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Weak-probe absorption and dispersion spectra in a two-level system driven by a strong trichromatic field

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Résumé. — On étudie à l'aide d'un champ sonde l'évolution temporelle d'un système à 2 niveaux soumis à un champ quasi résonnant à trois fréquences différentes. Les équations de Bloch correspondantes sont résolues par une méthode d'exponentielle de matrice qui permet de se passer de fractions continues. On montre que lorsque la fréquence de Rabi est suffisamment grande, le profil d'absorption du champ sonde ressemble à un profil de dispersion. On met en évidence l'existence d'un nouveau type de résonances paramétriques qui ne sont pas liées à des résonances de Rabi.

Abstract. — With the help of a probe-field method the time-behaviour of a two-level system subjected to a strong trichromatic near-resonant field is studied. The suitable Bloch equations are solved by a matrix exponent method which allows us to do without the continued fractions. It is shown that, when the Rabi frequency is sufficiently large, the profile of the weak-probe field absorption resembles a dispersion profile. The existence of a new type of parametric resonances not connected with the Rabi resonances is shown.

1. Introduction.

The probe-field method is widely adopted in studying the behaviour of a quantum system excited by a strong resonant field. This method consists in studying the absorption or the dispersion of the field depending on its frequency provided the quantum system is excited by a resonant field at the fixed frequency. From the absorption (dispersion) line form of the probe field one can draw the conclusions of the nonlinear phenomena in the quantum system. Rautian and Sobelman [1] were the first to investigate the absorption of a weak in the presence of a strong wave. They showed that the absorption of the weak (probe) wave changes into the gain when the strong field amplitude and the frequency difference between the strong and weak field reach certain values. This problem was also considered in [2-6] by the solution of equations for the density matrix of a two-level system which allows us to take into account the relaxation in contrast to the solution of equations for the wave function amplitude [1]. It has also been obtained that the weak-probe field absorption may be negative (the parametric gain effect) when the saturation field is sufficiently strong. This gain has a maximum when the frequency difference between the strong and weak field is equal.
\( \Omega = \pm \Omega_R \), \( \Omega_R \) is the nutation frequency in the system (the Rabi frequency). The negative absorption effect was experimentally confirmed in a large number of papers both in the rf-range and in optics [7-11]. Subsequently the probe-field method was developed in two directions: 1) the case of equal saturating and probe field intensities (so-called strong modulated exciting field) [12-24], 2) the case of a number of equal probe fields with the frequencies which are symmetric relative to the saturating field frequency. The amplitudes of these probe fields may be both weak [25-28] and equal to the strong pump amplitude [29]. It was shown that there are the resonances at \( \Omega = \pm \Omega_R \) and \( |\Omega| = \Omega_R/n \), \( n = 1, 2, 3, ... \) (the so-called sub-radiation structure [30]) in their absorption spectra when the probe wave amplitudes grow. In addition in [22, 28] was obtained that the dispersion profile resembles an absorption one while the absorption profile resembles a dispersion one when the probe wave amplitude is sufficiently large.

Now we describe briefly the mathematical methods which have been used by the above-mentioned authors. It is notorious that the time dependence of the density matrix of a two-level system interacting with the resonant radiation is described by the Bloch equations (see, for example, [31]). When the exciting amplitude and frequency are constant the Bloch equations are simultaneous linear inhomogeneous equations with constant coefficients and they can be solved [32]. If an exciting field is monochromatic but its amplitude is a periodic function of time, then the Bloch equations are equations with periodic coefficients. Such a form of the field amplitude modulation is equivalent to the presentation of the exciting field by a set of the monochromatic components with the same amplitudes and equidistant frequency spectrum. Up to now these equations have been solved by the following methods: 1) the expansion on a small parameter (in the case of weak probe fields) [4, 5, 25, 26]; 2) the Laplace (or Fourier) transformations [2, 5, 27] with some simplifying assumptions; 3) the introduction of a trial solution [29]; 4) the Floquet theorem method which allows us to present the solution by an infinite set of the intermode frequency sidebands. The amplitudes of these sidebands are expressed in terms of continuous fractions which are needed in the numerical calculation. Feneuille et al. [12], Toptygina and Fradkin [15] were the first to apply the Floquet theorem and now this method is widely applied in studying the Bloch equations with periodic coefficients and constant inhomogeneous terms [16-24].

