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Vibrations in regular and disordered fractals: from channeling waves to fractons

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Résumé. — On présente des résultats sur les propriétés de vibrations d’un tapis de Sierpinski obtenus à partir de simulations numériques. Du calcul de la densité d’états on obtient la dimension spectrale $d = 1.6$ ; on observe aussi la présence de singularités associées aux bords de zones de Brillouin du réseau de Bravais sous-jacent. Puis, on présente des cartes de modes où l’on voit un nouveau type d’onde « canalisé » à faible désordre qui devient localisé (au sens d’Anderson) à fort désordre. Ceci est reflété par la variation non monotone du taux de participation en fonction du désordre. De l’analyse des cartes de modes, nous n’avons trouvé aucun indice en faveur de la conjecture de « superlocalisation ». Cette étude montre que le concept de fracton « universel » ne peut décrire les deux types de vibrations observés.

Abstract. — Computer simulation of the vibrational properties of a Sierpinski carpet are reported. From the computed density of states, we find the spectral dimension $d = 1.6$ and the presence of singularities attributed to edges of the Brillouin zones of the underlying Bravais lattice. Then, we present mode patterns showing a new form of « channeling wave » at weak disorder becoming Anderson-localized at strong disorder. This is reflected in a non-monotonic variation of the participation ratio as a function of disorder. Analysing the mode patterns, we find no evidence supporting the conjecture of superlocalization. This study shows that the concept of a universal fracton is not appropriate to describe the two types of vibrational excitations we observe.

1. Introduction.

Atomic vibrations in disordered networks have received considerable attention this last decade. In the case of a lattice with a random distribution of masses, one observes basically two features: the low frequency spectral density of modes still varies as $\omega^{d-1}$ ($d$: the dimension of the embedding space) like in a perfect crystal whereas the Van Hove singularities are rounded off. At low frequencies, this insensitivity to disorder of the spectral density is attributed to the wavelength becoming longer than the length scale associated with the disorder. In spite of this, mode patterns are affected in a very dramatic way. For most cases — in one and two dimensions or above a frequency threshold in three dimensions — the eigenmodes of vibration become localized in the sense that the spatial envelope of their...
amplitudes decreases exponentially with a characteristic length known as the localization length \( \xi \).

In fractals, disorder exists at any length scale affecting more deeply the spectral density. The presence of holes of all sizes changes the spectral density function into a power law of the type \( \omega^{d-1} \) where \( d \) is the spectral dimension \([1, 2]\). Indeed, when the fractal structure exists only inside blobs of size \( L \), one expects a cross-over at low frequency from \( \omega^{d-1} \) to \( \omega^{d-1} \) when the wavelength becomes comparable to \( L \). Concerning the three characteristic lengths of the vibrating modes in disordered networks — namely the amplitude correlation distance \( \lambda (\omega) \) or « wavelength », the elastic mean free path \( \ell (\omega) \) and the localization length \( \xi (\omega) \) — it has been conjectured \([3]\) that they are of the same order of magnitude and obey the same scaling law. The concept of fractons has emerged in this context: a rarefied spectral density in \( \omega^{d-1} \) at low frequency (\( d < d' < d \); \( d' \) the Hausdorff dimension), a pseudo-dispersion relation \( \lambda (\omega) \) and a localization length \( \xi (\omega) \). Actually, it has also been suggested \([4, 5]\) that the fractons are « superlocalized » instead of simply localized. This effect of superlocalization would be embedded in a peculiar spatial variation as \( \exp \left\{ \left( -\frac{r}{\xi (\omega)} \right)^{a} \right\} \) instead of the standard exponential \( a = 1 \) associated with the Anderson localization. The physical justification of this conjecture is that the asymptotic exponential decay follows the tortuous path on the fractal network instead of the straight lines of the underlying embedding euclidean space.

