New aspects in the theory of the integer quantum Hall effect
J. Riess

To cite this version:
New aspects in the theory of the integer quantum Hall effect

J. Riess

Centre National de la Recherche Scientifique, Centre de Recherches sur les Très Basses Températures, B.P. 166X, 38042 Grenoble Cedex, France

(Reçu le 5 octobre 1989, accepté sous forme définitive le 3 janvier 1990)

Résumé. — A partir d'une approche de la théorie de l'effet Hall quantique entier (IQHE) proposée récemment, un modèle explicite de désordre faible est étudié. On montre, que la dynamique d'un électron conduit à des phases aléatoires dans le développement temporel de la fonction d'onde, ce qui permet de décrire l'évolution d'un électron en termes de probabilités associées à des chemins différents dans le plan de l'énergie-temps du processus de diffusion. Cette analyse conduit à une explication détaillée du mécanisme microscopique, qui est à l'origine de l'IQHE du système. Les principaux aspects nouveaux de notre approche théorique générale sont discutés et confrontés à des théories antérieures basées sur des modèles similaires.

Abstract. — In the framework of our recently proposed new theoretical approach to the integer quantum Hall effect (IQHE) an explicit weak disorder model is investigated. Here the dynamics of a single electron causes randomization of phases in the time dependent wave function of the electron, which permits the description of the time evolution of the electron in terms of probabilities associated with different paths in the energy-time plane of the scattering process. This leads to a detailed explanation of the microscopic mechanism, which is responsible for the formation of the Hall conductance plateaus in this system. The principal new aspects of our general theoretical approach are discussed and confronted to previous theories based on similar models.

1. Introduction.

It is generally accepted that the integer quantum Hall effect [1] (IQHE) originates from a localization-delocalization process due to a disorder potential in the two-dimensional electron system. In the last decade considerable theoretical effort has been devoted to this phenomenon and to transport properties in the presence of disorder and of a strong magnetic field [2].

Very recently we have developed a new point of view for investigating two-dimensional systems of charged particles in crossed electric and strong magnetic fields in the presence of a disorder potential [3, 4]. We considered the time dependence of the single-particle Schrödinger functions of a large class of 2-dimensional systems. It was found on general grounds that adiabatic states are dc-insulating and that the macroscopic Hall current is carried by the non-adiabatic states [4]. Hall conductance plateaus correspond to values of the magnetic field for...
which the Fermi energy is situated in a region associated with adiabatic states. The plateaus disappear completely when the Hall field (and hence the Hall current) has reached a sufficiently high value, where all states have become non-adiabatic. As an illustration we have presented an explicit model system [4] with a substrate potential composed of homogeneous disorder and of a sequence of smooth barriers and wells. Here the non-adiabatic states could be calculated explicitly in a weak disorder approximation whenever the Hall field is sufficiently low. In this case the quantum mechanical scattering process induced by disorder could be understood in detail. In particular it was shown that those of the non-adiabatic states, which are intermediate between fully adiabatic and fully non-adiabatic give rise to a special kind of (disorder induced) non-classical particle propagation and that this is the microscopic origin of the so-called compensating currents which lead to integer quantization of the Hall conductance in this system.

In the present paper we further analyze the nature of the scattering process for the model system considered in reference [4], which will be briefly reviewed in section 2. In section 3 we will show that the disorder induced scattering process leads to phase randomness, which causes additivity of the scattering probabilities originating from different paths in the energy-time plane of the scattering process. (In Ref. [4] this additivity has been assumed without proof.) In section 4 we discuss the basic features of the time dependent electronic wave functions, which are responsible for the IQHE in our systems. With the results of section 3 the microscopic process of the Hall plateau formation in the model system of section 2 becomes transparent and understandable in detail. In section 5 we stress the salient new aspects of our general theoretical framework. We critically examine previous theories of the IQHE, which are based on similar models.

2. Description of the model.

We consider a system of independent particles of charge \( q \) situated on a long strip in \( x \)-direction (with a width \( L_y \)) and subject to a static substrate potential \( V(x, y) \), a magnetic field \( B = (0, 0, B) \) and an electric field \( E = (0, E_y, 0) \). We consider potentials \( V(x, y) \) which do not describe a macroscopic electric field, i.e. their integrals over \( x \) and over \( y \) vanish. The one-particle Hamiltonian has the form:

\[
H = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 + \left[ \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{q}{c} \left( B x + \phi(t) / L_y \right) \right]^2 + V(x, y),
\]

(1)

where \( \phi(t) = -cE_y L_y t \). As in [4], we neglect edge effects and chose the boundary conditions

\[
\psi(x, y, t) = \psi(x, y + L_y, t).
\]

(2)

If \( \phi \) in (1) is considered as a parameter and if \( V(x, y) \) has no symmetry (e.g. in the presence of disorder), the eigenvalues of \( H \) are periodic with \( \phi \) with period \( \hbar c/q \) (cf. Refs. [3-5]). It further follows [3, 4] that in the adiabatic regime, i.e. where \( E_y \) is sufficiently small such that \( \phi(t) \) in \( H[\phi(t)] \) can be considered as a parameter, the single particle currents are periodic in time with period \( \tau = \hbar/(qE_y L_y) \). Since \( \tau \) is very small for realistic values of \( E_y L_y \) (e.g. \( \tau = 4 \times 10^{-12} \) s for \( E_y L_y = 1 \text{ mV} \)), this means that adiabatic states are dc-insulating and further, that any dc-current must result from non-adiabatic states (including states intermediate between fully adiabatic and fully non-adiabatic).