However, when the exciting field modulation is nonperiodical (for example, the phase of a probe field is a stochastic function [33], or the field spectrum is non-equidistant), the Floquet theorem is inapplicable because the Bloch equations for this case have a nonperiodical coefficients or a time-dependent inhomogeneous term (the validity of the application of the Floquet theorem for studying the differential equations have been discussed, for example, in [34]). Such equations can be solved with the help of the matrix exponent method which was used in [33]. We show below that this method allows us to obtain analytic solutions of the Bloch equations with both periodical and nonperiodical coefficients and without using the continuous fractions.

The present paper is devoted to the calculation of the weak-probe absorption and dispersion profiles in a two-level system saturated by a strong trichromatic field in an analytic form, i.e. we apply the probe-field method to study the nonlinear effects in a two-level system driven by a strong resonant amplitude-modulated field. In section 2 we obtain (with the help of the matrix exponent method) an analytic solution of the Bloch equations with periodic coefficients and constant inhomogeneous terms. These equations describe the behaviour of the two-level system density matrix in the strong trichromatic field with three components of equal amplitude. In section 3 we obtain the response of such a two-level system to the weak probe field which adds to the Bloch equations the time-dependent inhomogeneous terms. These equations are also solved by the matrix exponent method. Because of the expressions
obtained for the absorption coefficient and dispersion are very complex, the graphs for these values are calculated numerically.

2. Two-level system in the resonant trichromatic field.

The trichromatic field with the components of the same amplitude and with the equidistant spectrum and a probe field can be written in such a form:

\[
E(t) = E_0[\cos(\omega t + \cos(\omega + \Omega)t) + \cos(\omega - \Omega)t] + \varepsilon \cdot \cos(\omega + \delta)t
\]

where \(E_0\) is the amplitude of the saturating field component, the frequency \(\omega\) is supposed to be equal to the natural frequency \(\omega_0\) of the two-level systems, \(\varepsilon\) is the probe field amplitude, \(\Omega\) is the difference between the neighbouring component frequencies, \(\delta = \omega_w - \omega\), \(\omega_w\) is the probe field frequency.

The equations for the density matrix elements of a two-level system interacting with the field (1) in the rotating wave approximation (the optical Bloch equations) read then:

\[
\frac{d\alpha(t)}{dt} = -\alpha(t) \Gamma_2 - \frac{\varepsilon}{E_0} \Omega_R \sin \delta t \cdot n(t)
\]

\[
\frac{d\beta(t)}{dt} = -\beta(t) \Gamma_2 + \Omega_R \left(1 + 2 \cos \Omega t + \frac{\varepsilon}{E_0} \cos \delta t\right) n(t)
\]

\[
\frac{dn(t)}{dt} = (n_0 - n) \Gamma_1 - \Omega_R \left[1 + 2 \cos \Omega t + \frac{\varepsilon}{E_0} \cos \delta t\right] \beta(t) - \frac{\varepsilon}{E_0} \sin \delta t \cdot \alpha(t)
\]

where \(\alpha(t) = 2 \text{Re} \sigma_{21}(t)\); \(\beta(t) = 2 \text{Im} \sigma_{21}(t)\); \(n(t) = \rho_{22}(t) - \rho_{11}(t)\); \(\sigma_{ij}(t)\) are the slowly varying components of density matrix elements \(\rho_{ij}(t)\); \(n_0\) is the level population difference per unit volume in the absence of an external field; \(\Omega_R = dE_0/h\) is the Rabi frequency; \(d\) is the dipole moment of a two-level system; \(\Gamma_2^{-1}, \Gamma_1^{-1}\) are the relaxation times, collisional and radiative, respectively. In deriving (2) we have supposed the validity of the rotating wave approximation for all components of the trichromatic field, i.e. we have assumed that the condition \(\Omega \ll \omega\) is valid.

