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The arrangement of cells in 3- and 4-regular planar networks formed by random straight lines

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Résumé. — La relation entre le nombre moyen, m_i, de côtés dans des cellules adjacentes aux cellules avec i-côtés, et i, a été étudiée dans deux types de réseaux planaires engendrés par une distribution aléatoire (Poisson) de lignes droites. Si les lignes sont continues, le réseau est 4-régulier. Si les lignes sont interrompues, de façon à avoir des jonctions en T, on obtient un réseau 3-régulier. Dans le premier cas, on peut définir deux types d'adjacence entre cellules, et, pour chaque type, on trouve une variation linéaire de m_i avec $1/i$. Le même résultat s'applique dans le réseau 3-régulier. Les relations linéaires (loi de Aboav) peuvent être mises sous une forme générale, en faisant intervenir le second moment de la distribution de polygonalités des cellules et un seul paramètre, a, comme Weaire l'a suggéré.

Abstract. — The relation between m_i, the average number of sides in cells adjacent to i-sided cells, and i, was investigated in planar networks of two types, generated by random, Poisson distributed, straight lines. When the lines are continuous, a 4-regular network results. If the lines are interrupted to form T-junctions, a 3-regular network is obtained. In the first case, two types of cell adjacency can be defined and for both m_i varies approximately linearly with $1/i$. The same applies to the 3-regular network. The linear relations (Aboav's law) can be put in a general form, involving the second moment of the distribution of cell polygonalities and a single parameter, a, as suggested by Weaire.

Introduction.

In planar networks, formed by various cells, each with a number i of edges (sides), it is possible to define an adjacency relation between cells, in terms of common topological elements (edges or vertices) in the cells. In 3-regular (or trivalent) networks (Fig. 1a) there is only one type of adjacency since two cells that have a common edge also have a common vertex. In 4-regular (or tetravalent) networks we distinguish between edge-adjacency (common edge) and vertex-adjacency (common vertex, with no common edge). In figure 1b the cells A, B, C and D are edge-adjacent to cell 0, while cells A′, B′, C′, D′ are vertex-adjacent. When refering to edge- and vertex-adjacent cells collectively we shall use the term adjacent cells.

The average number, m_i, of sides in cells adjacent to i-sided cells is a quantity that has deserved considerable attention in the literature, since Aboav's [1] observation of a nearly

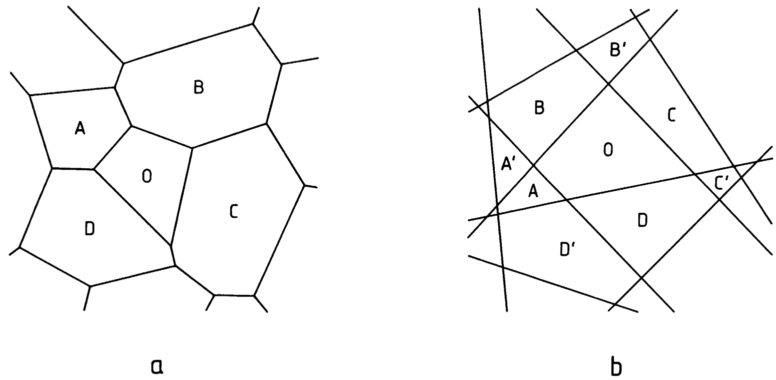

Fig. 1. — Cells adjacent to a cell 0 in a 3-regular (a) and 4-regular (b) network. In the 4-regular network two types of adjacency can be defined.

linear variation of m_i with $1/i$ in the 3-regular network of grain boundaries in a plane section of a MgO polycrystal. The linear relation found by Aboav in this case was

$$m_i = 5 + \frac{8}{i}. \tag{1}$$

The linear dependence of m_i on $1/i$, although not always with the same constants, was subsequently found to hold approximately for other random 3-regular planar networks, including sections of soap froths [2], Voronoi networks [3-5], planar sections of cork [6] and of plasticine polycrystals [7], thin films of chalcogenide alloy glasses [8] and other natural networks [9]. All such networks are similar in their geometrical aspect, in that the cells are fairly regular (equi-angle) polygons ; the second moment, μ_2, of the distribution of cell polygonalities is also relatively small, usually below 2, indicating a small dispersion of polygonalities around the average value of 6. We shall refer to these networks as Voronoi-like networks.

