tensor is diagonal and

$$\mu_{11} = \mu_{22} = \mu_\perp \ll \mu_{12} = \mu_{21} = \mu_\parallel.$$ Both perpendicular and parallel masses scale in the same way with $(t/J)$: $\mu^{-1}$ goes like $t^2/j^2$ at small $t$ and becomes of order $Ja^2$ at large $t$ [6]. It is worth noting that this first term represents hopping events where the hole remains on the same sublattice and as such should be affected by Aharonov-Bohm effects discussed by many authors [7], which originate from the complex overlap between neighbouring spins. However, the phases we are going to discuss do not involve the creation of any fictitious flux because they have no skyrmions, in contrast for instance to the double spiral phases discovered in [8]. Therefore for our present purpose, we can forget about this complication.

The second term is the central part of Shraiman-Siggia’s theory since it deals with the problem of coherent hopping from one sublattice to the other, as is made clear formally by the presence of the $\sigma_x$ or $\sigma_y$ matrices: the unit vectors $\hat{p}_i$ are directed along the momenta $k_i$ within each valley to leading order in $n$. As said in the introduction these vectors appear like dipole moments which couple to the « electric fields » $\sin \theta \nabla \varphi$ and $\nabla \theta$. In the small $t$ limit, this term is obtained by writing the hopping part of the hamiltonian in terms of Schwinger bosons and spinless holes operators and expanding the resulting expression to lowest order in gradients. One gets $g = 2 \sqrt{2} St$ (note that our definition of $g$ differs by a factor $\sqrt{2}$ from the one used in I). It was argued in I that in the large $t$ limit $g$ should saturate to some value of order $J$ which seems unfortunately very difficult to calculate microscopically. Note that in (1) we discarded the coupling of the hole density to the local magnetization present also in I because it is higher order in $n$ for wave vectors concentrated near $k_1$ or $k_2$.

Equation (1) constitutes the starting point of our analysis. We can simplify things further by rotating the spinors like $\chi = e^{-i(\pi/4)\sigma_y} \tilde{\chi}$ (as was also done in II) in such a way that the $x$ direction in the spinorial space is brought onto $z$. The first and third terms are left unaffected in such an operation, while the second transforms into (leaving aside the twiddles)

$$\sum_{i=1,2} g a \chi_i^+ (\sin \theta \partial_z \sigma_z - \partial_z \theta \sigma_y) \chi_i.$$ (2)
It is observed that the first term in (2) is diagonal in spin space in this new representation. It is now easy to understand the nature of the spiral phase. We imagine a phase where the spins stay in a plane which can be conveniently chosen to correspond to $\theta = \pi/2$. Each valley splits into two subbands with an effective spin index $\uparrow$ or $\downarrow$ and the conditions for reaching a stationary energy minimum are given by

$$\langle \partial_\uparrow \varphi \rangle = -\frac{g_d}{\rho} (n_{i \uparrow} - n_{i \downarrow}).$$

The spiral phase corresponds to the appearance of a finite but uniform value of $\partial_\uparrow \varphi$ or $\partial_\downarrow \varphi$.

It is simple exercise to calculate the energy of a polarized phase ($n_{i \uparrow} \neq n_{i \downarrow}$). One obtains for each valley and per unit area

$$\frac{1}{2} (n_{i \uparrow} + n_{i \downarrow})^2 \frac{\pi}{(\mu_\perp \mu_\parallel)^{1/2}} + \frac{1}{2} (n_{i \uparrow} - n_{i \downarrow})^2 \left[ \frac{\pi}{(\mu_\perp \mu_\parallel)^{1/2}} - \frac{g^2 a^2}{\rho} \right].$$

Thus, provided that $g > g_c = \left[ \frac{\pi \rho}{(\mu_\perp \mu_\parallel)^{1/2} a^2} \right]^{1/2}$, it is advantageous to fully polarize each valley i.e. $n_i = n_i \uparrow, n_i \downarrow = 0$. This indeed maximises $|n_{i \uparrow} - n_{i \downarrow}|$ at a given value of the density $n_i$ carried by each valley. The energy of a fully polarized valley is

$$\frac{1}{2} n_i^2 \left[ \frac{2 \pi}{(\mu_\perp \mu_\parallel)^{1/2}} - \frac{g^2 a^2}{\rho} \right] = \frac{\pi n_i^2}{(\mu_\perp \mu_\parallel)^{1/2}} [1 - \alpha]$$