The first analysis of fractons were performed on the infinite percolation cluster which is an archetype of random fractal structures. By analogy to the classical random walker on percolating networks, the spectral density of the vibrations of such structures was obtained as \( \omega^{d-1} \) where \( d \) is very close to \( 4/3 \) for any embedding space dimension. Recently the percolation model has received great attention \([6, 7]\) in order to test the main predictions on fractons. In a numerical study of the percolation model in 2 dimensions, deVries et al. \([6]\) have obtained the frequency dependence of the participation ratio and the pseudo-dispersion relation \( \lambda (\omega) \propto \omega^{-d'/d} \) with \( d' \) close to \( 4/3 \). Considering the spatial decay, they found that the « superlocalization » exponent cannot be larger than 1.1. In another study, Yakubo and Nakayama \([7b]\) have performed numerical experiments on the percolating elastic networks in 2 and 3 dimensions. They also found a value of \( d' \) very close to \( 4/3 \). Concerning the envelope they found a value of \( a \approx 2.3 \) for the ensemble averaged fracton which is not consistent with any theoretical prediction.

The concept of fracton \([1]\) was born in the context of the percolation model: its generality or universality is questionable. In view of testing this concept in a different random fractal network, we have studied the vibrations of a Sierpinski carpet embedded in 2 dimensions. Considering a series of such networks for which we increase the mass disorder, our method offers the possibility of studying the cross-over from a fractal network (with equal mass on each site) to one dominated more by the strong local disorder (corresponding to the Anderson model of localization) than by its fractal aspect. These networks differ markedly from the random bond networks of the percolation model in the sense that all the elastic path through bonds are closed and reticulated. The absence of dangling bonds or dead ends, which are very numerous close to the percolation threshold, indicates that the Sierpinski carpets are better suited for a description of the structure of very porous materials like the silica aerogels. In fact, from the electronic microscope observation \([8]\) and the low temperature thermal properties \([9]\) it has been proposed to model the structure of silica aerogels as a hierarchical sponge of Mengel type with full reticulation and fractal structure. The two dimensional Sierpinski carpet can be considered as a simplified version of this kind of porous material which retains the main features of the structure: hierarchy and full reticulation.
A second important simplification is introduced in this numerical study: it concerns the scalar elasticity; only the scalar displacement of the atoms are considered. One knows that the true elasticity described by elastic tensors can change the value of the spectral dimension $\bar{d}$ [10, 11]; however, since this study is rather devoted to qualitative description of fractons than numerical determination of elastic parameters we restricted ourselves to the approximation of scalar elasticity.

The model that we used will be presented in section 2. It is based on the numerical integration of the equations of motion of the atoms under the effect of externally applied forces at given frequency $\Omega$ (see Williams and Maris [12]). Two variations of the algorithm have been used:

- a global algorithm where a random force is applied on each particle. Under certain conditions the rate of increase of the total energy will be proportional to the spectral density $g(\Omega)$. The exact procedure and the results will be presented in section 3;
- a new local algorithm where an external force is applied on a single site during a finite time. The localization of the energy around the excited site can be studied by this local algorithm. Procedure and results will be presented in section 4.

Sections 5 and 6 will be devoted to the analysis of the mode patterns obtained with the local algorithm. In section 5 we will analyse the patterns in terms of a participation ratio and show how this ratio varies with disorder for different systems. In section 6 we will analyse the envelope of the patterns for strongly disordered systems. We will compare the classical exponential localization law to the « superlocalization » law.

The main results obtained by the « global » algorithm on the spectral density of states are the following. The spectral dimension of the Sierpinski carpet (Fig. 1) is obtained; $\bar{d} = 1.6 \pm 0.1$. Steps in the curve $g(\Omega)$ subsist even with important mass disorder. They are signatures of the underlying partial « Bravais » lattice built-up with punctured cells. These Van Hove structures in the spectral density are located at the edge frequencies of this Bravais lattice. The steps are relatively robust and disappear only for strong disorder.

Fig. 1. — The Sierpinski Carpet used in our computer simulation. The size of the sample is shown by the dotted line. Periodic boundary condition are used.
With our local algorithm we found some surprising results. For a relative mass fluctuation $\sigma$ less than 0.1, we observed a strong channeling of the excited mode packets along the lines of dense matter of the network. Inside the «tubes», the mode packets are delocalized as it is exhibited by the patterns presented in the figures. This behaviour is perhaps the most important result of this work and does not seem to have been reported previously. Another unexpected feature is the non-monotonic variation of the participation ratio as a function of the disorder parameter $\sigma$. With increasing disorder, the participation ratio first increases, passes through a maximum at $\sim 0.1$ then decreases sharply. In the regime of increasing participation ratio, it is observed that the energy spreads out of the «tubes» into the transverse direction. When the disorder becomes large enough to localize the vibration in the arms or «channels», the energy becomes isotropically distributed leading to an Anderson like localization. This behaviour has no analogue in the case of euclidean lattices.