An argument due to Weaire [10], based on the partition of the 2 π angle at a vertex among the three cells at the vertex, leads to a linear relation between m_i and $1/i$. His result is

$$m_i = 5 + \frac{6}{i}. \tag{2}$$

The same result was obtained by Blanc and Mocellin [11], based on the analysis of the effects on m_i of random unit topological operations in a 3-regular network (triangle elimination and cell neighbour switching). Both derivations use somewhat loose averaging procedures to obtain m_i. In fact, the linear relation is known to be not more than a good approximation for the 3-regular planar networks so far studied, meaning that an exact analysis will certainly not lead to a linear relation.

The purpose of this paper is two-fold. Firstly we will show that Aboav's law still holds for 3-regular networks of a completely different geometry from that of the Voronoi — like networks so far analysed. Secondly, it will be shown that the linear law is also applicable to 4-regular networks, when either type of cell adjacency is considered to obtain m_i. The networks to be analysed with this purpose are both based on a Poisson distribution of straight lines in the plane. Before describing how these networks are generated and indicating the results obtained for m_i, we shall derive, following Weaire [10], a more correct and general form of the linear relation between m_i and $1/i$ valid for both 3- and 4-regular networks.

The general form of linear relation between m_i and $1/i$.

An edge that belongs to (adjacent) cells with i and k sides is termed and ik-edge. The fraction of ik-edges in an arbitrary network is denoted f_{ik}. In 4-regular networks we shall consider also vertex-adjacent cells. In such networks, a vertex is identified by two pairs ik and jl which are the polygonalities of the cells on opposite sides of the vertex, i.e., of the cells that are vertex-adjacent at that vertex, such as cells 0 and A′ in figure 1b. The fraction of ik-pairs at the vertices of a 4-regular network is also denoted by f_{ik}. In both cases, no distinction is made between ik and ki pairs ($f_{ik} = f_{ki}$).

In general, the average polygonality, m_i, of the cells edge- or vertex-adjacent to i-sided cells is given by

$$m_i = \frac{\sum_k k f_{ik}}{\sum_k f_{ik}} . \tag{3}$$

If m_{iE} and m_{iV}, respectively, are the values for edge- and vertex-adjacent cells in 4-regular networks, and m_i is the value for (all) adjacent cells we have

$$2\, m_i = m_{iE} + m_{iV} \tag{4}$$

since the total number of cells adjacent to an i-cell is $2\,i$ and there are i of each type (edge- and vertex-adjacent). In all cases under consideration, the sum $\sum_k f_{ik}$ is proportional to $i\,f_i$, where f_i is the fraction of i-sided cells in the network. We then have, from (3)

$$\langle i m_i \rangle = \sum i\, f_i\, m_i = \langle i^2 \rangle . \tag{5}$$

This equation can also be derived from the following argument. The quantity $i m_i$ is the average total polygonality of the cells adjacent to i cells. A particular k-cell enters k times (with a contribution k) when the total polygonalities of all cells are evaluated. Therefore the average $\langle i m_i \rangle$ equals $\langle k^2 \rangle$.

Introducing the second moment, $\mu_2 = \left\langle (i - \bar{i}\,)^2 \right\rangle$ of the distribution f_i, equation (5) becomes

$$\langle i m_i \rangle = \bar{i}^2 + \mu_2 \tag{6}$$

$\bar{i}$ being the average value of i. In 4-regular networks, equation (5) holds for m_{iE}, m_{iV} and m_i. The form of a linear relation between m_i and $1/i$ that satisfies the identity (5) is

$$m_i = \bar{i} - a + \frac{a\bar{i} + \mu_2}{i} \tag{7}$$

where a is a constant for a given network. In Voronoi-like networks (which have $\mu_2 < 2$) it is found that a is close to unity.

The form (7) of the linear relation between m_i and $1/i$ was first advanced by Weaire [10] for 3-regular networks ($\bar{i} = 6$). An argument due to Lambert and Weaire [12] shows that this relation, with $a = 1$, is exact in the limit $\mu_2 = 0$.

If the arrangement of the cells in a network were random and not restricted by space-filling requirements, one would expect that the fraction f_{ik} would be proportional to the total number of topological elements in cells i and k,

$$f_{ik} \propto i f_i \,.\, k f_k \,. \tag{8}$$

Under these conditions, equation (3) gives m_i independent of i :

$$m_i = \frac{\langle i^2 \rangle}{\bar{i}} = \bar{i} + \frac{\mu_2}{\bar{i}} \,. \tag{9}$$

The fact that m_i depends on i in actual networks is of course due to space-filling requirements, with the sum of the internal angles of the cells meeting at a vertex equal to 2π. Since the average internal angle in a cell with i sides is $\pi(1 - 2/i)$, there must be a correlation between m_i and i.

