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Non-linear behavior of convection in a vertical cylindrical cell at high aspect ratio

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Résumé. — Nous présentons ici des résultats expérimentaux sur la convection dans des cellules verticales cylindriques au-delà du seuil de convection. Nous avons étudié, dans un large domaine de différence de température $\Delta T$ entre le bas et le haut de la cellule, l'évolution de certaines structures sélectionnées. Lorsque $\Delta T$ augmente, le système présente d'abord des oscillations de relaxation à très basse fréquence, puis des oscillations périodiques de plus haute fréquence. Pour le premier type d'oscillations, nous proposons une explication basée sur une relaxation entre deux modes convectifs par un mouvement spiralé. Pour des valeurs de $\Delta T$ suffisamment élevées, le système transite vers un chaos temporel et, enfin, perd sa cohérence spatiale, caractéristique des deux régimes précédents.

Abstract. — We present here experimental results on convection in high aspect ratio vertical cylindrical cells well above the onset. We observe the evolution of some selected states in a wide range of the temperature difference $\Delta T$ between bottom and top of the cell. We bring evidence that, as $\Delta T$ increases, the system undergoes low frequency relaxation oscillations, then exhibits higher frequency periodic oscillations. For the first type of oscillations, we suggest an explanation based on a relaxation between two convective modes mediated by a spiraling movement. For sufficiently higher values of $\Delta T$, the system goes to a temporal chaotic state, then loses its spatial coherence, an essential feature of the two previous regimes.

1. Introduction.

Much experimental and theoretical work [1] has been done on convection instabilities, i.e. on convective systems run well above the convection onset and which present numerous time-dependent phenomena associated with the non-linearity. The geometry we choose to study here, a vertical cylinder with large aspect ratio $\eta$ (i.e. the ratio of height on radius), presents some interesting characteristics. Independently of its implication in many phenomena like geysers, atmospheric circulation, and crystal growth, it is an extended system and thus particularly adapted to the study of spatiotemporal chaos. The literature contains few experimental works in this geometry and most were done in systems with relatively low aspect ratio $\eta$ [2, 3]. In this paper, we work at values of $\eta$ equal to 11 and 32, which give complementary results.
Few theoretical studies [4, 5] were concerned with the geometry and non-linear conditions of our experiments. That is to be contrasted with the classical Rayleigh-Bénard geometry, a flat box with an aspect ratio smaller than 1, where much theoretical work was done both at high and low Prandtl numbers.

A characteristic feature of the geometry we chose consists in the possible occurrence of several convection modes for a given temperature gradient. Such a situation creates many experimental opportunities, but also difficulties. Nevertheless, one can observe as a function of the Rayleigh number the evolution of some selected states in a wide enough range. By increasing slowly the temperature gradient we were able to follow our system starting from the convective stationary state, undergoing relaxation oscillations between two states, then presenting periodic oscillations, and finally going to chaos. Such observations were made at two different aspect ratios $\eta = 32$ and $\eta = 11$ with water as the convecting fluid.

Some preliminary experiments were made on ethylene glycol, with $\eta = 32$ to study quantitatively the convection threshold.

2. Experimental set-up.

We have built two experiments: the first one was designed for visualization studies and the other for quantitative investigation of the dynamics of the convective flow. Both experimental systems are based on a vertical cylindrical cell of fluid limited by two high thermal conductivity copper plates; the upper plate is held at constant temperature $T_0$, while the lower one heated up at constant power (see Fig. 1) reaches at equilibrium $T_0 + \Delta T$ ($\Delta T > 0$).

2.1. VISUALIZATION EXPERIMENTAL CELL. — The cell consists of a vertical cylindrical transparent container made of highly conducting material (sapphire); its aspect ratio is equal to 21. We use water as the working fluid. On top of the sapphire cylinder, a massive copper piece is thermally regulated by thermostated water circulation. On the bottom of the cylinder, a thick copper piece is electrically heated. A couple of platinum resistances, located on top and bottom, allows for measuring the temperature difference between the copper plates. A solution of Kalliroscope (suspension of microscopic crystalline flakes) is added to the water in order to visualize the flow.