Assuming that the probe field is weak (\(\varepsilon/E_0 \ll 1\)) the solution of equations (2) is sought as a sum of two solutions — the solutions in the absence of a weak field (for \(\varepsilon = 0\)) and a small correction to the latter caused by a weak field, namely,

\[
\alpha(t) = U(t) + u(t); \quad \beta(t) = V(t) + v(t); \quad n(t) = W(t) + w(t);
\]

where \(|u, v, w| \ll |U, V, W|\). After substituting (3) into (2) and retaining the terms of the order \(\varepsilon/E_0\), we obtain two systems of equations for \(U, V, W\) and \(u, v, w\) instead of (2):

\[
\frac{dX(t)}{dt} = A(t) X(t) + L
\]

\[
\frac{dx(t)}{dt} = A(t) x(t) + \ell(t)
\]

where

\[
X(t) = \begin{bmatrix} U(t) \\ V(t) \\ W(t) \end{bmatrix} \quad x(t) = \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix} \quad A(t) = \begin{bmatrix} -\Gamma_2 & 0 & 0 \\ 0 & -\Gamma_2 & \Omega_R(1 + 2 \cos \Omega t) \\ 0 & -\Omega_R(1 + 2 \cos \Omega t) & -\Gamma_1 \end{bmatrix}
\]
The matrix equation (4) is the equation with the periodical coefficients and constant inhomogeneous term. It can be solved with the help of the Floquet theorem. However we shall show that the matrix exponent method allows us to obtain more easily both equation (4) and equation (5) which do not satisfy the Floquet theorem, since the period of its inhomogeneous term $\ell(t)$ is not equal to the period of $A(t)$.

So provided the commutator $[A(t), \exp B(t)]$ is equal to zero \cite{35} the solutions of equations (4), (5) can be written as

$$X(t) = e^{\beta(t)} \left\{ \int_0^t e^{-\beta(t')} L \, dt' + \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} \right\}$$

$$x(t) = e^{\beta(t)} \int_0^t e^{-\beta(t')} \ell(t') \, dt'.$$

To calculate $\exp B(t)$ we shall make use of Silvester's formula \cite{34}:

$$e^B = e^{\lambda_1} \frac{(B - \lambda_2 I)(B - \lambda_3 I)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + e^{\lambda_2} \frac{(B - \lambda_1 I)(B - \lambda_3 I)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + e^{\lambda_3} \frac{(B - \lambda_1 I)(B - \lambda_2 I)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}; \quad (8)$$

where $\lambda_{1,2,3}$ are eigenvalues of matrix $B(t)$, $I$ is the identity matrix.

Since $B(t) = \int_0^t A(t') \, dt'$ we can easily obtain (provided $\Omega_R^2 \gg \gamma^2$):

$$\lambda_1 = -\Gamma_2 t; \quad \lambda_{2,3} = -\nu t \pm i f(t)$$

where $f(t) = \Omega_R t + (2 \Omega_R / \Omega) \sin \Omega t$; $\nu = (\Gamma_2 + \Gamma_1)/2$; $\gamma = (\Gamma_2 - \Gamma_1)/2$. After substituting (9) into (8) and retaining the terms of the order $\gamma / \Omega_R$ we obtain:

$$e^{\pm B(t)} = \begin{bmatrix} 0 \\ e^{\pm \nu t} \left\{ \cos f(t) \mp \frac{\gamma}{\Omega_R} \sin f(t) \right\} \pm e^{\nu t} \sin f(t) \end{bmatrix}$$

$$\begin{bmatrix} \mp e^{\nu t} \sin f(t) \\ e^{\pm \nu t} \left\{ \cos f(t) \pm \frac{\gamma}{\Omega_R} \sin f(t) \right\} \end{bmatrix}.$$ (10)

The direct calculation shows that $\exp(B) \exp(-B) = 1$ within the bounds of our approximation.

We can now obtain the conditions for the validity of solutions (6), (7). Using (10), one obtains:

$$[A(t), e^{B(t)}] = 4 \gamma e^{-\nu t} \sin f(t) \cos \Omega t \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ (11)
The commutator is null in four cases

1) \( \gamma = 0 \); 2) \( \sin f(t) = 0 \); 3) \( \cos \Omega t = 0 \); 4) \( t \to \infty \).

Note that the state of the system at a given time \( t \) generally depends on its previous history. The commutator has then in principle to be null at any time \( t' \) in the range \([0, t]\). This is only achieved if \( \gamma = 0 \) (case 1). However the cancellation of the commutator for \( t' \to \infty \) (case 4) entails that the steady state solution, which does not depend on the initial conditions, can be obtained by the matrix exponent method, whatever \( \gamma \) is.

