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Critical currents in superconducting networks and magnetic field effects

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Résumé. On a étudié les solutions des équations des réseaux supraconducteurs en présence d'un courant extérieur. On calcule les courants critiques pour les géométries typiques. En présence d'un champ magnétique, le courant critique est très sensible à la topologie du réseau. Ceci est illustré de façon explicite sur un réseau carré infini. On compare nos calculs aux résultats de mesures préliminaires du courant critique effectuées sur un réseau carré de filaments d'indium submicroniques.

Abstract. Network equations for finite and extended superconducting arrays are studied in the presence of external currents. Critical currents are obtained for typical network geometries. In the presence of a magnetic field, the critical currents are shown to be very sensitive to the underlying topology of the network. This is illustrated through explicit calculations on an extended square network. Our results are compared with some preliminary critical current measurements performed on a square array made of submicronic indium wires.

1. Introduction.

Recent technical and theoretical advances have led to a revival of interest in the magnetic properties of superconducting networks [1]. One line of thought sees these structures as an appropriate tool for studying some specific frustration effects. Another one sees the diamagnetic superconducting currents as a sensitive tool for studying the topology of regular or irregular network structures [2-4]. Up to now the considered networks fall into three categories: simple finite geometries (single ring, double-loop, etc.); infinite regular arrays (square or honeycomb lattices, Sierpinski gasket or carpet, Penrose quasi-periodic patterns, etc.) and disordered structures (percolation clusters). In all these cases, attention has been focused on the following physical properties: i) the critical field for the onset of superconductivity \( H_{\text{c1}} \), ii) magnetization \( M(T, H) \) and derivatives \( \partial M / \partial T \), \( \partial M / \partial H \) at zero or finite field, iii) mixed state (current distribution, order parameter configurations, etc.). In this respect, the important problem of the critical current \( I_c \) has not been worked out in details. To our knowledge the only available calculation of \( I_c \) is that of reference [5] where the zero field limit of \( I_c \) has been considered, as well as the case a single ring in a magnetic field.

The main purpose of this paper is to report on some new results relative to the critical currents in a superconducting network at finite magnetic field. In addition to theoretical considerations, we present some preliminary critical current measurements performed on a square array made of submicronic indium wires. The paper is organized as follows. Section 2 is devoted to a brief summary of the mixed state properties and the associated supercurrent densities. The network equations with external currents are considered in section 3, with two illustrative examples. The precise calculation of the critical currents is addressed in section 4, where the variation of \( I_c \) as function of the magnetic field is studied in details for two geometries: single loop and infinite square array. In section 5, our results are compared with the experimental data of critical current measurements.

2. Supercurrents in the mixed state in absence of external currents

The properties of the mixed state (magnetization, order parameter configuration, etc.) of a superconducting network have been worked out recently in reference [6] (see Appendix A). In this section we summarize the main results so obtained for the supercurrents in the equilibrium state. This allows in particular a coherent set of notations.
The approach to understanding the superconducting properties of wire networks below the critical temperature $T_0$ is based upon the Ginzburg-Landau (GL) theory and follows the original treatment of the mixed state of type II superconductors [4, 6]. As a main approximation it is assumed that the superconducting phase is described by a complex order parameter (with neglecting fluctuations) and that the actual order parameter is, by continuity, very close to the solution of the linearized GL equation in the considered geometry.

The main step consists therefore in searching the solutions of the linear equation and treating by perturbation the higher order term in the GL free energy expansion.

Since the linear solutions will be used away from the phase transition line it is convenient to write the GL equation as a eigenvalue equation for the order parameter:

$$
\left( -i \nabla - \frac{2 \pi}{\Phi_0} A \right)^2 \Psi = \frac{1}{\xi_e} \Psi
$$

(2.1a)

where $A$ is the vector potential, $\Phi_0 = \hbar c / 2 e$ the flux quantum and $1/\xi_e^2$ (noted $E(H)$ in the Appendix A) is the eigenvalue which corresponds to the eigenfunction $\Psi$. Using this definition $\xi_e$ is a function of the magnetic field and the boundary condition, i.e. local minima of the linear GL free energy. Since this section is devoted to the equilibrium state, we will restrict here to the solution for which $1/\xi_e^2$ is minimum.