where $\alpha = \frac{1}{2} \frac{g^2}{g_c^2}$. If $\frac{1}{2} < \alpha < 1$, the coefficient of $n_i^2$ in equation (5) is positive. This means in particular that at a given density $n = n_1 + n_2$, it is energetically favourable to have $n_1 = n_2$, i.e. an equal distribution of the holes between the two valleys. Turning back to equation (3) we see that this result implies $|\partial_\uparrow \varphi| = |\partial_\downarrow \varphi|$. In other words, the uniform twist of the staggered magnetization characteristic of the spiral phase has to be directed along the $(1, 0)$ or $(0, 1)$ directions, as obtained in II. On the other hand, when $\alpha > 1$ the compressibility of each valley is negative and the spiral phase cannot be thermodynamically stable.

A more elaborate calculation was given in II where the renormalization of the spin wave propagator by particle-hole fluctuations was considered. One finds the renormalized static stiffness $\tilde{\rho}(q) = \rho - g^2 a^2 \chi_d(q, 0)$ where $\chi_d(q, 0)$ is the static susceptibility of 2D-free electrons. To leading order in $n$, a given valley contributes only to the renormalization of the $\sin \theta \partial_\uparrow \varphi$ or $\partial_\theta \varphi$ terms where $i$ is the index of the valley. At zero wave vector $\chi_d = \frac{\pi}{(\mu_\perp \mu_\parallel)^{1/2}}$ and one recovers the same criterion for the instability of the Néel phase. It is interesting to repeat the same calculation in the spiral phase. The spin fluctuations in the plane of the spiral, as described by $\nabla \varphi$, do not mix the bands $\uparrow$ and $\downarrow$ according to (2) and they are renormalized as in the Néel state except that $\chi_d$ has to be divided by two because we are now dealing with a fully polarized state. Therefore these fluctuations acquire a negative static stiffness for $g > \sqrt{2} g_c$. In contrast, the spin fluctuations out of the plane of the spiral induce interband transitions and they will be renormalized by the hole’s bubble-diagrams in a different way. A straightforward calculation gives

$$\tilde{\rho}_\theta(q) = \rho - 2 g^2 a^2 \sum_k \frac{n_{k \downarrow} - n_{k \uparrow}}{E_{k \uparrow} - E_{k \downarrow}}.$$
As before, only the valley carrying the index \( i \) contributes to the renormalization of the propagator of \( \beta_i \). By definition, the gap between the two subbands of a same valley in the spiral phase is \( 2 \Delta_i = 2 g a |\beta_i \phi| = + \frac{2 g^2 a^2}{\rho} n_i \). Carrying back this result in (6), one sees that \( \tilde{\rho}_\theta \) vanishes at zero wave vector, regardless of the value of the coupling constant \( g \). In fact, it can be shown that \( \tilde{\rho}_\theta(q) \) as defined above is a positive quantity for all values of \( q \), behaving like \( q^2 \) at small wave vectors. Since the same property should hold at finite frequency, we conclude that the transverse fluctuations of the order parameter \( \tilde{\Omega} \) are completely stable in the presence of a spiral distortion of \( \Omega \), even for \( \alpha > 1 \).

These considerations lead us to concentrate our search for structures appearing above \( \alpha = 1 \) on purely planar phases i.e. involving only variations of an azimuthal angle \( \varphi \) at a fixed value of \( \theta \). To motivate the next section, let us point out that the \( t-J \) model in the small \( t \) limit does indeed yield a coupling constant \( g \) which exceeds \( \sqrt{2} g_c \). In this limit all quantities can be calculated exactly. We already quoted in the text \( \rho = JS^2 \) and \( g = 2 \sqrt{2} t S \). The effective masses \( \mu_\perp \) and \( \mu_\parallel \) can be obtained from standard second order perturbation theory in \( t/J \) [9] or numerically from exact diagonalization methods on small clusters [3]. Both approaches agree to give \( \mu_\perp \sim 1.6 J t^{-2} a^{-2} \), \( \mu_\parallel \sim 4.2 J t^{-2} a^{-2} \). Combining these expressions, we find \( \alpha \sim 3.3 \). Even if the \( t-J \) model is not very physical in the small \( t \)-limit, this remark means that a numerical simulation of this problem for a sufficient large size of the lattice would not give a spiral phase but a different structure which we would like now to understand (see however the end of section 3 for a brief discussion of other possible instabilities in the \( t-J \) model). In the large \( t \) limit, \( \alpha \) must take on general grounds a value of order 1 independent of \( t/J \) but we cannot say whether it is bigger than 1 (or bigger than 1/2 corresponding to the first instability).