Substituting (10) into (6), we obtain the analytic solution for all components of the vector \( X(t) \):

\[
X(t) = n_0 \Gamma_1 \begin{bmatrix}
- I_1 e^{-\gamma t} \left[ \cos f(t) - \frac{\gamma}{\Omega_R} \sin f(t) \right] + \left( I_2 - \frac{\gamma}{\Omega_R} I_1 + \Gamma_1^{-1} \right) e^{-\gamma t} \sin f(t) \\
I_1 e^{-\gamma t} \sin f(t) + \left( I_2 - \frac{\gamma}{\Omega_R} I_1 + \Gamma_1^{-1} \right) e^{-\gamma t} \left[ \cos f(t) + \frac{\gamma}{\Omega_R} \sin f(t) \right]
\end{bmatrix}
\]

(13)

where \( I_1 = \int_0^t e^{\nu t'} \sin f(t') \, dt' \); \( I_2 = \int_0^t e^{\nu t'} \cos f(t') \, dt' \).

\[
\sin f(t) = \sum_{k=0}^{\infty} \left\{ c_k J_{2k}(2 \rho) [\sin (\Omega_R + 2 k \Omega) t + \sin (\Omega_R - 2 k \Omega) t] + J_{2k+1}(2 \rho) [\sin (\Omega_R + (2 k + 1) \Omega) t - \sin (\Omega_R - (2 k + 1) \Omega) t] \right\}
\]

(14)

\[
\cos f(t) = \sum_{k=0}^{\infty} \left\{ c_k J_{2k}(2 \rho) [\cos (\Omega_R + 2 k \Omega) t + \cos (\Omega_R - 2 k \Omega) t] - J_{2k+1}(2 \rho) [\cos (\Omega_R + (2 k + 1) \Omega) t - \cos (\Omega_R - (2 k + 1) \Omega) t] \right\}
\]

(15)

where \( J_n(2 \rho) \) are Bessel functions of the first kind, \( \rho = \Omega_R / \Omega \), \( c_0 = 1/2 \), \( c_1, c_2, \ldots \), \( c_n = 1 \). Taking into account (14), (15), solutions (13) can be written as (we write the steady-state parts only):

\[
\begin{pmatrix} V(t) \\ W(t) \end{pmatrix} = n_0 \Gamma_1 \sum_{k, \ell = 0}^{\infty} \sum_{i-1}^{6} \begin{pmatrix} A_{i}^{k, \ell} \\ B_{i}^{k, \ell} \\ P_{i}^{k, \ell} \\ Q_{i}^{k, \ell} \end{pmatrix} \sin s_{i}^{k, \ell} t + \begin{pmatrix} \cos s_{i}^{k, \ell} \end{pmatrix}
\]

(16)

where

\[
\begin{align*}
s_{1,2}^{k, \ell} &= 2(k \pm \ell) \Omega \\
s_{3,4}^{k, \ell} &= (2 \ell + 1 \pm 2k) \Omega \\
s_{5,6}^{k, \ell} &= (2k + 1 - 2 \ell) \Omega \\
s_{6}^{k, \ell} &= 2(k + \ell + 1) \Omega.
\end{align*}
\]

(17)

The explicit expressions for time-independent coefficients \( (A, B, P, Q)^{k, \ell} \) are greatly complicated. After introducing the following notation:

\[
\begin{align*}
F_{n}^{\pm} &= \frac{\nu}{\nu^2 + (\Omega_R - n \Omega)^2} \pm \frac{\nu}{\nu^2 + (\Omega_R + n \Omega)^2} \\
G_{n}^{\pm} &= \frac{\Omega_R - n \Omega}{\nu^2 + (\Omega_R - n \Omega)^2} \pm \frac{\Omega_R + n \Omega}{\nu^2 + (\Omega_R + n \Omega)^2}
\end{align*}
\]

(18)
these expressions can be written in such a form:

\[ A_{1}^{k,t} = c_{k}c_{t}J_{2k}J_{2t}F_{2k}^{-} ; \quad A_{2}^{k,t} = A_{1}^{k,t} + J_{2k+1}J_{2t+1}F_{2k+1}^{-} ; \]

\[ A_{3}^{k,t} = A_{4}^{k,t} - c_{t}J_{2t}J_{2k+1}F_{2k+1}^{+} ; \quad A_{4}^{k,t} = c_{k}J_{2k}J_{2t+1}F_{2k}^{+} ; \]