A quantitative illustration of these general properties of systems described by (1-2) is given by the explicit model system introduced in reference [4], where:

\[
V(x, y) = V(x) + V^1(x, y).
\]

(3)
Here $V_1(x, y)$ describes homogeneous disorder and $V(x)$ is a sequence of smooth barriers and wells (see Fig. 1) whose first and second spatial derivatives (piecewise) vary slowly with respect to the magnetic length $\lambda = (\hbar c/qB)^{1/2}$, which is of the order of 100 Å or less in high magnetic fields. In the absence of disorder $V_1(x, y)$, the adiabatic solutions of the time-dependent Schrödinger equation are then in very good approximation given by the Landau functions:

$$
\psi_{p,n}(x, y, t) = (L_y)^{-1/2} \exp(i2\pi py/L_y) \cdot u_{p,n}(x, t),
$$

where $u_{p,n}(x, t)$ is the product of a Hermite polynomial $\chi_{np}(x)$ and a Gaussian

$$
g_p(x, t) = \exp \left\{-\frac{[x-x_p(t)]^2}{2\lambda^2}\right\}
$$

with

$$
x_p(t) = c \hbar p / (qBL_y) - \phi(t) / (BL_y) - \frac{mc^2}{q^2 B^2} V'(x_p),
$$

where $V'(x)$ denotes $dV(x)/dx$. The energies are

$$
E_{p,n} = \hbar \omega (n + 1/2) + V(x_p) + \frac{mc^2}{qB} \left[\frac{cV'(x_p)}{qB}\right]^2
$$

($\omega = |qB|/(mc)$). In the following we consider a single Landau band and drop the band index $n$.

The adiabatic variation of $E_p$ is essentially given by the term $V[x_p(t)]$. Therefore, for states $\psi_p$, $\psi_p'$ with localization centres $x_p(t)$, $x_p'(t)$ on different sides of a barrier (or of a well) in

![Diagram showing the substrate potential $V(x)$ and the nature of single-particle states in the case where, in addition to $V(x)$ a weak disorder potential $V_1(x, y)$, an electric field $E_y$ and a strong perpendicular magnetic field $B$ are present. Under weak disorder conditions these states are characterized by the position of the localization centres $x_p$ of the corresponding Landau orbitals $\psi_p$ in the absence of $V_1(x, y)$: full line regions on the abscissa correspond to fully non-adiabatic (classically conducting) states, shaded regions to intermediate states, dashed regions to fully adiabatic (dc-insulating) states. The figure corresponds to the parameter values of reference [4] ($L_y = 0.01 \text{ cm}$, $B = 6 \text{ T}$, $E_y = 2.37 \times 10^{-7} \text{ V cm}^{-1}$). The unperturbed ($V_1 = 0$) energies $E_p$ are represented by $V(x)$ (see (7)). Outside the dashed regions these energies coincide with the perturbed energies (on the scale of the figure). $x_p$, $x_p'$ schematically show the localization centres of two unperturbed states $\psi_p$, $\psi_p'$ in a situation, where their energies $E_p$, $E_p'$ intersect.]}
In the case, where this perturbation is weak, i.e. where
\[ |V_{pp'}^1| = |\langle \psi_p | V^1 | \psi_{p'} \rangle| = f |E_p (\phi^* + hc/2 q) - E_{p'} (\phi^* + hc/2 q)| \quad \text{with} \quad f \leq 0.2 \quad (8) \]
the perturbed functions \( w_n(x, y, \phi) \) are in very good approximation expressed as a linear combination of the two unperturbed functions \( \psi_p, \psi_{p'} \) in the anticrossing interval

\[ \phi^* - \frac{hc}{4q} < \phi < \phi^* + \frac{hc}{4q}. \]

This is true for any anticrossing interval of length \( hc/2 q \) (with \( \psi_p, \psi_{p'} \) replaced by the two unperturbed functions associated with the considered interval). Further, in this weak disorder approximation a perturbed adiabatic function \( w_n(x, y, \phi(t)) \) is equal to an unperturbed function \( \psi_p(x, y, \phi(t)) \) whenever \( \phi(t) = \phi^* + (hc/4q) \times \text{odd integer} \), i.e. whenever \( \phi(t) \) is exactly halfway between two anticrossing values of \( \phi \).