Convexity imposes additional limitations on cell adjacency. For example in convex trivalent networks two triangular cells cannot be adjacent at an edge because such a « cluster » cannot be surrounded by convex cells. Other clusters of cells are also forbidden on the same grounds.

The exact determination of m_i from equation (3) would require equations for the f_{ik} as a function of i and k. Such equations are not known for any random network, implying that the m_i have to be determined by inspection. This is indeed how the m_i have been determined so far.

3- and 4-regular networks of random straight lines.

The networks to be analysed are based on a Poisson distribution of straight lines in the plane. The lines are drawn within a unit square with edges defining x and y axes. The coordinates x and y of a point P in the square are chosen by taking two numbers in the interval 0, 1 (all values are equiprobable). A straight line is drawn through P with a random orientation α ; α is the angle with the x axis, and can take any value in the interval $(0, \pi)$. A number, n, of lines is drawn in this way.

When the lines are continuous (Fig. 2), a 4-regular network is obtained, with $\bar{i} = 4$. Properties of this network were derived by Crain and Miles [13], both exactly and from computer simulations. The number of cells produced by n lines (n large) is $n^2/4$. The distribution f_i was determined in simulations [13] and is given in table I, second column. The value of f_3 is known exactly, as well as the second moment μ_2 [13] :

$$\begin{aligned} f_3 &= 2 - \pi^2/6 = 0.3551 \\ \mu_2 &= (\pi^2 - 8)/2 = 0.935 \,. \end{aligned} \tag{10}$$

Crain and Miles did not determine the fractions f_{ik}. In order to obtain the m_i, two networks with $n = 30$ were constructed. The edges at the sides of the square were counted as edges of the peripheric cells. The fractions f_i for one of these networks are given in table I and do not differ appreciably from those of Crain and Miles, based on a much better statistics.

The 3-regular network of random straight lines is generated similarly, by drawing successive straight lines in a unit square, but a new line is alternately interrupted between the points of intersection with previously drawn lines, forming T-junctions (Fig. 3). The segment containing the point P, used to generate the line, is kept in the line. Figure 3 shows a network generated in this way, with $n = 30$ lines ; the order in which the lines were drawn is indicated for the first few lines. The network has $\bar{i} = 6$. Its geometry is quite different from that of the Voronoi-like networks, in that a large number of π internal angles appear in the cells. Since the number of

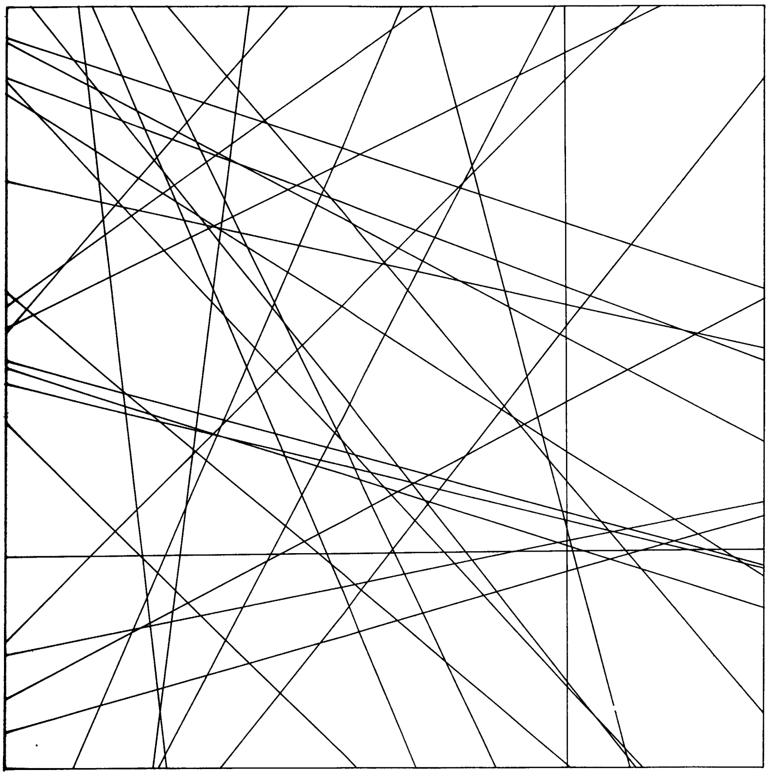

Fig. 2. — 4-regular network generated by randomly distributed and oriented straight lines in a square.