Concerning the energy distribution in strongly disordered fractal systems, our main conclusion is that the envelope is not different from the standard one observed in euclidean space.

In the conclusion, we will emphasize the differences between the fractons of the percolation network and the present observation of another type of fractons in weakly disordered reticulated structures which propagate along tubes of dense matter. This new type of excitations — guided or channeling modes — would be important in the analysis of the transport properties of fully reticulated structures like possibly the silica aerogels.

2. Model.

We consider a network of $N$ atoms connected by linear springs between nearest-neighbours [7, 12]. The mass at each site is $m_i$; random masses are used for studying disorder. Various networks have been considered in this study: linear chains and square lattices (for the purpose of testing our different procedures) as well as a fractal structure of the Sierpinski type (Fig. 1). The displacement of the $i$-th-particle is a scalar quantity $u_i$. On each site an external force $F_i(t)$ is applied leading to the following equations of motion:

$$m_i \ddot{u}_i(t) = - \sum_j \phi_{ij} u_j(t) + F_i(t)$$

$$\phi_{ij} = \begin{cases} -z_i \text{ for } i = j, \text{ coordination number at site } i, \\ 1 \text{ for } i \text{ nearest-neighbour of } j, \\ 0 \text{ otherwise.} \end{cases}$$

We then proceed to the numerical integration of these equations. In this paper we present results from two types of external forces $F_i(t)$.

First, the density of states can be obtained using an oscillating force of random amplitude on each site. The advantage of this method has been discussed in the original article of Williams and Maris [12]. Results and analysis are presented in section 3.

Second, localization properties can be investigated using an external force applied on a single site (Sect. 4). This second procedure can easily be adapted for the study of more complicated systems such as systems where the anharmonicity is taken into account.

The numerical studies were performed on samples of 4096 atoms. Although the size is relatively small, it turns out to be sufficiently large for exhibiting specific features of fractons.

The average mass $m_0$ is adjusted so that the period $T_0$ of the characteristic frequency $(\omega_0 = \sqrt{1/m_0})$ is about 100 units of time (1 unit of time correspond to one time-step in the integration of the equations of motion). The largest frequencies of our samples will be
\( \sqrt{8} \omega_0 \) (which is the Debye frequency for the ordered square lattice) so that the shortest period of oscillation will be about 35 time units.

Our simulations were performed on an experimental parallel computer built up at the CRTBT. The computer is composed of 40 interconnected MC68000 based processor boards each with 256 K of ram memory. The machine is driven via another similar board running the OS-9 operating system.


To obtain the density of states (DOS) we apply on each site a force \( F_i = F_0 \sqrt{m_i} \cos (\phi_i) \cos (\Omega t) \) with \( \Omega \) the excitation frequency, \( \phi_i \) a uniformly distributed random number, \( m_i \) the mass at site \( i \) and \( F_0 \) a constant.

As it is proven in [12], if one averages over different configurations \( \{ \phi_i \} \), the mean energy \( \langle E(\Omega, T) \rangle \) after time \( T \) will be related to the normalized density of states by

\[
g_T(\Omega) = \frac{8 \langle E(\Omega, T) \rangle}{\pi TF_0^2 N} \quad \text{where } N \text{ is the total number of sites}.
\]

\( T \) is a free variable in this expression. To this time \( T \) corresponds a spectral width \( \Delta \Omega = 2 \pi / T \) of the excitation force. The modes within the range \( \Omega \pm \Delta \Omega \) will be resonant. If the number of these modes is large, then a single choice of the random \( \{ \phi_i \} \) will give a representative value of \( \langle E(\Omega, T) \rangle \). Then \( g_T(\Omega) \) will be representative of the average density of states over the resonant frequency range \( \Omega \pm \Delta \Omega \). \( T \) should be short enough so that the number of excited modes in the range \( \Omega \pm \Delta \Omega \) be sufficiently large but also \( T \) should be long enough so that \( \Delta \Omega \) be small compared to \( \Omega \) itself. If both conditions are satisfied then the values taken by \( g_T(\Omega) \) become independent of \( T \). Averaging 20 configurations \( \{ \phi_i \} \), we found reliable results when \( T \) was such that the inequalities \( N g_T(\Omega) \Delta \Omega \geq 20 \) and \( \frac{\Delta \Omega}{\Omega} \leq \frac{1}{10} \) were satisfied (\( Ng_T(\Omega) \Delta \Omega \) is actually the number of resonant modes).