\[ A_{5}^{k,t} = -c_{t}J_{2t}J_{2k+1}F_{2k+1}^{+} ; \quad A_{6}^{k,t} = -J_{2k+1}J_{2t+1}F_{2k+1}^{+} ; \]

\[ B_{1}^{k,t} = c_{k}c_{t}J_{2k}J_{2t}G_{2k}^{+} ; \quad B_{2}^{k,t} = B_{1}^{k,t} + J_{2k+1}J_{2t+1}G_{2k+1}^{+} ; \]

\[ B_{3}^{k,t} = -B_{4}^{k,t} - c_{t}J_{2t}J_{2k+1}G_{2k+1}^{+} ; \quad B_{4}^{k,t} = -c_{k}J_{2k}J_{2t+1}G_{2k}^{+} ; \]

\[ B_{5}^{k,t} = -c_{t}J_{2t}J_{2k+1}G_{2k+1}^{+} ; \quad B_{6}^{k,t} = -J_{2k+1}J_{2t+1}G_{2k+1}^{+} ; \]

\[ P_{1}^{k,t} = -c_{k}c_{t}J_{2k}J_{2t}\left(G_{2k}^{+} - \frac{\gamma}{\Omega_{R}}F_{2k}^{-}\right) ; \quad P_{2}^{k,t} = P_{1}^{k,t} + J_{2k+1}J_{2t+1}\left(G_{2k+1}^{-} - \frac{\gamma}{\Omega_{R}}F_{2k+1}^{-}\right) ; \]

\[ P_{3}^{k,t} = P_{4}^{k,t} + c_{t}J_{2t}J_{2k+1}\left(G_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}F_{2k+1}^{-}\right) ; \quad P_{4}^{k,t} = -c_{k}J_{2k}J_{2t+1}\left(G_{2k}^{+} - \frac{\gamma}{\Omega_{R}}F_{2k}^{+}\right) ; \]

\[ P_{5}^{k,t} = c_{t}J_{2t}J_{2k+1}\left(G_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}F_{2k+1}^{-}\right) ; \quad P_{6}^{k,t} = J_{2k+1}J_{2t+1}\left(G_{2k+1}^{-} - \frac{\gamma}{\Omega_{R}}F_{2k+1}^{-}\right) ; \]

\[ Q_{1}^{k,t} = c_{k}c_{t}J_{2k}J_{2t}\left(F_{2k}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k}^{-}\right) ; \quad Q_{1}^{k,t} = Q_{1}^{k,t} + J_{2k+1}J_{2t+1}\left(F_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k+1}^{-}\right) ; \]

\[ Q_{2}^{k,t} = -Q_{4}^{k,t} - c_{t}J_{2t}J_{2k+1}\left(F_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k+1}^{-}\right) ; \]

\[ Q_{4}^{k,t} = -c_{k}J_{2k}J_{2t+1}\left(F_{2k}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k}^{+}\right) ; \quad Q_{5}^{k,t} = -c_{t}J_{2t}J_{2k+1}\left(F_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k+1}^{-}\right) ; \]

\[ Q_{6}^{k,t} = -J_{2k+1}J_{2t+1}\left(F_{2k+1}^{+} + \frac{\gamma}{\Omega_{R}}G_{2k+1}^{-}\right) . \]

(19)

For the sake of simplicity we omitted the argument 2ρ for all Bessel functions.

So we obtained the analytic solutions of the Bloch equations in the case of trichromatic resonant excitation without using the Floquet theorem. In contrast to [29], where the trichromatic excitation was also considered, we supposed the inequality of \( r_{2} \) and \( r_{1} \).

The following conclusions can be drawn from equations (16)-(19). The steady-state oscillation spectrum of \( W(t) \) (the spectrum of undamped nutation) contains the intermode distance \( \Omega \) sidebands. This result corresponds to the results obtained by the Floquet method for the differential equations with the periodical coefficients. The sideband amplitudes are maximum when \( \Omega_{R} = k\Omega \) \((k = 1, 2, 3, \ldots)\) i.e. when the Rabi resonance conditions are valid. These amplitudes decrease with increasing \( k \).