As a consequence, the adiabatic evolution of each of the corresponding perturbed functions \( w_n[x, y, \phi(t)], w_{n+1}[x, y, \phi(t)], \ldots \) in figure 2 describes an elementary charge, which « oscillates » between the two sites \( x_p(\phi^*) \) and \( x_{p'}(\phi^*) \) situated on two different sides of the barrier (well) \[3, 4\]. This motion is periodic with a time period \( \tau = |h/(qE_y L_y)| \). Such a
displacement of charge is completely non-classical, since the charge disappears at
\( x = x_p(\phi^*) \) and simultaneously reappears at \( x = x_p(\phi^*) \) where it stays for a time
\( \Delta t = \tau/2 \) before it disappears and reappears at \( x = x_p(\phi^*) \) and so on (this corresponds to a
discontinuous motion of a mass point from \( x = x_p(\phi^*) \) to \( x = x_p(\phi^*) \). For more details, see
references [3, 4]. If \( \phi(t) \) varies slowly, i.e. if \( E_y \) is sufficiently small, all states are adiabatic
functions \( \psi_p[x, y, \phi(t)] \) (These states form a complete set in the Hilbert space).

If \( E_y \) increases some of the perturbed time dependent functions become non-adiabatic. This
means that at each anticrossing of two energies complete Zener tunnelling [7] between
adjacent adiabatic levels occurs, and the wave functions essentially behave as unperturbed
functions \( \psi_p[x, y, \phi(t)] \), i.e. with localization centres moving with the constant velocity
\( v = cE_y/B, \) and with energy expectation values following the unperturbed curves of figure 2.

For a constant electric field \( E_y \) the Zener tunneling probability \( P \) decreases exponentially
with increasing \( |V_{pp}|^2 \), which itself increases exponentially with decreasing distance
\( (x_p - x_{p'}) \) of the two unperturbed orbitals \( \psi_p(t) \) and \( \psi_{p'}(t) \), of which the perturbed adiabatic
orbital \( \psi_n[x, y, \phi(t)] \) is composed in the neighbourhood of the intersection of the two
unperturbed levels \( E_p[\phi(t)] \) and \( E_{p'}[\phi(t)] \) (see Refs. [3, 4]). As a consequence each energy
band (with a fixed Landau level index \( n \)) of our model system contains a central zone of non-
adiabatic \( (P \neq 0) \), i.e. conducting states (contained between \(-E_b\) and \(+E_b\), in Fig. 1), which
is sandwiched between two zones of fully adiabatic \( (P = 0) \), hence dc-insulating states.
Usually for the highest and lowest adiabatic states of a band condition \( (8) \) is not fulfilled and
the states are more complicated superpositions of unperturbed functions as in the weak
disorder case. But since these states carry no dc-current their explicit structure is not needed
in the present investigation.

At the edges of the non-adiabatic zone there is a small zone of intermediate states for which
\( P \neq 0, \neq 1 \), i.e., which are neither fully adiabatic \( (P = 0) \) nor fully non-adiabatic
\( (P = 1) \). By writing \( P = 1 \) or \( P = 0 \) we mean here \( P \) close to 1 or close to 0, such that in the
course of the scattering process the effect due to the very small deviation from
\( P = 1 \) or \( P = 0 \) is negligible. (In the model system of Ref. [4], the parameter values have
been chosen such that all non-adiabatic (including intermediate) states can be expressed in the
weak disorder formalism.) At each level anticrossing the intermediate wave functions split
into two parts: one part develops according to the anticrossing (adiabatic) branch with
probability \( 1 - P \), the other part according to the intersecting, i.e. unperturbed branch (non-
adiabatic branch) with probability \( P \) (see Fig. 3).

3. Phase randomness.

Let us consider a wave function, which belongs to the fully non-adiabatic energy range
between \( E_a \) and \( -E_a \) in figure 1. At \( t = 0 \), we assume this function to be an unperturbed
Landau function \( \psi_s(x, y, t) \) with a localization centre \( x_s(t = 0) \) situated e.g. on the left of a
barrier, say between \( x_3 \) and \( x_4 \) in figure 1. Since \( P = 1 \) in the fully non-adiabatic energy range,
the modulus of this wave function continues to have the form of an unperturbed function,
which moves in x-direction with the velocity \( v \) (\( v \) is positive if \( B \) and \( E_y \) are positive). After
some time its centre enters the intermediate zone between \( x_3 \) and \( x_4 \) in figure 1, corresponding
to the energy interval \([E_a, E_b]\), where adiabatic processes become important (we remember that
\( P \) decreases exponentially with \( |x_p - x_{p'}| \) for \( E_p = E_{p'} \), i.e. with \( x_p \) and \( x_{p'} \) on opposite sides of
a barrier or well, see Ref. [4]).

During the passage through the intermediate energy levels the wave function undergoes a
sequence of « splittings » (according to Fig. 3), as a result of which it is transformed into a sum
\[ \sum_p c_p^2(t) \psi_p(x, y, t) \] of unperturbed functions \( \psi_p \). We choose the origin of \( t \) such that
Fig. 3. — Splitting between adiabatic and non-adiabatic time evolution in the vicinity of $t = t^* \mod T$. $P_{n,n+1}$ denotes the Zener tunnelling probability for the non-adiabatic processes (dashed lines), $1 - P_{n,n+1}$ the probability for the corresponding adiabatic behaviour (full lines). The expansion coefficients $d_n(t_0)$, $d_{n+1}(t_f)$ of an orbital at $t = t_f$ are obtained from those at $t = t_0$ by relation (12).