2.2. EXPERIMENTAL CELL FOR DYNAMICAL STUDY. — In the second experiment, the convection cell is realized in a cylinder of low thermal conductivity material (P.V.C.), its external diameter is 50 mm and the working fluid fills an internal bore of 10 mm diameter. The top and bottom of the cells are copper circular flanges. On the top, an interchangeable copper plunger allows for varying the height of the cell, leading to an aspect ratio $\eta$ (height over radius) ranging from 10 to 40. The temperature of the top copper piece, 30 mm thick, is imposed by means of an internal circulation of thermally regulated water coming from a cryothermostat of high capacity, whose temperature is maintained near 28 °C. The regulating system of the water bath imposes a thermal stability of $1.5 \times 10^{-3}$ °C for the top plate. On the bottom plate, 20 mm thick, a metal-embedded type heating resistance is inserted. The bottom heater current is supplied by an operational power supply amplifier driven by a 8086 based microcomputer. A platinum resistance is glued in a hole neighboring to the bottom plane of the cell and a similar one is also glued in the top copper piece of the cell. Values of the top and bottom temperatures are measured by platinum resistances inserted in a three wires A.C. bridge.

The top copper piece of the cell is securely fixed to the cover of a much larger cylindrical concentric container. This vessel is permanently evacuated to a pressure of approximately
$10^{-6}$ torr. Its external wall is plunged in a water bath regulated at temperature $T_0$ to limit and keep constant the radiated power. To insure a linear mean temperature gradient between top and bottom of the cell, a copper cylinder of 1 mm thick and 55 mm in diameter is mounted between top and bottom copper pieces. It also acts as a thermal screen. The transmitted power $P$ is equal to $k \Delta T S/L$ where $k$ is the thermal conductivity of the material, $S$ its transverse section, $L$ its length and $\Delta T$ the temperature difference. For the copper screen in the case of the $\eta = 32$ cell, $P$ is of the order of 4 W, when $\Delta T$ is $10$ °C. The power radiated

![Diagram](image_url)
by the cell in these conditions is of the order of 4 mW. Note that the power transmitted by the convection cell itself is 3 mW and that by the polyvinyl chloride part is 12.5 mW.

The thermal gradient is thus determined by the high thermal conduction in the screen. On the lateral walls of the PVC cylinder, 1.5 mm diameter holes are drilled to locate the sensors (represented schematically in Fig. 1). These sensors are NTC thermistors of nominal resistance 5 000 Ω at 25 °C. They are embedded in epoxy resin and machined in such a way that no hydrodynamic perturbation is introduced in the cell. The resistance of a pair of thermistors can be simultaneously measured by two A.C. bridges; outputs of the bridges are recorded and spectrally analyzed by means of a Hewlett Packard digital signal analyzer HP 5420A.

In order to avoid the presence of air bubbles in the convecting fluid, the cell is first evacuated to a pressure of 10⁻² torr and then filled with liquid.

3. Visualization study.

Increasing the temperature gradient, we observe the following regimes of flow:

— first, at a temperature difference slightly higher than that of the convection threshold, a vertical non axisymmetric roll is clearly visible. As the temperature difference increases, this roll is no more stable, it spirals out and, at the end of the process, two rolls are present, one over the other. Such an evolution is indicated in figure 2a. A picture of the spiraling intermediate pattern, predicted by Normand [4], is shown in figure 2b. Although it is difficult to analyse clearly the transition between the spiraling roll and the two superimposed rolls, the spiraling regime is probably the intermediate state from the basic state to a higher rolls number convection structure.

When increasing the power delivered to the bottom heater two rolls, then four, six and finally eight are successively obtained. Odd number of rolls have also been observed. All these rolls are normally of equal size and these configurations are stable. One sometimes sees, always starting from the vertical non axisymmetric roll state, other configurations with from one to four small superimposed rolls of equal size with a bigger roll above. These latter configurations seem to be unstable. Figure 3 shows such a behavior, obtained in the case when the temperature has been increased rapidly from the onset. Figure 4 shows a picture of a stable configuration of superimposed rolls of equal size.