Figure 2 shows the result of this procedure for numerical determination of the DOS of a square lattice along with a plot of the analytical solution [13]. The accord between the two curves is excellent except at low frequency where the number of modes in the frequency window becomes too small. This proves the validity of the procedure itself as well as the validity of the numerical integration of the equations of motion.

Next we show the result for the DOS of a Sierpinski carpet (black squares in Fig. 3). Although more irregular than the preceding curve (despite an average made over 20 configurations \( \{ \phi_i \} \)) some global features are preserved: the high frequency cut-off is close to \( \omega / \omega_0 = \sqrt{8} \) which is the cut-off of the square lattice. Also the DOS passes by a maximum for \( \omega / \omega_0 \) close to 2. We note however that the cut-off is smoother for the case of the carpet and that the « bump » corresponding to the logarithmic Van Hove singularity is wider.

A new feature of the Sierpinski carpet DOS is the presence of « edge-singularities ». We can compare these « edges » with the ones that appear in the DOS of a Bravais lattice built-up with unit-cells identical to the lowest level Sierpinski carpet cell (a \( 3 \times 3 \) square cell with a hole in the middle) (solid line Fig. 3). We attribute these singularities to edges of the Brillouin zone of this Bravais sub-lattice.

From the low-frequency DOS we can obtain an estimate for the \( \tilde{d} \) of this lattice. By fitting a power law \( \omega^{d-1} \) up to \( \omega / \omega_0 = 0.62 \) we get \( \tilde{d} = 1.6 \pm 0.1 \) which is different from the percolation \( \tilde{d} = 4/3 \).
Fig. 2. — Density of states of a square lattice (dots). $T$ is adjusted so that $\Delta \Omega \left(= \frac{2\pi}{T}\right)$ is equal to $\frac{\omega_{\text{max}}}{40}$. The vertical scale is proportional to the normalized density of states $g_T(\Omega)$. We have actually plotted $Ng_T(\Omega)\Delta \Omega$ which is the number of resonant modes associated with each point on the curve. At low frequency the observed deviation is due to the small number of modes in the frequency window. Solid line is the analytic solution [13] for the density of states of the infinite network

$$\frac{\omega}{\pi^2} K \left( \frac{\omega^2(8 - \omega^2)}{16} \right)$$

where $K$ is the complete elliptic integral of the 1st kind.

Fig. 3. — Density of states of Sierpinski carpet (black squares). Solid line is the DOS of the Bravais sub-lattice (see text). Edge-singularities in the carpet DOS are due to edges of the Brillouin zone of the Bravais sublattice.

Fig. 4. — Density of states for a) square lattice with $\sigma = 0$ (+ signs) and $\sigma = 0.10$ (square dots) and b) Sierpinski carpets with $\sigma = 0$ (dotted squares) and $\sigma = 0.10$ (round dots). The effect of disorder is to levels the peaks and valleys.
We have also studied the effect of disorder. From the comparison of the DOS of ordered and disordered square lattice and Sierpinski carpets (Figs. 4a and b) we see that disorder effectively levels off the peaks and valleys in the DOS. We note however that this effect is much more effective at high frequencies and that, despite the disorder, small peaks persist at low frequencies.

4. Localization properties: the local algorithm.

We have used a new method in which a single site experiences a force at frequency $\Omega$. It is expected that the excitation will spread on the lattice sites up to a distance comparable with the localization length $\xi$.

The exact time profile of the force is $\exp(-t^2/2\sigma^2) \cos(\Omega t)$ with $\sigma = 3\sqrt{2}\pi/\Omega$. It implies that a mode packet of relative width $\frac{\sigma\Omega}{\Omega} = \frac{1}{3\sqrt{2}\pi} \approx \frac{1}{10}$ around $\Omega$ is excited. Thus, there are about $N/10$ modes within this frequency window. However, for localized systems, not all those $N/10$ modes will be excited by the local force. If the localization length at frequency $\Omega$ is $\xi$, then a mode encompasses $\xi^d$ sites on which the amplitude is non-negligible. The probability for a mode to be excited by a local force is then $\xi^d/N$ (assuming a uniform distribution of the localized modes in space). The number of excited modes becomes:

$$\frac{N \xi^d}{10N} = \frac{\xi^d}{10}$$ independent of $N$.