$\phi(t = 0)$ lies exactly between two anticrossings. At a time $T = m\tau/2$ ($m$ integer) the coefficients $c_p(T)$ result from a unitary matrix transformation which is composed of complex $2 \times 2$ matrices each of which describes a particular splitting process (indicated in Fig. 3). As a consequence each $c_p(T)$ can be expressed as a sum of contributions $c[j(s, p)]$ due to different sequences of splittings, i.e. originating from different splitting paths (see Fig. 4), which we number by the index $j(s, p)$, with:

$$c[j(s, p)] = \prod_{r=1}^{m} (R[j(s, p)]) \exp[i \xi[j(s, p)]] = R[j(s, p)] \exp[i \xi[j(s, p)]]$$

(9)

where $(R[j(s, p)])^2$ is the probability for the $r$-th splitting on the $j$-th path « from $s$ to $p$ ». The final probability for the occupation of the component $\psi_p$ at the time $T$ originating from the initial function $\psi_s(x, y, t = 0)$ is therefore:

$$c_p^*(T) c_p(T) = \sum_{j=1}^{J(s,p)} (R[j(s, p)])^2 +$$

$$+ \sum_{j \neq j'} R[j(s, p)] R[j'(s, p)] 2 \cos (\xi[j(s, p)] - \xi[j'(s, p)]) ,$$

(10)

where $J(s, p)$ is the total number of paths « from $s$ to $p$ ». We will show below that the phases $\xi[j(s, p)]$ are random. Therefore the second term in (10) vanishes if $J(s, p) \gg 1$. Now $J(s, p)$ is at least equal to $m - 1$ except for the two coefficients $c_h$ and $c_l$ belonging to the two
functions with the highest and the lowest energy in the expansion, see figure 4. But for these particular coefficients one has $J(s, p) = 1$ (hence the second term in (10) does not exist). This means that for large $m$, due to phase randomness, the final occupation numbers $c_p^*(T) c_p(T)$ do not depend on the phases of the matrix elements but only on the square of their absolute values, i.e. they only depend on the individual splitting probabilities \((|R[j(s, p)]|^2)\):

$$
c_p^*(T) c_p(T) = \sum_{j=1}^{J(s, p)} \prod_{r=1}^{m} (|R[j(s, p)]|^2).
$$

(11)

The phase randomness of the amplitudes $c[j(s, p)]$ originating from different splitting paths «between $s$ and $p»$ in figure 4 is proved as follows. Consider a single splitting process.

---

Fig. 4. — Shown are different splitting paths, which contribute to the transformation of an unperturbed function $\psi$, with coefficient $c_i(0)$ at $t = 0$ into a linear combination $\sum_p c_p^i(m\tau/2) \psi_p$ at $t = m\tau/2$. In the figure $m$ equals 6 (whence $c_i^0 = c_i(6)$ and $c_i^f = c_{p-3}(6)$). $E_s$, $E_p$, $E_p'$ denote the unperturbed ($V^1 = 0$) energies. Note that in this weak disorder approximation at times $t = m\tau/2$, $m$ integer, each perturbed adiabatic orbital $w_n(m\tau/2)$ coincides with a corresponding unperturbed orbital $\psi_p(m\tau/2)$. 

---
(Fig. 3) by which the complex coefficients $d_n(t_0)$, $d_{n+1}(t_0)$ are transformed into $d_n(t_f)$, $d_{n+1}(t_f)$. Such a non-adiabatic process between weakly perturbed levels occurs in various areas of physics. A process analogous to ours has been studied in the context of one-dimensional conduction [8-10]. Here explicit equations for the time evolution of the corresponding coefficients $d_n(t)$, $d_{n+1}(t)$ have been obtained [8, 10]. The derivation of these equations can be extended in a straightforward way to our two-dimensional system described by the Hamiltonian (1). As a result, in the expressions of reference [8] one simply has to replace $x$ by $y$ and $\partial / \partial x$ by $\partial / \partial y - iqBx/(hc)$ and to make the identifications $e = q$, $F = E_y$, $L = L_y$, $\alpha(t) = -(q/hc)\phi(t)$, $u_{k+\alpha/L} = \psi_p(\phi)$, $u_{-k+\alpha/L} = \psi'_p(\phi)$, $U_{k,-k} = V_{p,p'}$. One obtains two coupled first order differential equations for $d_n(t)$ and $d_{n+1}(t)$, which can be solved asymptotically [8, 10]. The result is of the form [10]

$$
\begin{pmatrix}
    d_{n+1}(t_f) \\
    d_n(t_f)
\end{pmatrix} =
\begin{pmatrix}
    e^{i(\alpha_{n,n+1} - \varphi_{n,n+1})/2} & e^{i(\beta_{n,n+1} - \varphi_{n,n+1})/2} \frac{1}{2} \\
    e^{i(\gamma_{n,n+1} + \varphi_{n,n+1})/2} & e^{i(\delta_{n,n+1} + \varphi_{n,n+1})/2} \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
    d_{n+1}(t_0) \\
    d_n(t_0)
\end{pmatrix}.
$$