3. Modulated spiral phases.

From the discussion of the preceding section, it should be clear that in order to restabilize the spin waves one must find a way to lower the hole susceptibility. This can be achieved by opening gaps in the hole dispersion relations, or in other words creating localized holes states. We thus shall consider inhomogeneous distributions of the gradients of \( \varphi \) by choosing \( \varphi(x, y) \) of the form \( F(x_1) + F(x_2) \) we decouple the two valleys (to the order of approximation of Hamiltonian (1)). Since we want the two valleys still to be fully polarized, we take \( F \) to be a continuously increasing function of \( x_1 \) or \( x_2 \). According to (1), in that case, only the lower subbands will be populated at zero temperature within each valley. The spiral phase can be described by \( F(x) = Qx \). For the valley 1, we have to find the eigenstates of the Hamiltonian

\[
-\frac{1}{2 \mu_\perp} x_\perp^+ \delta_{x_\perp}^2 x_\perp - \frac{1}{2 \mu_\parallel} x_\parallel^+ \delta_{x_\parallel}^2 x_\parallel - g a x_\parallel^+ x_\parallel (\partial_{x_\parallel} F)
\]

with the condition that \( \partial_{x_\parallel} F = \frac{g a}{\rho} \langle x_\parallel^+ x_\parallel \rangle \) where the average is taken over all occupied states (of negative energy).

We first study the case of one single localized structure or domain wall across the \( x_1 \) direction, uniform along \( x_2 \) and carrying an arbitrary number of holes \( N_h \). This number divided by the total length in the \( x_2 \) direction defines a linear density of hole within the defect which we call \( n_\parallel \). The hole eigenstates are still plane waves in the \( x_2 \) direction but we assume that they share the same wave function perpendicular to the domain wall. According to equation (7) and the self-consistency condition, this wave function should be the lowest
energy solution normalized to unity of the stationary non-linear Schrödinger equation

\[-\frac{1}{2 \mu_\perp} \partial_x^2 \psi - \frac{g^2 a^2}{\rho} n_I |\psi|^2 \psi = E \psi.\]  

(8)

Rescaling the wave function and the coordinate as \(\psi(x_1) = \frac{1}{\sqrt{\ell}} \tilde{\psi}(x = x_1/\ell)\) with the length \(\ell\) given by \(\ell = n_I^{-1} \left[ \frac{1}{8} \frac{\rho}{g^2 \mu_\perp} a^2 \right]\), we obtain the following equation for \(\tilde{\psi}\) (with the condition that \(\tilde{\psi}\) be of unit norm)

\[-\partial_x^2 \tilde{\psi} - \frac{1}{4} |\tilde{\psi}|^2 \tilde{\psi} = \tilde{E} \tilde{\psi} \quad E = \frac{\tilde{E}}{2 \mu_\perp \ell^2}.\]  

(9)

The solution of this problem is well known: to obtain it simply, we identify \(u = 1/\sqrt{|\tilde{E}|} \tilde{\psi}\) and \(\tau = \sqrt{|\tilde{E}|} X\) as respectively a position and a time and treat equation (9) as the classical motion of zero energy of a unit mass point in the potential \(V(u) = u^4 - \frac{u^2}{2}\). In this way we get the localized solution \(u(\tau) = \frac{1}{\sqrt{2} \cosh \tau}\), whence follows the value of the energy \(\tilde{E} = - \left[ \int_{-\infty}^{\infty} u^2 d\tau \right]^{-2} = -1\). However, to avoid counting twice the interaction energy, we have to add to this result \(\frac{1}{2} \int_{-\infty}^{\infty} |\tilde{\psi}|^4 dx = \frac{2}{3} (\tilde{E})^{3/2} = \frac{2}{3}\) so that the global energy per hole coming from the motion perpendicular to the soliton line is \(-1/3\) in reduced units. Reexpressing this result in real energy units and combining it with the kinetic energy in the \(x_2\) direction, one finds the total energy of this structure, divided by the number of holes to be given by

\[E_w = \frac{n_I^2}{6} \left[ \frac{\pi^2}{\mu_\perp} - \frac{1}{4} (g^2/\rho)^2 \mu_\perp a^4 \right] = \frac{n_I^2 \pi^2}{6 \mu_\perp} [1 - \alpha^2].\]  

(10)

Thus \(E_w\) becomes negative at exactly the same threshold as the spin wave stiffness.