We obtain information on the localization properties of a packet of modes at frequency $\Omega$ located close to the excited site. We see that the study of localized excitations does not require large networks since the number of excited modes is independent of $N$. We also note that, as the modes become more localized, the number of excited ones becomes smaller. Our method differs from the one proposed by Williams and Maris [12] where a single resonating mode is isolated. But since we expect $\xi$ to be a smoothly varying function of frequency, some relevant information can be obtained by this method. In fact the characteristic length of the wave packet should reflect the localization length of the individual modes present in the packet. Moreover, properties of wave packets should be more robust than properties associated with a single mode.

In order to study the spreading excitations, we define an energy per site $i$ at the $u$-th period $e_i(u)$ which is the average over one period of $\Omega$ of the instantaneous energy of the site. This average insures that, for instance, for a pure plane wave $e_i(u)$ is constant for all $i$. However, for our excitations, the $e_i(u)$ can fluctuate in time due to beating between the excited modes. In order to smooth out these fluctuations, we average additionally the $e_i(u)$ over 100 periods after a steady state regime has been obtained. From these $\langle e_i \rangle$ we calculate either a participation number (Sect. 5) or a localization length (Sect. 6); we will also draw 3 dimensional « maps » (2 dimension for site space and a third for the energy) of the energy distribution in space.

Let us first start with selected maps of square lattices and Sierpinski carpets. For disordered systems we give the value of $\sigma$ the rms of the mass distribution. All our samples are submitted to periodic boundary conditions. The actual size of each sample is indicated in the figures by the dotted lines. Except figure 7 the excitation site is located in the centre of the map. For figure 7 the excited site is seen 4 times, one of them indicated by the arrow.

We first present an extended mode packet on the square lattice for the frequency $\Omega = 2.26$ when $\sigma = 0$ (Fig. 5a). For the disordered square lattice with $\sigma = 0.4$ at the same frequency (Fig. 5b), one has an Anderson-localized state.
Fig. 5. — a) Extended excitation for an ordered square lattice ($\sigma = 0$) at frequency $\frac{\Omega}{\omega_0} = 2.26$.

b) Anderson-localized state for a disordered square lattice ($\sigma = 0.4$) at the same frequency.

For the Sierpinski carpet at the same frequency, figure 6a exhibits the mode pattern for $\sigma = 0$. The striking feature of this map is the anisotropy. The mode packet is not fully localized in space since important region are forbidden. We call these types of excitations «channeling modes». The energy is distributed along dense lines or «tubes» of the
Fig. 6. — Excitations for Sierpinski carpets at $\frac{\Omega}{\omega_0} = 2.26$ with increasing disorder. The excitation site is located in the middle of the figures. a) $\sigma = 0$, channeling wave; b) $\sigma = 0.01$, spreading of the energy into the transverse directions; c) $\sigma = 0.046$, isotropic excitation; d) $\sigma = 0.4$ localization sets in.
structure. If the excited site is away from the centre of the maps (Fig. 7), the «channeling» subsists in corridors or even in closed loops of the samples.

When we put a small amount of disorder \( \sigma = 0.01 \) at the same frequency one can observe (Fig. 6b) that the energy has leaked into directions transverse to the «tubes» and is now more evenly distributed. This «spreading» of the energy can be understood as the effect of scatterings of the waves by the mass defects introduced in the primary channels.
For $\sigma = 0.046$, scattering starts to dominate channeling and the energy distribution becomes isotropic, centred around the excited site (Fig. 6c). In this intermediate regime, the distribution of the energy is more uniform (than in the previous cases) : this is a paradoxal effect of the disorder.

For higher values of $\sigma$ we see that the localization phenomenon has settled in ($\sigma = 0.4$, Fig. 6d). Disorder has killed channeling. This last picture is not very different from the map of a disordered square lattice as in figure 5b.