Table I. — *Fractions, f_i, of i-sided polygons in networks formed by random straight lines.*

	4-regular network Crain and Miles	4-regular network $n = 30$	3-regular network $n = 30$
f_3	0.35561	0.348	0.225
f_4	0.37790	0.396	0.157
f_5	0.19183	0.179	0.152
f_6	0.05922	0.0625	0.141
f_7	0.01318	0.0149	0.0785
f_8	0.001958	0	0.0785
f_9	0.000248	0	0.0524
f_{10}	0.000038	0	0.0209
f_{11}			0.0314
f_{12}			0.0157
f_{13}			0.0157
f_{14}			0.0052
f_{15}			0.0105
f_{16}			0.0105
f_{17}			0
f_{18}			0.0052
μ_2	0.942	0.91	9.11

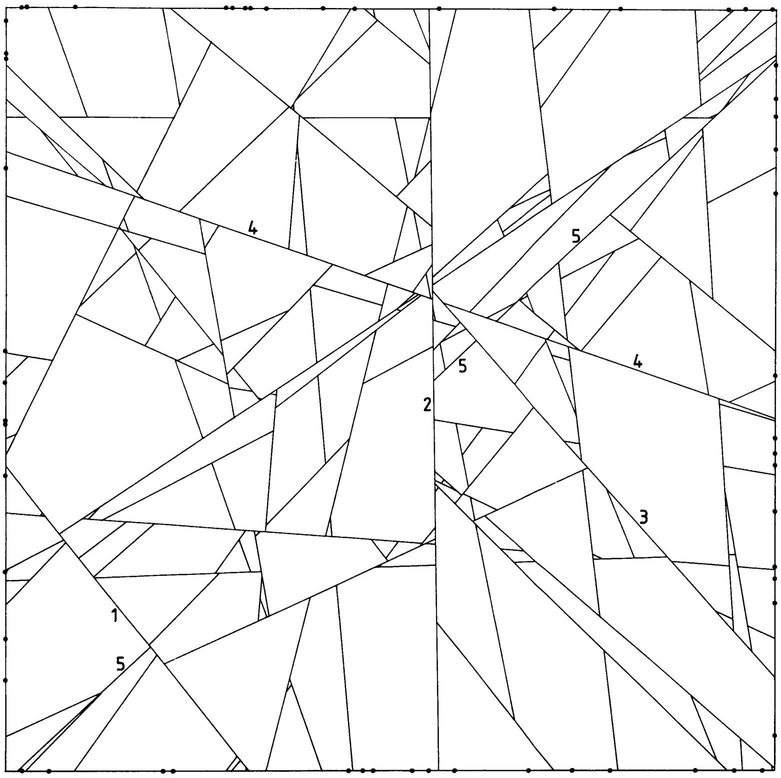

Fig. 3. — 3-regular network generated by randomly distributed and oriented straight lines each of which is alternatively interrupted when it meets pre-existing lines. The figures give the order in which the first 5 lines were drawn. The dots at the periphery are additional vertices introduced to obtain periodic boundary conditions.

vertices in the network is three times the number of cells, and each vertex contributes a π internal angle, it follows that the fraction of these angles is 1/2. The number of cells formed by n straight lines (n large) is now $n^2/16$. In fact, for the same n, the number of vertices in this network is 1/2 of the number in a 4-regular network constructed with the same lines ; and the number of vertices in each network is respectively F and $F/2$, where F is the number of cells in each.

The fractions f_i of i-sided cells in the 3-regular network of figure 3 are given in table I, fourth column. The calculated second moment is

$$\mu_2 = 9.11 \tag{11}$$

which is considerably larger than that for the 3-regular networks so far analysed for the m_i. Additional vertices were placed at each side of the square in correspondence to the vertices, on the opposite side, that result from intersection with the straight lines of the distribution. In this way the peripheric cells can be used in the determination of f_i and m_i.

m_i results.

Figure 4a shows a plot of im_i as a function of i for the 3-regular network of figure 3. Probably because of the poor statistics there is some deviation from a linear relation at large i. The best fit straight line of the form (7) is

$$m_i = 5.65 + 11.23/i \tag{12}$$

and corresponds to

$$a = 0.35 . \tag{13}$$

It is the straight line drawn in figure 4a.