A periodic array of such soliton lines with an inverse spacing \(n_\perp = n/n_I\) is characterized by an energy per unit area \(nE_w\), if we assume that the solitons are so widely spaced that they negligibly interact. A comparison of \(nE_w\) to the energy of the spiral phase (Eq. (5)) shows at once that the modulated phase is of lower energy, as soon as \(\alpha > 1\) and \(n_\perp > n\left(\frac{\mu_\parallel}{\mu_\perp}\right)^{1/2}\). It is also clear that the lowest energy state among these modulated phases is reached when \(n_I\) takes its maximal value \(1/a\) corresponding to one hole per site along the soliton lines. Note that even if we were assuming that the Hamiltonian (1) is correct down to the scale of the lattice spacing, there should be corrections to (10) when \(\ell\) becomes comparable to \(a\). In particular we do not expect \(E_w\) to scale like \(a^2\) in the large \(\alpha\) limit but rather like \(\alpha\). Evidently, we cannot trust our initial Hamiltonian well before reaching such short length scales. But the nature of the instability is undoubtedly correctly understood by our purely large scale approach since all the calculations are consistent for \(na^2, n_I a \ll 1\) and \(\alpha \sim 1\). Up to now, we restricted our discussion to one valley but it applies without difficulty to the case of two valleys. The second valley will develop similar solitons in the perpendicular direction. The
overall physical picture is a periodic pattern of square cells along the diagonal directions of the square lattice, with an area scaling like $n^{-2}$ and an almost uniform orientation of the Néel order parameter within each cell. By each crossing of a soliton marking the boundary between cells, the angle $\varphi$ of the spins in the plane $x, y$ rotates by a finite amount $\frac{g}{\rho} an_1$. This quantity becomes of order 1 when $n_1 \sim 1/a$.

The tendency of holes to form lines can be simply understood from the following classical argument: we have seen in section 2 that each hole can be assigned a dipole moment $\vec{p}_i$, depending on the valley it occupies. Spin waves mediate between two holes a static interaction which takes the form of the 2D electrostatic dipolar interaction with a minus sign i.e. $-\frac{1}{2\pi} \frac{g^2}{\rho} \frac{|\vec{p}_i \cdot \vec{p}_j - 2(\vec{p}_i \cdot \vec{r}_{ij})(\vec{p}_j \cdot \vec{r}_{ij})|}{r^2}$. The change of sign comes from the fact that in the energy enters, besides the magnetic term which corresponds to the usual electrostatic energy, the interaction term between the dipole moments of the holes and the spin current. From the above expression, it is clear that two holes carrying the same dipole moment $\vec{p}$ attract when their relative position $r_{ij}$ is perpendicular to $\vec{p}$ and repel when they lie parallel to $\vec{p}$. It follows that holes belonging to the same valley gain energy by aligning themselves perpendicularly to $\vec{p}$.

We wish now to discuss the question of the interaction between soliton lines. The formation of walls screens completely the dipolar interaction on a classical level (in electrostatics, the electric field outside an infinite capacitor vanishes). This is no more true on a quantum mechanical level where we expect a weak repulsion due to the small overlap between neighbouring self-trapped states. If we trust Hamiltonian (1) down to the lowest length scales, then the problem of finding one-dimensional stationary periodic solutions of equation (7) is equivalent to the Peierls-Fröhlich problem, for which exact solutions are known [10, 11]. In the small density limit ($n \ll a^{-2}$), the energy of an individual soliton inside a periodic array of self-trapped states, is given by [11]

$$E = E_w(1 - 6 \exp - 2L/\ell)$$  \hspace{1cm} (11)