3. Absorption and dispersion spectra of the weak-probe field.

It is well known that a two-level system linear polarization induced by a monochromatic field (the average dipole moment of a volume unit) is equal to:

\[ P(t) = d[\rho_{12}(t) + \rho_{21}(t)] = \frac{1}{2} d[u(t) \cos \omega t + v(t) \sin \omega t] \]

(20)
where \( u(t) \), \( v(t) \) are the proper Bloch-vector components. On the other hand the polarization can be expressed in terms of the medium polarizability \( \chi(\omega) \):

\[
P(t) = \varepsilon_0 \varepsilon \left[ \text{Re} \chi(\omega) \cos \omega t + \text{Im} \chi(\omega) \sin \omega t \right]
\]

(21)

where \( \varepsilon_0 \) is the vacuum electric permittivity, \( \varepsilon \) is a field amplitude. If the exciting field is nonmonochromatic, the total polarization is the sum of the polarization at both the frequencies containing the exciting field and the combination frequencies:

\[
P(t) = \varepsilon_0 \varepsilon \sum_q \left[ \text{Re} \chi_q(\omega_q) \cos \omega_q t + \text{Im} \chi_q(\omega_q) \sin \omega_q t \right]
\]

\[
= \frac{1}{2} \varepsilon \sum_q \left[ u_q \cos \omega_q t + v_q \sin \omega_q t \right].
\]

(22)

When a two-level system is excited by the strong trichromatic field the polarization induced by the probe-field can be obtained at the first order in \( \varepsilon \) by substituting the solutions of equations (5) into (22) and by selecting the items at the frequency \( \omega_q = \omega_w \) only.

With the help of (7) and (10) we obtain:

\[
u(t) = \frac{d \varepsilon}{\hbar} e^{-\Gamma_2 t} \int_0^t e^{\Gamma_2 t'} W(t') \sin \delta t' \, dt'
\]

(23)

\[
v(t) = \frac{d \varepsilon}{\hbar} e^{-\nu t} \left[ \left( I_3 + I_5 + \frac{\gamma}{\Omega_R} I_4 \right) \cos f(t) + \left( I_4 - I_3 - I_6 + \frac{\gamma}{\Omega_R} I_5 \right) \sin f(t) \right]
\]

(24)

where

\[
I_3 = \int_0^t e^{\nu t'} W(t') \cos f(t') \cos \delta t' \, dt'
\]

\[
I_4 = \int_0^t e^{\nu t'} W(t') \sin f(t') \cos \delta t' \, dt'
\]

\[
I_5 = \int_0^t e^{\nu t'} V(t') \sin f(t') \cos \delta t' \, dt'
\]

\[
I_6 = \int_0^t e^{\nu t'} V(t') \cos f(t') \cos \delta t' \, dt'.
\]

In (23), (24) we may use the only steady-state parts of \( V(t) \) and \( W(t) \), which are written in (16), since the substitution of their damped parts into (23), (24) results in damped parts also. Consequently, since one of conditions (12) \( t \to \infty \) is valid, we may ignore the rest conditions. Substituting (16)-(19) into (23) and (24) we obtain:

\[
\begin{aligned}
\frac{d}{dt} \varepsilon_0 \varepsilon \Gamma_1 \sum_{k, l} \sum_{i=1}^6 \left[ \Gamma_2 Q_{k, l}^i - \left( \delta + s_i^{k, l} \right) P_i^{k, l} \right] \frac{\sin \left( \delta + s_i^{k, l} \right) t}{\Gamma_2^2 + \left( \delta + s_i^{k, l} \right)^2} + \\
+ \frac{\Gamma_2 Q_{k, l}^i + \left( \delta - s_i^{k, l} \right) P_i^{k, l}}{\Gamma_2^2 + \left( \delta - s_i^{k, l} \right)^2} \sin \left( \delta - s_i^{k, l} \right) t - \frac{\Gamma_2 P_i^{k, l} - \left( \delta + s_i^{k, l} \right) Q_i^{k, l}}{\Gamma_2^2 + \left( \delta + s_i^{k, l} \right)^2} \cos \left( \delta + s_i^{k, l} \right) t + \\
+ \frac{\Gamma_2 P_i^{k, l} + \left( \delta - s_i^{k, l} \right) Q_i^{k, l}}{\Gamma_2^2 + \left( \delta - s_i^{k, l} \right)^2} \cos \left( \delta - s_i^{k, l} \right) t \right]
\end{aligned}
\]
where are determined in (25) and (26). The coefficients $a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij}$ in (26) are linear combinations of $a, b, c, d, e, f, g, h$. We shall not write these coefficients in an explicit form.