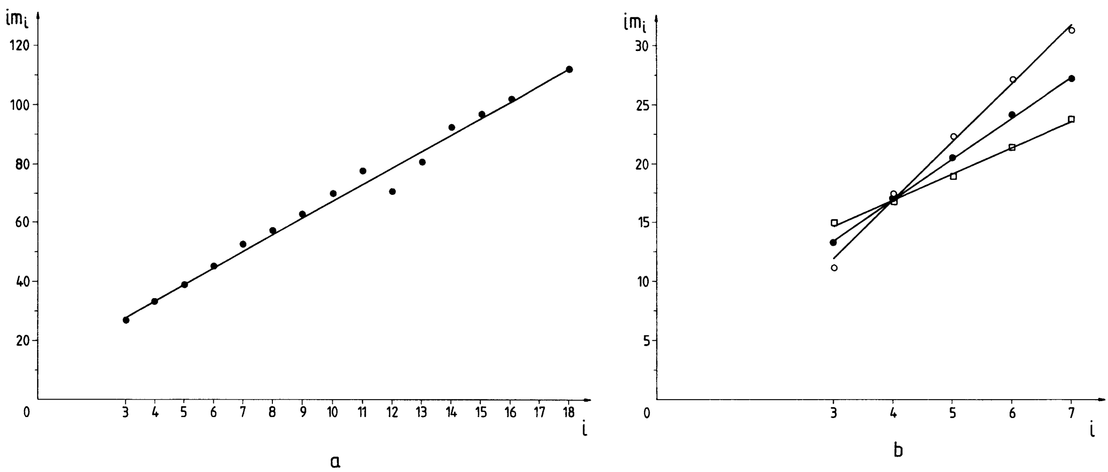

Fig. 4. — Plot of im_i as a function of i : a) for the 3-regular network of figure 3 ; b) for the 4-regular network of figure 2, respectively for edge-adjacency (○), vertex-adjacency (□) and edge or vertex adjacency (●). The straight lines are the best fit lines of the form of equation (7) and their equations are given in the text (Eqs. (12) and (14) respectively).

In figure 4b are plotted the values of im_i as a function of i for one of the 4-regular networks constructed, and considering the different types of adjacency. The best fit straight lines of the form (7) are drawn in figure 4b and are as follows

$$\begin{aligned} m_i &= 3.51 + 2.86/i \\ m_{i\mathrm{E}} &= 2.24 + 7.96/i \\ m_{i\mathrm{V}} &= 4.98 - 3.00/i \end{aligned} \tag{14}$$

which satisfy approximately equation (4). The corresponding values of a are

$$\begin{aligned} a &= 0.49 \\ a_{\mathrm{E}} &= 1.76 \\ a_{\mathrm{V}} &= -0.98 . \end{aligned} \tag{15}$$

Discussion.

The applicability of equation (7) (Aboav's law) to networks based on Poisson distributions of straight lines, confirms that this equation is probably of general applicability to random networks.

The argument of Weaire leading to equation (2) can be repeated to obtain an equation for m_i in 4-regular networks. The argument goes as follows. The average internal angle in an i-sided cell is $\pi(1-2/i)$. The average internal angle in the adjacent cells is then $(1/3)[2\pi - \pi(1-2/i)]$, since at each vertex there are 3 adjacent cells. The sum of the internal angles in a polygon with m_i sides is $\pi(m_i - 2)$. Since the average number of sides in the cells is 4 we may write

$$\frac{\pi(m_i-2)}{(1/3)[2\pi - \pi(1-2/i)]} = 4 \tag{16}$$

which leads to

$$m_i = \frac{10}{3} + \frac{8}{3\,i} = 3.33 + 2.67/i \tag{17}$$

not much different from the equation previously written for m_i (first of Eq. (14)). Note that a similar argument cannot be given to obtain $m_{i\mathrm{E}}$ or $m_{i\mathrm{V}}$ separately. The derivation of the linear relation given by Blanc and Mocellin [11] for 3-regular networks cannot be applied to 4-regular networks since unit operations in such networks cannot easily be defined.

The evidence so far accumulated suggests that Aboav's law is indeed quite general. However an argument that shows its general applicability to random networks is still lacking. Such an argument is necessarily very general and approximate and could possibly allow the generalization of Aboav's law to aggregates of polyhedra, with an approximate linear relation between $1/F$ and the average number of faces, m_F, in polyhedra adjacent to F-faced polyhedra. Such a general argument will be given elsewhere.

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