(12)

Here $\varphi_{n,n+1}$ is the phase of $V_{p,p'}^1$ (where $\psi_p$ and $\psi'_p$ are the unperturbed functions involved in the considered splitting), and $P_{n,n+1} = P_{p,p'}$ is the corresponding Zener tunnelling probability. The phase terms $\alpha_{n,n+1}$, $\beta_{n,n+1}$, $\gamma_{n,n+1}$, $\delta_{n,n+1}$ depend on the energies $E_p[\phi(t)]$, $E'_p[\phi(t)]$ in the considered time interval and on $|V_{p,p'}^1|$, but are independent of $\varphi_{n,n+1}$.

In reference [10] a term $A_{n,n+1}$ is common to all the phases $\alpha_{n,n+1}$, $\beta_{n,n+1}$, $\gamma_{n,n+1}$, $\delta_{n,n+1}$. It is different for different splitting energies, and it was shown to be of order $\pi$ or larger in the examples considered in reference [10]. Generically (i.e. except for very special cases) it is not equal to an integer multiple of $\pi$. This means that generically the products of the factors $\exp(A_{n,n+1})$ originating from subsequent applications of equation (12) along different splitting paths from a given initial point to a given final point in figure 4 constitute a quasi random set of values on the complex unit circle [10].

In reference [9, 10] a perturbation potential in the form a delta function was used, which means that all the phases $\varphi_{n,n+1}$ are equal. In our case of a random perturbation $V^1(x, y)$, however, the phases $\varphi_{n,n+1}$ are randomly distributed as a function of $n$. This leads then immediately to the conclusion that the total phases originating from the different splitting paths are random, since they are composed of different sums of the random phases $\varphi_{n,n+1}$ (plus a term independent of the $\varphi_{n,n+1}$).

We remark that it is highly probable that an analogous phase randomness also holds in the general case (« strong disorder »), where each non-adiabatic function develops into a linear combination of more than two basis functions $\psi_p(x, y, t)$ in a time interval $\tau/2$. As we have seen this phase randomness considerably simplifies the calculation of the time evolution in $x$-direction of the total particle density of a wavepacket and makes the scattering process understandable in terms of probabilities. In the weak disorder case numerical calculations of such scattered particle densities are in progress.


We consider now a situation, where the Fermi energy $E_F$ is situated in a range of adiabatic levels such that all non-adiabatic (including intermediate) levels below $E_F$ are fully occupied and all non-adiabatic levels above $E_F$ are empty. We suppose as an initial condition at $t = 0$, that all occupied non-adiabatic, including intermediate states have the form of unsplit, i.e. unperturbed states $\psi_s(x, y, t)$. We thus have $|c_s(t = 0)|^2 = 1$, where $s$ runs over all non-
adiabatic states (for each band with occupied non-adiabatic states). We further assume $t = 0$ to be in the middle between two consecutive splittings (see Fig. 4). In the course of time each of these functions $\psi_s(x, y, t)$ develops according to the multiple splitting process described above. (The adiabatic functions are not split. This means that they are not mixed with the functions of the non-adiabatic zone.) At a time $t = T = m\tau/2$, $m$ integer, $m \gg 1$, the total occupation number $|c_p(T)|^2$ of a component $\psi_p(x, y, t)$ of a non-adiabatic state of a band is now the \textit{sum} of probability terms (11) coming from all the $\psi_s$ which span the non-adiabatic subspace of that band at $t = 0$:

$$|c_p(T)|^2 = \sum_s |c_p^s(T)|^2 = \sum_s |c_s(t = 0)|^2 \sum_{j=1}^{J(s, p)} \prod_{r=1}^{m} (\mathcal{R}[j(s, p)])^2,$$

(13)

where $|c_s(t = 0)|^2 = 1$. Here $J(s, p)$ is the total number of splitting paths «from $s$ to $p$» in figure 4 and $j(s, p)$ is the index of such a path. As we have seen, equation (11), hence (13) follow from phase randomness, as a consequence of which only the splitting \textit{probabilities} $(\mathcal{R}[j(s, p)])^2$ have to be taken into account for the calculation of the final occupation numbers $|c_p(T)|^2$.