5. Participation number.

The participation number $\mathcal{P}$ is a standard parameter for the description of localization in disordered systems. It is defined by the ratio

$$\mathcal{P} = \frac{\left( \sum \varepsilon_i \right)^2}{\left( \sum \varepsilon_i^2 \right)},$$

where the sums run over all the sites of the network. For instance, if the total energy is concentrated on a single site then $\mathcal{P} = 1$ ; on the other hand, if the energy is distributed among all sites then $\mathcal{P} = N$. Analysis in terms of a participation number is easier to compute and is always valid whatever the shape of the wave function is. In particular, one does not need an assumption of isotropy of the wave function as in the case of a localization length analysis. Indeed, the participation number does not give any information on the geometry of the wave function.
Figure 8 shows the participation number $\mathcal{F}$ versus $\sigma$ the rms of the mass distribution for a square lattice and for a Sierpinski carpet at different excitation frequencies $\Omega$.

For the square lattice, $\mathcal{F}$ decreases monotonically from a delocalized state at weak disorder to a localized one at large values of $\sigma$.

Now for the fractal structure we observed that the introduction of a small amount of disorder creates a spreading of the energy from the channels to secondary tubes of the network. This is reflected in the non-monotonic variation of the $\mathcal{F}$ vs. $\sigma$ curves. Starting at $\sigma = 0$, where the excitations are « channeled » and $\mathcal{F}$ small, one sees that the introduction of a small amount of disorder tends first to increase $\mathcal{F}$. Then, as $\sigma$ gets larger, $\mathcal{F}$ passes through a maximum which corresponds to a transition to isotropy. The standard localization of the Anderson type is recovered at larger values of $\sigma$ where $\mathcal{F}$ decreases strongly. Note that the values of $\mathcal{F}$ on the carpet are smaller than the values of $\mathcal{F}$ on the square lattice in the corresponding case ($\Omega = 2.26$). Obviously, the holes of the fractal network introduce additional disorder as compared to the euclidean lattice. But ultimately when $\sigma$ gets very large so that the localization length becomes of the order of a few interparticle distances one does not expect any difference between fractal and euclidean structures.

![Participation number vs. sigma](image)

Fig. 8. — Participation number $\mathcal{F}$ vs. $\sigma$ for square lattice and Sierpinski carpets at different frequencies (the number of sites on the networks in 4096 for both lattices). $\mathcal{F}$ for the carpet is non-monotone.

6. Strong disorder and « Superlocalization ».

In this section we do a quantitative analysis of our mode patterns obtained by the local algorithm of section 4.

We have seen that the mode patterns of the Sierpinski carpet at weak disorder are not localized but channeled along « dense » tubes of the structure. This means that we cannot define an envelope of the amplitude squared depending only on $r$ the distance from the centre of excitation. Thus, for a weakly disordered carpet, the envelope of the amplitudes squared is not an Anderson-type localization law of the form $\Psi(r)^2 = A_0^2 e^{-2(r/\xi)}$ by reason of anisotropy.
However, at strong disorder the energy distribution is localized and isotropic. This is reflected in all the samples we have studied. We can then average the energy distribution (amplitudes squared) over different samples. In what follows we have done averages of 20 to 40 samples of the same \( \sigma \). Let us call the averaged energy on each site \( \langle e_i \rangle \).

We can then construct a function \( \Psi (r)^2 \) representing the envelope of this averaged energy distribution. We proceed as follows: we count the number of sites \( N_C \) for which the average energy \( \langle e_i \rangle \) is larger than some threshold value \( C \). From \( N_C \) we evaluate an effective radius \( r_{eff} \) which is equal to

\[
- \frac{N_C}{2} \quad \text{for a linear chain ,}
- \sqrt{\frac{N_C}{\pi}} \quad \text{for a square lattice and}
- (N_C)^{1/d} \quad \text{for a Sierpinski carpet .}
\]

Inverting these relations we get \( C(r_{eff}) \) which is then assimilated with \( \Psi (r)^2 \) (under the assumption of isotropic energy distribution). The resulting modes or envelopes are analysed in terms of a generalized exponential law

\[
\Psi (r)^2 = A_0^2 e^{-2(r/\xi)^a} .
\]

This type of « super-localization » law with \( a > 1 \) has been proposed [4, 5] in the context of the problem of localization of eigenmodes on fractals. For \( a = 1 \) one gets the classical Anderson-type localization law \( \Psi (r)^2 = A_0^2 e^{-2(r/\xi)} \) valid for a euclidean lattice.