These visualization experiments clearly indicate the existence of transitions between various states of flow, involving helical modes.

4. Dynamical study.

4.1 CONVECTING STEADY STATE. — Let us recall that, for a convection cell with aspect ratio \( \eta = h/r \) larger than one (\( h \) and \( r \) are the height and radius of the cell respectively), the Rayleigh number \( \text{Ra} \) is defined by

\[
\text{Ra} = g \beta \Delta T r^4/(h \nu \kappa)
\]

(1)

where \( g \) is gravity acceleration, \( \beta \) the coefficient of volume expansion, \( \Delta T \) the temperature difference between the bottom and the top of the cell, \( \nu \) the kinematic viscosity and \( \kappa \) the thermal diffusivity of the fluid.

Let us remark that this expression of \( \text{Ra} \) differs from that used in Rayleigh-Bénard geometry, i.e. \( \text{Ra} = g \beta \Delta T h^3/(\nu \kappa) \). Indeed, the time scales associated to heat diffusion \( \tau_h \) and vorticity diffusion \( \tau_v \) in this case depend on \( r^2 \), not \( h^2 \), while the destabilizing time scale due to buoyancy \( \tau_b \) remains unchanged. As \( \text{Ra} \sim \tau_r \tau_v/\tau_b \), one obtains equation (1).
Above a critical value $\Delta T_c$, corresponding to the critical value $Ra_c$ of the Rayleigh number, we observe the onset of convection. According to theory [5], the critical Rayleigh number $Ra_c$ for the onset of stationary convection mode depends on the thermal boundary conditions at the cylindrical wall through the parameter $\chi = k_f/k_w$ where $k_f$ stands for the thermal conductivity of the fluid, $k_w$ for that of the cell walls. For ethylene glycol as the convecting fluid and walls of polyvinyl chloride, $\chi = 2.37$. Whereas with water as the fluid and the same wall material, $\chi = 5.53$. Ostroumov's theory [5], quoted by Gershuni [6], shows that in both cases we can use the approximation of walls conductivity much lower than those of fluid, i.e. $k_f/k_w \approx \infty$. On the other hand, the critical Rayleigh number also depends on the aspect ratio $\eta$; this effect has been estimated by Verhoeven [7], using Hales equations (8) with modified
boundary conditions, i.e. isolating walls. For an aspect ratio $\eta = 11$, he found $Rac = 70.83$ while for $\eta = 32$, $Rac = 68.41$, values very close to the infinite tube value of 67.97. (See also Catton and Edwards [9]). Using these theoretical values, we find, for a cell of $\eta = 32$ and $r = 5$ mm, $\Delta T_c = 1.19$ °C for water and $\Delta T_c = 4.83$ °C for ethylene glycol.

Experimentally, we observe for ethylene glycol at $\eta = 32$ a critical value $\Delta T_c = 4.8 \pm 0.5$ °C, in excellent agreement with theory. For water, the theoretical values of convection onset are $\Delta T_c = 1.2$ °C with $\eta = 32$ and $\Delta T_c = 0.41$ °C with $\eta = 11$.

From a theoretical point of view, the first steady convective state corresponds to a non axisymmetric mode with one roll in a vertical plane, which was investigated, in the scope of linear stability theory, by Rosenblat [10], using as mechanic lateral condition a null tangential vorticity. In that framework, Normand [11] finds a critical Rayleigh number given by:

$$Ra_c = \frac{1}{k_n^2} \left[ k_n^2 + \left( \frac{m \pi}{\eta} \right)^2 \right]^3.$$  

For the mode $(n, m)$, the integer $n$ characterizes the angular mode of flow and $m$ the number of vertical cells. $n = 0$ corresponds to the axisymmetrical modes and $n = 1$ to the first diametrically non axisymetrical modes. $\eta$ is the aspect ratio and $k_n$ is the adimensionalized radial wave vector of the flow. To obtain an evaluation of $Ra_c$ versus $\eta$, we use somewhat arbitrarily in the former expression, as values of $k_n$, the solutions of the equation:

$$I_n(k_n) J'_n(k_n) + J_n(k_n) I'_n(k_n) = 0$$

where $J_n$ and $I_n$ are Bessel functions of first order and which is strictly valid in the infinite cylinder case with insulating lateral cell walls and with realistic mechanic lateral conditions. As we are principally interested by the asymptotic part of the solution, such a process is satisfying.