The right hand side of (13) is obtained explicitly by calculating in each splitting interval simultaneously all the occupation numbers $|c_p|^2$ of the considered band at a time $t = m\tau/2$ from those of the previous time $(m-1)\tau/2$, by means of the matrices (12), but where the phases of the matrix elements are suppressed. Starting with the initial condition $|c_s(t = 0)|^2 = 1$ for all $s$ associated with the non-adiabatic states of the band and using the fact that the transition probabilities associated with the four possible paths of an individual splitting process pairwise add up to one (see Fig. 4), one immediately sees that all the occupation numbers $|c_p|^2$ of the functions $\psi_p(x, y, t)$ which span the subspace of the non-adiabatic states of the band at $t = m\tau/2$ (m integer) are equal to one:

$$|c_p(m\tau/2)|^2 = \sum_s |c_p^s(m\tau/2)|^2 = 1, \quad m \text{ integer}.$$  

(14)

Equation (14) can be extended to the general case, which is not based on the weak disorder assumption and on phase randomness. In the general case one obtains [11]:

$$|d_n(t)|^2 = \sum_s |d_n^s(t)|^2 = 1.$$  

(15)

Here $|d_n(t)|^2$ is the total occupation number of the $n$-th of the adiabatic basis states $w_n(x, y, t)$, which span the subspace of the non-adiabatic states of the considered band at the time $t$. Our derivation of (14) constitutes an immediate verification of (15) in the special case of our weak disorder model. Equation (15) follows directly from the unitarity of the time evolution operator and from the special form of the adiabatic time evolution. Further it can be shown [11] that these general mathematical properties (which imply (15) and the periodicity of the modulus of the adiabatic functions $w_n(x, y, t)$) have the following physical consequences (provided $E_F$ lies in a range of adiabatic energies for the considered value of $E_y$ and the temperature is sufficiently low, such that the non-adiabatic states are not affected by inelastic interactions with the surrounding heat bath):

a) The total charge density of the system is periodic in time with period $\tau$, i.e. it is constant over macroscopic time intervals much larger than $\tau$. This means that despite the presence of the electric field $E_y$ there is no charge redistribution which could modify the potential
\( V(x, y) \) in the course of time (in particular no macroscopic field \( E_x \) is created), and all adiabatic energies \( \varepsilon_n(t) \) remain periodic with period \( \tau \).

b) There is no dissipation (the total energy associated with the occupied non-adiabatic states remains equal to the ground state energy in the course of time). Hence

c) \( \sigma_{xx} = \sigma_{yy} = 0, \ I_y = 0, \ I_x = \text{constant} \) (\( I_x \) and \( I_x \) denote dc-currents). The last equation means that there exists a Hall conductance plateau as long as \( E_F \) is in a range of adiabatic states.

Let us now return to our explicit weak disorder model. Here (14) means that all unperturbed states \( \psi_p(x, y, t) \) representing the fully non-adiabatic states are occupied with probability \( |c_p|^2 = 1 \) for all times \( t_N = N\tau, \ N \text{ integer} \). This means that, being fully non-adiabatic states, they are fully occupied for all times. (Note that the set of indices \( p \) corresponding to these states depends on \( t : p = s + N = s + t_N / \tau \). Further we remark again that the probabilities \( |c_p(t)|^2 \) result from the projection on the unperturbed states \( \psi_p(x, y, t) \) of all the different individual single particle functions at the time \( t \). These particular unperturbed functions \( \psi_p(x, y, t) \) representing the fully non-adiabatic states are spatially localized in the fully non-adiabatic zones between barriers and wells (see Fig. 1).

Their localization centres move with velocity \( v \). This means that the distance between two neighbouring centres \( x_s(t) \) and \( x_{s+1}(t) \) is passed by the unit charge \( q \) in the time interval \( \tau \). Therefore in these spatial regions occupied by the fully non-adiabatic states a dc-current \( I_x \) (per band) flows in \( x \)-direction, with:

\[
I_x = q / \tau = (q^2 / h) E_x L_y.
\]  

\( I_x \) flows at all times. This gives a contribution to the Hall conductance of \( q^2 / h \) per band.

The dc-current of the fully adiabatic states is zero \([3, 4]\). In figure 1 the localization centres of these states are situated between the spatial regions occupied by the localization centres of the fully non-adiabatic states, which move with the constant velocity \( v \) towards the next intermediate region to their right. Since there is conservation of the total charge in the system, one charge \( q \) (per band) must disappear per time interval \( \tau \) in the intermediate region on the left of a barrier (well) and one charge \( q \) must reappear in the same time interval on the right of a barrier (well). Otherwise no dc-current (16) could be possible on the spatially distinct fully non-adiabatic regions between the barriers and wells. This non-classical particle propagation across the fully adiabatic spatial regions in the center of the barriers and wells is the result of the time-dependent scattering process in the intermediate energy zone, which includes scattering between states with localization centres on different sides of a barrier (well). This process is responsible for the so-called compensating currents (see Ref. [4] for a more detailed discussion).

If \( E_F \) is situated within a range of intermediate or fully non-adiabatic levels, equations (14) and (15) no longer hold. In this case some initial coefficients \( |c_s(t = 0)| \) corresponding to states above the Fermi energy are zero in equation (13), i.e. the corresponding spatial sites \( x_s(t = 0) \) are empty. But due to the discussed time evolution some of the corresponding coefficients \( c_{s+N}(0 + N\tau) \) will be different from zero after a time interval \( N\tau \). (Further, occupied sites at \( t = 0 \) may be unoccupied at \( t = N\tau \). This may lead to macroscopic charge rearrangements, since \( x_{s+N}(N\tau) \) is equal to \( x_s(0) \), which was not occupied at \( t = 0 \). Hence an additional electric field in \( x \)-direction may be created. Further, since these new occupied components \( \psi_{s+N}(t = N\tau) \) have an energy above \( E_F \), states which are composed of such components will be modified (their energy will be reduced) by inelastic scattering (due to interaction with the environment) after an average time \( \tau_{in} \), where \( \tau_{in} \) is the average time...
between two inelastic events. This means there is dissipation, hence non-zero diagonal conductivity.