Equations (25), (26) enable us to reach the following conclusions. First, the weak probe field results in the appearence of another type parametric resonances than the Rabi-resonances, namely, the resonances at $\delta = 0$ and $s_{i}^{k,l} = 0$ for $u(t)$ and at $\Omega_{ijkl} = 0$ for $v(t)$. These resonances are not the Rabi-resonances since, for example, $s_{i}^{k,l} = 0$ for $k = l = 0 (i = 1)$, for $k = l = 1 (i = 2)$, i.e. these resonances do not depend on the Rabi frequency $\Omega_{R}$. Second, the weak-probe field results in the undamped oscillations of the density matrix elements both at the intermode distance sidebands and at the combination sidebands of $f_{i}^{2}$ and $f_{i}^{2R}$. This result is a consequence of the fact that equation (5) does not satisfy the Floquet theorem. Substituting (25) and (26) into (20) and taking into account (22) we obtain:

$$v(t) = \frac{d}{\hbar} \cdot e_{n_{0}} \Gamma_{1} \sum_{k,l,m=0}^{\infty} \sum_{i=0}^{6} \sum_{j=1}^{16} \frac{1}{\nu^{2} + (\Omega_{ijkl})^{2}} \times$$

$$\times \left[ a_{ijkl}^{k,l} + \gamma \right] \sin (\Omega_{ijkl} t - \Omega_{R} t + 2 \rho \sin \Omega t) +$$

$$+ \left( b_{ijkl}^{k,l} + \gamma \right] \sin (\Omega_{ijkl} t - \Omega_{R} t - 2 \rho \sin \Omega t) +$$

$$+ \left( c_{ijkl}^{k,l} + \gamma \right] \cos (\Omega_{ijkl} t - \Omega_{R} t + 2 \rho \sin \Omega t) \right]$$

where $(P, Q, s_{i}^{k,l})$ are determined in (17) and (19),

$$\Omega_{ijkl}^{k,l} = \begin{cases} \Omega_{R}^{k,l} & \text{for} \ k = l = 0 (i = 1), \text{for} \ k = l = 1 (i = 2), \text{i.e. these resonances do not depend on the Rabi frequency} \Omega_{R}. \end{cases}$$

The coefficients $(a, b, c, d, e, f, g, h)$ in (26) are linear combinations of $(A, B, \gamma)$ multiplied by $\Omega_{ijkl}^{k,l}$. We shall not write these coefficients in an explicit form.
Equations (28), (29) describe the total linear polarization of a two-level medium induced by the weak-probe field. Selecting from these equations the items which are attached to \( \sin \omega_c t \) and \( \cos \omega_c t \), we obtain the real and imaginary parts of the medium polarizability at the weak-probe field frequency, respectively. Because the coefficients \((a, b, c, d, e, f, g, h)\) are cumbersome, this selection is more convenient in numerical