Figure 5 shows the variations of $Ra_c$ with aspect ratio $\eta$ according to the above theory for the three lowest modes of $n$, $n = 0, 1, 2$ and the first three values of $m$. For aspect ratios greater than 5, one can immediately infer that the axisymmetric mode $n = 0$ and the non
axisymmetric mode \( n = 2 \) are not responsible for convective motions appearing at Rayleigh numbers just above critical and that the only reasonably foreseen modes in our experiments are the non axisymmetric ones with \( n = 1 \), the first to become marginally unstable being the one with one vertical cell, i.e. \( m = 1 \).

Another feature of the linear stability problem is that, for large aspect ratios (\( \eta \geq 10 \)), modes with different numbers \( m \) of vertical cells have extremely close values of \( \text{Ra}_c \); therefore one can expect a competition between several modes in this region of parameters [11].

4.2 TIME DEPENDENT REGIMES. — Successive temporal temperature signals at one of the thermistors are shown in figure 6 for increasing Ra.

4.2.1 Relaxation oscillatory state. — The first signature of the existence of a time-dependent regime is the onset of a low frequency oscillation of frequency \( f \) at \( \text{Ra}/\text{Ra}_c = 2 \) (see Figs. 6, 8). This transition is hysteretic, which, together with the occurrence of non-zero

Fig. 4. — Picture of a typical stable pattern of superimposed rolls of equal size at a higher temperature difference.
Fig. 5. — Results of theoretical calculations of the critical Rayleigh number \( R_{ac} \) versus aspect ratio \( \eta \) for the three first values of \( n \) and of the number \( m \) of vertical cells (Ref. [11]).

oscillation amplitude, indicates that this bifurcation is subcritical. The time record (see Fig. 6a) shows relaxation oscillations of the local temperature between two values: the upper one, which is very close to that of the basic stationary convection pattern, corresponds to the state 1 of conductivity of the cell, while the lower one corresponds to a state 2 of higher conductivity. Indeed, as a constant heat flow is applied at the bottom of the cell, an increase of the local temperature can be associated to a decrease of thermal conduction. State 2 is thus characterized by a higher thermal diffusivity than state 1. We propose thus that, as state 1 corresponds to the basic convective mode (the non axisymmetric one with one convection roll), the second state should be a convective state of higher geometrical complexity, such as a non axisymmetric configuration with several rolls, which is presumably transferring heat more effectively.

In order to estimate the period of the observed relaxation-like phenomena let us note that, since the Prandtl number \( P \) is high (\( P = 7 \) for water), the dynamics of the system is governed by the thermal diffusion time: \( \tau_\kappa = L^2 / \kappa \) where \( L \) is a characteristic length.

If we take the height of cell \( h \) for a characteristic length and calculate the corresponding thermal diffusion time \( h^2 / \kappa \), we find: \( \tau_\kappa = 1.8 \times 10^5 \) s for \( \eta = 32 \) and \( \tau_\kappa = 2 \times 10^4 \) s for \( \eta = 11 \).

These values turn out to be in disagreement with the experiment, since, in both cases, the longer oscillation period is 2 000 s. If we now take as a characteristic length the cell diameter \( d \), we obtain \( \tau_\kappa = 4 200 \) s, which is close to the experimental result for the two values of \( \eta \). This result can be understood by invoking the fact that spirals — with characteristic length \( d \) — play a role in the transition between modes.