5. Discussion.

In our weak disorder model each scattering event in the intermediate energy zone simultaneously gives rise to an adiabatic splitting corresponding to non-classical motion with change of the side of the barrier (well), hence either to forward (i.e., to the right in our case) or backward scattering, and to a non-adiabatic splitting corresponding to classical motion without changing the side of the barrier (well), hence to forward scattering only. Now consider the case of a wave function $\psi(x, y, t)$ which at $t = 0$ is an unperturbed Landau orbital with localization centre in the fully non-adiabatic zone to the left of a barrier (or well), moving to the right with velocity $v$. After some time the orbital reaches the intermediate zone and is subject to a sequence of splitting processes as described above. Once a component $\psi_p(t)$ belonging to the fully non-adiabatic zone localized on the right hand side of the barrier (well) is occupied with non zero probability $c_p^*(T) c_p(T)$, it is no longer split for some time. It remains on this side and its modulus moves with the classical velocity $v$ away from the barrier (well) until it reaches the next intermediate zone. As a consequence (and because of individual charge conservation) the probability for occupying a component with centre $x_p$ on the opposite side of the barrier (well) increases with time whereas that for occupying a component on the original side decreases. Finally, after a sufficient number of time intervals $\tau/2$ practically the entire wave function $\psi(x, y, t)$ is localized on the opposite side of the barrier (well).

After the passage of the first barrier (well) an originally unperturbed Landau function has thus developed into a function $\psi(x, y, t)$ which is somewhat delocalized in x-direction. This delocalization is increased after each passage across a subsequent well or barrier, since each of its components $c_p(t) \psi_p(x, y, t)$ is decomposed again in the described manner by the scattering events in the intermediate zone of every subsequent barrier or well. But in the fully non-adiabatic zones between a barrier and a well each of the components $c_p(t) \psi_p(x, y, t)$ of the wave function $\psi(x, y, t)$ has an unchanged modulus which moves in x-direction with the common constant velocity $v$.

The scattering process which brings an incoming Landau wave packet from one side of a barrier (well) to the other side requires a time typically of the order of $\tau$ times the number of intermediate states (in the extreme, hypothetical case where there is only one intermediate state per barrier or well, a time equal to $\tau/2$ is required, see Ref. [4]). During its time evolution through the intermediate states a single-particle wave function is a time dependent sum of non-adiabatic and adiabatic components. This means, that in each time interval of length $\tau$ the wave function contains a part, which is periodic in $\phi = - cE_y L_y t$ (or, in the extreme case mentioned above, which at least represents half of a period). To describe this periodicity by an expansion with respect to $\phi$ or $E_y$ all orders would have to be included. Hence a linear approximation is not adequate, it would only be a good approximation for time intervals appreciably smaller than $\tau/2$. This is equally true for a perturbed one-dimensional conductor (loop), since the corresponding mathematical developments are the same (see Sect. 3). For the case of one-dimensional conductance it has already been emphasized previously [8, 12]) that linear response theory fails to describe the time evolution of the scattered states in cases where the Zener probabilities noticeably differ from unity, i.e. in cases where, in our terminology, the time-dependent wave function passes through intermediate states.

Quite generally in the large class of two-dimensional systems investigated in this and our previous [4, 11] articles the disorder induced time-dependent scattering process of the
adiabatic (i.e. non-conducting) and intermediate (i.e. non-classically conducting) single-particle wave functions is non-linear in the electric field on general grounds [11], and we have shown that the macroscopic Hall current $I_x$ is generated by the currents of the non-adiabatic states and that the intermediate states constitute a substantial part of this current, since they carry the so-called compensating currents [4, 11]. Further, in the special case, where the intermediate states are only weakly perturbed by the disorder potential, we have shown explicitly, that the totality of the time evolution (for times exceeding $\tau/2$) of each intermediate state is needed for the proof of the integer quantization of the Hall conductance. Therefore linear response theory (which is applicable only for times much smaller than $\tau/2$) cannot give a correct microscopic quantum mechanical explanation of the IQHE in these systems.

Our systems are characterized by periodic boundary conditions in the direction of the macroscopic Hall field. Now if the IQHE is a bulk effect, linear response theory should also fail to describe the IQHE in systems with other boundary conditions. Recently work based on Dirichlet boundary conditions in the direction of the macroscopic Hall field and generalized periodic boundary conditions (depending on an auxiliary parameter) in the direction of the Hall current did succeed in deriving quantization of the Hall conductance in the framework of the Kubo formalism [13-16]. However the fulfilment of a series of additional assumptions was indispensable to achieve this result. On the other hand, in our theory no such additional assumptions (and no auxiliary parameter) are necessary to obtain our conclusions. We hope that further work will clarify the relation between these different approaches.