\[
\frac{8 \hbar \epsilon_0}{d^3 \pi n_0 \Gamma_1} \sum_q \text{Im} \chi_q(\delta) \sin \omega_q t = \sum_{k,l} \sum_{i=1}^{6} \left\{ \frac{[\Gamma_2 Q_i^{k,l} - (\delta + s_i^{k,l}) P_i^{k,l}]}{\Gamma_2^2 + (\delta + s_i^{k,l})^2} \times \right\} \\
\times [\sin (\omega_w + s_i^{k,l}) t + \sin (\omega_w - 2 \omega + s_i^{k,l}) t] - \frac{[\Gamma_2 Q_i^{k,l} + (\delta - s_i^{k,l}) P_i^{k,l}]}{\Gamma_2^2 + (\delta - s_i^{k,l})^2} \\
\times [\sin (\omega_w - s_i^{k,l}) t + \sin (\omega_w - 2 \omega - s_i^{k,l}) t] \\
+ \frac{1}{4} \sum_{k,l,m,n} \sum_{i=1}^{6} \left\{ \frac{1}{(\Omega_{ij}^{k,l})^2} \right\} \\
\times [\sin (\Omega_{ij}^{k,l} + \Omega_R + \omega + 2 n \Omega) t + \sin (\Omega_{ij}^{k,l} + \Omega_R + \omega - 2 n \Omega) t] \\
- \sin (\Omega_{ij}^{k,l} + \Omega_R - \omega + 2 n \Omega) t - \sin (\Omega_{ij}^{k,l} + \Omega_R - \omega - 2 n \Omega) t] \\
+ J_{2n+1} \left[c_i^{k,l} + \frac{\gamma}{\Omega_R} g_i^{k,l}\right] [\sin (\Omega_{ij}^{k,l} + \Omega_R + \omega + (2 n + 1) \Omega) t] \\
- \sin (\Omega_{ij}^{k,l} + \Omega_R + \omega - (2 n + 1) \Omega) t - \sin (\Omega_{ij}^{k,l} + \Omega_R - \omega + (2 n + 1) \Omega) t] \\
+ c_n J_{2n} \left[d_i^{k,l} + \frac{\gamma}{\Omega_R} h_i^{k,l}\right] [\sin (\Omega_{ij}^{k,l} - \Omega_R + \omega + 2 n \Omega) t] \\
+ \sin (\Omega_{ij}^{k,l} - \Omega_R + \omega - 2 n \Omega) t - \sin (\Omega_{ij}^{k,l} - \Omega_R - \omega + 2 n \Omega) t] \\
- \sin (\Omega_{ij}^{k,l} - \Omega_R - \omega + 2 n \Omega) t] \\
+ J_{2n+1} \left[d_i^{k,l} + \frac{\gamma}{\Omega_R} h_i^{k,l}\right] [\sin (\Omega_{ij}^{k,l} - \Omega_R - \omega + (2 n + 1) \Omega) t] \\
- \sin (\Omega_{ij}^{k,l} - \Omega_R + \omega - (2 n + 1) \Omega) t - \sin (\Omega_{ij}^{k,l} - \Omega_R + \omega + (2 n + 1) \Omega) t] \\
+ [\sin (\Omega_{ij}^{k,l} - \Omega_R + \omega - (2 n + 1) \Omega) t] \\
+ J_{2n+1} \left[d_i^{k,l} + \frac{\gamma}{\Omega_R} h_i^{k,l}\right] \\
\right\}.
\]

Equations (28), (29) describe the total linear polarization of a two-level medium induced by the weak-probe field. Selecting from these equations the items which are attached to \( \sin \omega_c t \) and \( \cos \omega_c t \), we obtain the real and imaginary parts of the medium polarizability at the weak-probe field frequency, respectively. Because the coefficients \((a, b, c, d, e, f, g, h)\) are cumbersome, this selection is more convenient in numerical
Figures 1 and 2 show the absorption and dispersion profiles (Im $\chi_w(\delta)$ and Re $\chi_w(\delta)$) respectively at the following parameters: $\Gamma_2/\Gamma_1 = 10^3$; $\Omega_R = 10$ $\Gamma_2$ (a) and 20 $\Gamma_2$ (b); $\Omega_R = \Omega$. Summation on $k, \ell, m, n$ are carried on from 0 to 7. It is expected that

Fig. 1. — The absorption profiles of the weak-probe field for the cases: $\Gamma_2/\Gamma_1 = 10^3$; $\Omega_R = \Omega$; $\Omega_R = 10$ $\Gamma_2$ (a) and 20 $\Gamma_2$ (b).

Fig. 2. — The dispersion profiles of the weak-probe field for the same choice of parameters used in figure 1.
this restriction does not affect significantly the results. It is seen that the absorption and dispersion manifest the extreme behaviour when $\delta/\Omega_R$ is near by $0; \pm 1; \pm 2; \pm 3; \pm 4; \pm 5$ (note that $\text{Im} \chi_w(\delta) = \text{Im} \chi_w(-\delta)$; $\text{Re} \chi_w(\delta) = -\text{Re} \chi_w(-\delta)$). In the vicinity of these points the dispersion profile resembles the absorption profile of the monochromatic field. This result was obtained in [28] for trichromatic exciting field at $\Omega_R/\Gamma_2 = 50$ and 100, and also in [22] for strong bi-chromatic field at $\Omega_R/\Gamma_2 = 4, 6$.

So we showed that the application of the matrix exponent method to the solution of the Bloch equations allows us to obtain the analytic solutions in most general cases as compared with the Floquet theorem method. Note that there is another method of the solution of the Bloch equations. Prants [36] developed recently the theoretic-group method for solving the Bloch equations. He obtained the analytic solutions for arbitrary time modulation of the field amplitude and phase but in the case of equal relaxation times $\Gamma_1 = \Gamma_2$.

References

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