where $L$ is the distance between successive solitons, which scales like $(an)^{-1}$ and $\ell$ is their width (of order $a$). This result shows that the domain wall structure has an exponentially weak but positive compressibility. On the other hand, distortions of each individual line are controlled by energies of order $J(n_1 a)^2 \sim J$ for $n_1 a \sim 1$. Therefore the modulated spiral phase should be seen as a gas of weakly interacting almost rigid soliton lines. We shall not expand further on the physics of this phase since we lack a microscopic understanding of its short scale features. It is worth pointing out that such charged solitons have already been found in the Hubbard model from different approaches. In the weak coupling limit, Schulz [12] found incommensurate antiferromagnetic phases close to half filling: in that case solitons form in the $(1,0)$ or $(0,1)$ directions. In the large $U$ limit, motivated by a previous work on the two-band Hubbard model [13], Poilblanc and Rice [14] also found numerically soliton like solutions within Hartree-Fock theory, this time along the diagonal directions like in the present work. In both approaches, soliton lines separate domains of opposite direction of the Néel order parameter. In the present theory, the jump $\Delta \varphi$ in the orientation of the spins is $a$ priori not quantized. However, it is conceivable that in the limiting case $n_1 a \sim 1$, our soliton lines become in fact equivalent to the previous ones or in other words that they correspond to a jump $\Delta \varphi = \pi$.

Besides the instability towards domain wall formation, the long range dipolar interaction may also pair holes together. The coexistence of spiral order and superconductivity has been explored recently [15] in a mean field theory valid in the weak coupling limit $(a \ll 1)$ when
the pairs strongly overlap. In the moderate coupling limit ($\alpha \sim 1$) and small density of concern to us in this paper, it is more adequate to look directly at the pairing between two isolated holes. In the case of an isotropic mass tensor, the problem can be solved analytically since it becomes separable in the variables $\log r$ and $\theta$ (where $r$, $\theta$ are the relative polar coordinates of the two holes). For holes in the same valley and with the same spin index, the orbital wave function must be odd and we find a finite threshold for the formation of a bound state $\alpha_c \sim \frac{3}{\sqrt{2}} > 1$. On the contrary for holes belonging to different valleys or carrying a different spin index, the orbital wave function can be an $s$-wave state (with an admixture of higher even angular momentum components) and there is no threshold anymore. The dipolar interaction by itself cannot form bound states but only « collapsing » states which continuously decrease their energy by shrinking their size to zero. This is because the dipolar interaction scales exactly like the kinetic energy in $1/r^2$. To regularize the problem one has to restore a short range cut-off and the energy of the bound states will be very dependent on the physics at short scales. So, in view of all these uncertainties it is very difficult to make a general statement on the relative stability of a superconducting phase or an insulating one at $\alpha > 1$. We leave this question for further work.

4. Discussion of long range Coulomb effects.

In the previous section we have identified the nature of the instability which affects the spiral phase at $\alpha \gg 1$. The troublesome feature of our result is that it unavoidably leads to short scale structures with highly non-uniform distribution of charges. In this section we would like to see how a long-range Coulomb repulsion can fight this tendency and whether it restabilizes the spiral phase. We write the Coulomb interaction like $\frac{e^2}{2\pi r}$ and define $J_c = \frac{e^2}{2\pi a}$. The dipolar interaction between holes takes the form $-J_d a^2 \left( \hat{p}_i \cdot \hat{p}_j - 2 (\hat{r}_i \cdot \hat{r}_j)(\hat{p}_j \cdot \hat{r}_i) \right)$ with $J_d = \frac{g^2}{2\pi \rho}$ of the same order as $\frac{1}{\mu a^2}$. The length $a_0 = a (J_d/J_c)$ determines the scale beyond which the Coulomb repulsion dominates both dipolar interaction and kinetic energy which scale in the same way and are of the same order of magnitude. Within the present approach, the most plausible assumption is $J_c > J_d$ since $J_d$ is typically of order $J$ whereas $J_c$ can easily be some sizeable fraction of $U$. If consequently $a_0$ is of the order of a or less, then a Wigner crystal will form at low density and the question we ask loses its sense. We shall rather assume that $J_d \gg J_c$ and that a spiral phase has formed.

Let us reconsider the renormalization of the spin wave stiffness. We take the twist in the $(1, 0)$ direction and look first at the spin fluctuations in the plane. The field $\delta_y \phi$ couples to the holes like the difference of density between the two valleys (see Eq. (3)). Therefore its inverse propagator is not affected by Coulomb effects and remains therefore negative at $\alpha > 1$. On the contrary, $\delta_x \phi$ couples to the density and may be screened. Performing the usual RPA calculation, one finds for its inverse propagator

$$D_{xx}^{-1}(q, \omega) = \left[ \rho - g^2 a^2 \frac{\chi_d(q, \omega)}{1 + 2 \chi_d(q, \omega) \frac{e^2}{|q|}} \right].$$

(12)

To be complete, we also analyzed the transverse spin fluctuations. Since they are renormalized by particle-hole bubbles carrying two different spin indices, they are not
screened in the long wavelength limit by the Coulomb interaction which conserves the spin indices (they are corrected only by ladder diagrams). However, we have argued in section 2 that these transverse fluctuations are stable in the spiral phase. So they do not raise any trouble.