In our weak-disorder-model the expectation value of the velocity operator in $x$-direction of a non-adiabatic state is equal to $cE_y/B$ if the orbital is fully non-adiabatic, i.e., located in the full line interval before a barrier (or well), and it is again equal to this value once it has been scattered through the barrier (or well), if one waits a sufficiently long time, as we have seen. In the meantime this expectation value is higher (in general by many orders of magnitude, see [4]). If at a given time one sums over these time-dependent velocity expectation values of all the non-adiabatic states (part of which are non-linear in $E_y$), one obtains again the result (16), i.e., an expression, which is time-independent and linear in $E_y$.

If the electric field $E_y$ is sufficiently high (i.e. if the Hall current $I_x$ is sufficiently large), such that all adiabatic states become almost fully non-adiabatic, linear response theory becomes again applicable (in complete analogy to the corresponding one-dimensional conductance [8]). This corresponds to the situation where the Hall conductance plateaus have disappeared (breakdown of the quantum Hall effect).

The results presented in this and our previous [4, 11] articles open new perspectives for the theory of the IQHE. In particular they elucidate the microscopic scattering mechanism in the quantum Hall regime. The nature of this mechanism has been illustrated by our explicit model system (Fig. 1) in the weak disorder case. Our results contrast with certain aspects of previous theories, where similar Hamiltonians as defined by (1) and (2) have been used. Here we briefly examine the so-called gauge invariance argument. The essence of this theory is reviewed in [17] (based on a more complete form of the original argument). The considered systems are topologically equivalent to our systems defined by (1) and (2). As in our case the electric field is created by an increasing magnetic flux $\phi (t)$ (through a hole). The arguments of the gauge invariance approach are entirely based on the adiabatic wave functions. It is claimed that, if the Fermi energy is in a mobility gap, an adiabatic change of the flux by an amount $hc/q$ transports exactly an integer number of electrons across the disordered system in a direction perpendicular to the electric field. If we apply this statement to our explicit model system represented by figure 1 this would correspond to the adiabatic transfer of an integer number of electronic charges from a full line region on the left hand side of a barrier.
(or well) to the full line region on the right hand side of this barrier (or well). According to our results such an adiabatic process is impossible in such systems, since in the presence of disorder the adiabatic wave functions are dc-insulating (the adiabatic particle densities and currents are periodic in time) and therefore do not contribute to the dc-Hall current perpendicular to the electric field. We have seen that the macroscopic Hall current is carried by the non-adiabatic states, and in particular, that those of the non-adiabatic states, which are intermediate between fully non-adiabatic and fully adiabatic, carry the so-called compensating currents, which compensate the loss of current of the dc-insulating states (i.e., of the fully adiabatic states) and of the dc-insulating fractions of the dc-conducting states (i.e., of the fully adiabatic components of the intermediate states).

Next we analyze another well known theory, where the Hall conductivity is expressed as a topological invariant [18] and which may be considered as an extension of the gauge invariance argument. In this theory the conductivity is expressed in a version of the Kubo formula (i.e. by linear response theory), which is then averaged over the unit cell of two flux parameters (each of which describes an electric field in one of the two spatial directions). This average can be shown to have the mathematical form of a Berry phase, which describes the increment (in addition to the dynamical phase factor) of the global phase factor of the adiabatic wave function along a closed loop in the two-dimensional space of the two flux parameters. It is then claimed that the unit cell of the parameter space represents a torus. From this it follows, according to general results of differential geometry, that the flux averaged Kubo Hall conductivity times $h/q^2$ is a topological invariant (equal to an integer). The value of this integer is not determined by this general argument, but the idea is that it be different from zero.

According to our results such a theory cannot describe the conductivity associated with the macroscopic Hall current, since it is based on the adiabatic wavefunction and, in addition, on linear-response theory. Moreover, since the adiabatic time evolution is involved, which in the presence of disorder leads to periodic currents, as we have seen, we would expect that the averaged conductivity thus obtained is actually zero. From a recent mathematical analysis [19] it appears, that the flux averaged Kubo Hall conductivity is not a (non-trivial) topological invariant. This will be the subject of a forthcoming publication.

We conclude with some summarizing remarks. The class of models which we have considered are free of contacts and do not give rise to edge states. This supports the view that the IQHE observed in large two-dimensional systems [1] is not a contact or edge state effect. This is consistent with experiments performed without contacts [20]. In our systems the microscopic origin of the IQHE is due to the characteristic time dependence in the presence of disorder of the single-particle wave functions, where non-linear effects play an important role. The time-dependent, microscopic scattering mechanism can be illustrated in detail by an explicit model system. Our results provide a new picture for the understanding of the IQHE.

**Acknowledgment.**

The author thanks O. Viehweger for sending a copy of reference [15] prior to publication.

**References**


[18] For a review see THOULESS D. J., chapter 4 of reference [2].