The spectra and coherence of two bolometers signals are shown in figure 7. The frequency of the mode is sharply defined and the major fact is that the coherence signal obtained between these two thermistors is equal to 1 for the fundamental and its next ten following harmonics (Fig. 7c). The spatiotemporal correlation of the oscillation of relaxation is thus extremely high. This supports our interpretation of relaxation oscillations between two well defined global structures. In the visualization study, no stable mode competition was
Fig. 6. — Temporal dependence of the local temperature $T_{loc}$ of a point of the cell for successive values of $Ra/Ra_c$. a) $Ra/Ra_c = 5.65$ i.e. for a temperature difference between bottom and top of the cell $\Delta T = 6.72$ °C, « relaxation » oscillatory mode ; original state 1 is at upper level of oscillation, state 2 at lower ; the oscillation frequency is 1.3 mHz. b) $Ra/Ra_c = 11.71$ i.e. $\Delta T = 13.93$ °C, states 1 and 2 are no more « stationary » but oscillate at high frequency (state 2 with larger amplitude). c) $Ra/Ra_c = 16.56$ i.e. $\Delta T = 31.46$ °C, high frequency oscillation of state 2 of b is now dominant : there is just a mode, the oscillatory one. Here, the frequency is 18 mHz. The mean amplitude value corresponds to the local temperature of state 2.

observed, only transient mode, probably due to the insufficient thermal stability of this apparatus.
An increase of the Rayleigh number leads to a smooth and small increase of the low frequency oscillation (Fig. 8).
As temperature difference is increased, state 1 and then state 2 begin to oscillate at a much higher frequency than the relaxation oscillation. Typical high amplitude oscillations on state 2 are seen is figure 6b. Increasing $Ra$ further ($Ra \approx 12$) drives the system towards a single state : the fast oscillatory one (Fig. 6c). The transition between the slow relaxation oscillatory state and the fast oscillatory one is hysteretic and characteristic of a subcritical bifurcation.

4.2.2 Oscillatory mode. — This new state (Fig. 6c) has the following characteristics : a mean thermal conductivity value close to that characterizing state 2 ; around this value, we observe oscillations of the local temperature ; their amplitude is almost constant, but their frequency rapidly increases with $\Delta T$ (Fig. 8). On the other hand, the frequency of this mode is very well
Fig. 7. — Spectral study of a « relaxation » oscillatory mode. The working point is at $\Delta T = 2.98^\circ C$, i.e. $\frac{Ra}{Ra_c} = 2.5$. The signal spectrum of two thermistors is shown in figures a and b. The frequency ranges from 0 to 7.0 mHz. The coherence signal between the two thermistors shown in figure c is unity for the fundamental frequency and its seven harmonics at the less.

defined, corresponding to a sharp peak in the Fourier spectrum (Fig. 9a); just after the threshold value of $Ra$ for this transition, the coherence is practically 1 (see Fig. 9c).

4.2.3 Ultimate phase: chaos. — For values $\frac{Ra}{Ra_c}$ ranging between 15 and 20 (for the case of $\eta = 32$), the system undergoes a transition from the oscillatory state to a temporally
chaotic state with only one spatial mode (coherence equal to one), becoming soon a spatiotemporally chaotic one. Several well defined routes to chaos have been observed:

- subharmonic routes of various orders (1/2, 1/3, 1/4, 1/5);
- intermittent route.

For example, for $Ra/Ra_c = 14.37$, we have observed on one of the bolometers the occurrence of $f/5$ spectral lines in the frequency spectrum. This is clearly seen in the spectrum presented in figure 9b'; the coherence signal is also affected as can be seen in figure 9c'. An other situation is shown in figures 10a, b, c, where at $Ra/Ra_c = 15.90$, one observes $f/4$ spectral lines with conservation of coherence 1 up to harmonic 16; the phase difference between two bolometers is found proportional to the frequency, this can be related to a propagative nature of the observed phenomena.

Then, for a small increase of $Ra/Ra_c$ (2 %), the system transits to chaos, as one can see in figure 10a', b', c'. It is interesting to remark that the curves of coherence and phase still keep their characteristics, which means that the spatial mode is coherent, i.e. there is a transition to low-dimensional (temporal) chaos. The spectral amplitude of the signals of the two bolometers presents a decrease versus $f$ that one could observe on the peak values of the precedent curves (Fig. 10a).