The instability with respect to fluctuations of $\partial_y \phi$ can be cured by emptying completely one of the two valleys. This also amounts to rotate the twist of the spiral from the $(1, 0)$ to the $(1, 1)$ direction for instance, as previously noted in II. Then, only the $\partial_x \phi$ component still couples to the holes and it is strictly equivalent to a density fluctuation: so its stiffness is given by formula (12) with a factor 2 missing in the denominator. If $n^{-1/2} \gg a_0 \gg a$, then for $q \leq k_F$, $D_{x_1 x_1}^{-1}(q, 0)$ simplifies into $\rho [1 - |q| a_0]$, a clearly positive quantity. Under these conditions the spiral phase is stable unless the density is so low that a Wigner crystal forms. If on the contrary $a_0 \gg n^{-1/2} \gg a$, we see from equation (12) that spin waves are stable at scales larger than $a_0$ but unstable at intermediate scales between $a_0$ and $n^{-1/2}$. In that case, there is place for the formation of domain walls.

Although it is not clear to us which kind of structure realizes the best compromise between the Coulomb repulsion and the dipolar interaction we adopt the premise that a regular array of soliton lines can still form. Our arguments in favour of this choice are two-fold. First we have estimated the total cost in Coulomb energy accompanying the accumulation of holes along lines and have found that it is generally lower than the gain in dipolar energy. For this purpose, we borrow from [16] the expression of the classical interaction energy between holes occupying sites of a 2-D Bravais Crystal

$$E_1 = -\frac{2 e^2}{2 \pi (a_c)^{1/2}} \left( 2 - \sum_{l \neq 0} \varphi_{-1/2} \left( \frac{\pi}{a_c} x^2(I) \right) \right) \quad (13)$$

where $\varphi_{-1/2}(z) = \int_1^{+\infty} dt t^{-1/2} e^{-zt} \pi a_c$ is the area of an elementary cell. The summation is to be carried over all lattice sites $x(I)$ different from the origin. Viewing the domain wall structure as a rectangular Bravais lattice with a spacing $(na)^{-1}$ in one direction and $a$ in the other, one easily gets in this strongly anisotropic situation the singular contribution $E_1 \sim -e^2 \log na^2$. The Coulomb energy cost per hole is equal to $E_1/2$. On the other hand, the gain in dipolar energy per hole is of the form $\beta J_d$ (with $\beta = \pi^2/6$) in a strictly classical model or generally a constant of order 1. If the hole density satisfies $na^2 \gg \exp -\beta a_0/a$ (a less stringent condition in fact than $\sqrt{n} a_0 \gg 1$ for values of $\beta$ around 1), it is energetically favourable to form a modulated spiral phase.

Within the same simple-minded classical model, we can also study the local stability of the modulated phase and this leads to exactly the same conclusions. We looked in particular at the transversal phonon modes of such an array of dipole lines for wave vectors parallel to the chains. One single chain is found to destabilize against transversal distortions at scales of the order $a \exp \beta a_0/a = \ell_0$. Physically, beyond this characteristic length, the cost in Coulomb energy to add a supplementary hole to a preexisting segment of $\ell_0/a$ holes overcomes the gain in dipolar energy. However, for a regular array of lines, the singular Coulomb contribution $\sim J_c(qa)^2 \log qa$ is cut-off at wave vectors $q \ll na$ like $J_c(qa)^2 \log na^2$. The dipolar interaction provides a restoring energy $\sim \beta J_d(qa)^2$ and one recovers the stability of the 2-D structure under the same condition as before. So, in the interesting (but somewhat unlikely) limit $J_d \gg J_c$, we have established in a crude way but, we feel, qualitatively correct, that a
modulated spiral phase is a metastable state of lower energy than the homogeneous spiral phase, provided the hole density is not too low.

5. Conclusion.

In this work, we have shown the possibility of domain wall formation within Shraiman Siggia phenomenological Hamiltonian, for suitable strength of the effective dipolar interaction between holes. Clearly, it would be interesting to know the real value of $\alpha$ in the strong coupling limit $t \gg J$, to see whether this instability can occur. Also, quantum spin fluctuations around the various « mean field » solutions found up to now, remain to be worked out and may seriously modify our views.

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References