In our approach of this problem, it is important to note that we study the localization of a wave packet. \textit{A priori} the exponent « \( a \) » associated to a packet could differ from the one of the constituent eigenmodes for two reasons: first, because of the finite spectral width, an second because of the spatial distribution of the centres of the eigenmodes. However, at large distance from the excited site, the spatial decay of the wave packet must reflect the decay of the components of the packet. In other words, if the amplitudes of each individual modes of a packet decreases asymptotically faster than an exponential \( (a>1) \) then the superposition of those modes must also necessarily decrease in the same way.

We have tried to fit all three parameters \( A_0, \xi \) and \( a \) from our \( \Psi (r)^2 \). Now, this law is valid only asymptotically for \( r/\xi \gg 1 \). But since there is no « \textit{a priori} » indication about the interval of validity, one has to choose arbitrarily a starting point for \( r/\xi \). We have found that this starting point influences greatly the outcome parameters. We did not succeed at finding a coherent scheme to manage a three parameter fit of the superlocalization law.

However, when one fixes \( a = 1 \), the starting point becomes easy to identify on a log-log plot of \( \Psi (r)^2 \). Moreover the two outcome parameters \( A_0 \) and \( \xi \) are much less dependent on this starting point. We found that \( (\Psi (r/\xi)/A_0)^2 \) obeys a universal law which is valid for the linear chain, the square lattice as well as the Sierpinski carpet (Fig. 9). This law is nothing else than the exponential envelope of the « classical » localized modes.

Our results can be understood as evidence against the idea of superlocalization on a fractal of the Sierpinski type. For a possible interpretation of this result, we return to one of the original argument invoked to support the idea of superlocalization [4]. The argument is that the damping of the mode envelope follows paths \textit{on the network} and not on the embedding space. This causes an increased length of the paths which would be responsible for an additional attenuation leading to \( a > 1 \). The exponent \( a \) would then be related to the chemical length exponent \( \xi_c \) which itself relates the length between 2 points when one travels on the
network to the euclidean distance between these points. Now for a geometrical fractal of the Sierpinski type, the paths must go around holes which are never larger than the path lengths themselves; this leads to $\xi_c = 1$ [14] thus to $a = 1$ [5, 15]. Superlocalization probably does not exist on regular fractals like the Sierpinski carpet.

7. Conclusion.

The main originality of this work lies in the discovery of the strong anisotropy of the distribution of the energy for the mode packets in the Sierpinski carpet at weak disorder. Some previous claims concerning the fractons must be revised after this work in particular that the fractons have universal behaviour whatever the fractal network is. Here, we have found that the excitation modes of a Sierpinski carpet can be divided in two regimes of localization:

— at weak disorder, a regime of channeling of the excitation energy along the « tubes » or closed loops of dense matter. In this regime the wave propagates with finite group velocity but only within certain corridors in space. We have not studied the frequency dependence of the width of the channels although we have observed that this width increases as the frequency decreases. This variation could be related to some pseudo-dispersion relation $\lambda(\omega)$ mentioned in the introduction;

— at strong disorder, an isotropic energy distribution is restored. This corresponds qualitatively to the standard picture of fractons as strongly localized waves.

A consequence of these observations is that the participation ratio does not decrease monotonically with disorder. The passage from channeling modes to fractons increases the participation ratio due to the « isotropisation » of the energy distribution.

As far as we know, both these effects had not been reported until now (except for some of our preliminary results [16]). We believe that they are general and could exist for vectorial
displacement fields as well as 3 dimensional Sierpinski-type networks. The importance of this remark is related to the possible modelization of silica or mineral aerogels as a hierarchical structure of Sierpinski type. The description of the phonons in terms of channeling modes should throw new light on the transport properties of these materials. Since the group velocity of these modes is well defined as the derivative of the dispersion relation and that the mean free path of the « guided » modes can be derived by an argument of the Casimir type, the transport coefficients (such as the thermal conductivity) obtained from a standard kinetic theory of gases give a good description of the relevant experiments. Moreover, a new law of scattered Raman intensity can be deduced from this new class of excitations. We believe that this study will help to deepen the analysis of the transport measurements of this class of materials.

References