When the chaos is well established, the coherence falls to zero, and the spectral amplitude tends to follow a crudely $1/f$ law. At higher $\Delta T$, corresponding to $17.5 \leq Ra/Ra_c < 20$, there is a reentrant transition from the spatiotemporal chaotic phase to an ordered phase and the coherence becomes again equal to unity for the four first spectral lines, but now their amplitude decreases with $f$ (Figs. 10a, b, c). For greater values of $\Delta T$, the chaotic state is
Fig. 9. — Spectral study of oscillatory modes. Frequencies range from 0 to 120 mHz. a, b, c) The working point is at $\Delta T = 16.8 \, ^\circ C$, i.e. $Ra/Rac = 14.10$. The spectrum of two different bolometers are shown respectively in figures a and b. The coherence signal is unity (Fig. c). a’, b’, c’) The working point is at $\Delta T = 17.1 \, ^\circ C$, i.e. $Ra/Rac = 14.36$. The spectrum of two different bolometers are shown respectively in figures a’ and b’. One observes on one of the bolometers (Fig. b’) the occurrence of $f/5$ in the frequency spectrum. The coherence signal is no more unity (Fig. c’).

recovered with a coherence null between two differently located bolometers. The amplitude of the chaotic « oscillations » increases with $\Delta T$, but the spectral characteristics do not change more sensibly in our range of study ($Ra/Rac \leq 25$). It must be noted that the occurrence of chaos can sometimes be observed suddenly during the oscillatory mode. In that case, the
Fig. 10. — Spectral study of oscillatory mode and chaotic state. Frequencies range from 0 to 120 mHz.

a, b, c) One observes a $f/4$ oscillation (Fig. a) that can be attributed to two successive period doublings. The working point is at $\Delta T = 18.92 \, ^\circ \text{C}$, i.e. $Ra/Ra_c = 15.90$. The coherence signal is unity (Fig. c). $a'$, $b'$, $c'$) Spectral study of a chaotic state. The working point is at $\Delta T = 19.35 \, ^\circ \text{C}$, i.e. $Ra/Ra_c = 16.26$. The amplitude of the spectrum of the bolometer follows a decreasing law versus frequency (Fig. $a'$) similar to that observed on the peak values of the precedent spectrum (Fig. a). Coherence signal remains unity (Fig. $c'$). The phase differences (Figs. b, b') are linear versus frequency for the values where coherence is unity (Figs. c, c').
system can sometimes come back to the oscillatory mode, sometimes stay in the chaotic mode.

Up to now, all the results we have presented were obtained with an experimental cell with \( \eta = 32 \). With a cell of smaller length, same diameter and \( \eta = 11 \), we typically found similar behavior with greater experimental difficulties due to a greater separation between modes that seems to prevent the system from leaving easily the state of stationary convection.

**Conclusion.**

All these results suggest the occurrence, with increasing Ra, of two competing structures in phase 1:

- one large scale structure with a single roll;
- one small scale structure with superimposed convection rolls.

Abernathey and Rosenberger [2] did not observe any competition between structures comparable to ours. Such a fact can perhaps be explained by pointing out that the Prandtl number in their experiment was 0.64, ten times smaller than that of water. But it is probably the relatively low aspect ratio in their experiments (\( \eta = 6 \)) that inhibits the possibilities of transitions between different structures.

The great richness of modes in a narrow range of Ra, characteristic of large boxes (large aspect ratios), allows us to understand the dynamics of relaxation oscillations at relatively low Ra. At higher Ra, it becomes more difficult to give a satisfying interpretation; in the chaotic range, this complexity does not allow us to observe very definite routes. We then observe the phenomenology associated with the transition to chaos, but not in the order quoted in low aspect ratio experiments.

The dynamics of our system seems to be dominated by a few spatial degrees of freedom, allowing for the existence of definite behavior over well-ordered ranges. This is surprising because, for large aspect ratio cells, one could expect disordered chaotic states just above the convective onset as in the Rayleigh-Bénard geometry [12, 13]. This suggests that phase dynamics phenomena would be difficult to observe in convective systems of large vertical aspect ratios.

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**References**