When specialized to a wire network, the above equation reduces to the following set of finite difference equation [2, 4] :

$$
\sum_j \left[ -\Psi_i \cot \frac{\ell_{ij}}{\xi_e} + \Psi_j \frac{e^{-i\gamma_{ij}}}{\sin (\ell_{ij}/\xi_e)} \right] = 0.
$$

(2.1b)

The considered network is supposed made of strands $(i,j)$ of lengths $\ell_{ij}$ connecting the nodes $i$ where the order parameter assumes the value $\Psi_i$. In equation (2.1b), $\gamma_{ij}$ denotes the phase factor:

$$
\gamma_{ij} = 2 \frac{\pi}{\Phi_0} \int_{s} A \cdot ds
$$

and $s$ is the curvilinear coordinate along the strand $(i,j)$ (see Fig. 1 for notation).

Equation (2.1b) has been successfully used to determine the phase transition line of superconducting networks in various geometries (see Ref. [1] and reference therein). Here the critical temperature $T_c(H)$ is determined by the minimal eigenvalue:

$$
T_c(H) = T_0 \left( 1 - \xi^2(0) \min \frac{1}{\xi_e} \right)
$$

(2.2)

where $\xi(T) = \xi(0) / \sqrt{1 - T/T_c}$ is temperature dependent coherence length, and $T_0 = T_c(H)_{H=0}$.

The determination of the supercurrent induced by the magnetic field requires an explicit knowledge of the order parameter over the network. For a given solution of equation (2.1b), the order parameter value and the supercurrent between nodes $i$ and $j$ are given as a function of the $\Psi$ values at the nodes by the following expression:

$$
\Psi_i(s) = \frac{e^{i\gamma_{ij}}}{\sin (\ell_{ij}/\xi_e)} \times \left[ \Psi_i \sin \frac{\ell_{ij} - s}{\xi_e} + \Psi_j e^{-i\gamma_{ij}} \sin \frac{s}{\xi_e} \right]
$$

(2.3a)

$$
J_{ij} = \frac{2 e \hbar}{m^* \xi_e \sin (\ell_{ij}/\xi_e)} \Im \left[ \Psi_i^* \Psi_j e^{-i\gamma_{ij}} \right]
$$

(2.3b)

where $m^*$ refers to the mass of an electron pair. $\gamma_{ij}$ is the phase factor $\gamma_{ij} = 2 \frac{\pi}{\Phi_0} \int_{s} A \cdot ds$.

For a finite network with $N$ nodes, it is convenient to write $\Psi_i$ as $\Psi_i = \Psi_0 f_i$ where $f_i$ is the normalized order parameter at node $i$ and $\Psi_0$ is the mean square of the amplitude over the whole network. In regular networks $(\ell_{ij} = \ell)$, the normalization condition simply writes $\sum |f_i|^2 = 1$ ($N$ (see Appendix B).

Similarly we define the normalized current $j_{ij}$ by $J_{ij} = J_0 j_{ij}$ where $J_0$ is given by:

$$
J_0 = \frac{2 e \hbar}{m^* \xi_e \sin (\ell/\xi_e)}
$$

(2.4)

In equation (2.4) we have assumed that $\ell_{ij} = \ell$ is the same for all strands. This is appropriate for regular structures such as the square network. The normalized current $j_{ij}$ becomes:

$$
j_{ij} = \Im (f_i^* f_j e^{-i\gamma_{ij}}).
$$

(2.5)
Because of the linearity of equation (2.1), \( \Psi_0 \) and therefore \( J_0 \) have to be formed by considering the non linear terms in the GL equations. This perturbative approach to the mixed state is outlined in appendix A. Briefly it consists in minimizing the fourth order term in the GL free energy expansion as function of both the order parameter and the magnetic induction. At equilibrium (no external current) \( 1/\xi^4_0 \) is minimum and \( \Psi_0 \) and \( F_0 \) can be expressed in terms of the phase transition line \( T_c(H) \) which is defined by equation (2.2). In this situation only we have:

a) The mean amplitude of the order parameter \( \Psi^2_0 \) is linear in \( T_c(H) - T \):

\[
\Psi^2_0 = \frac{\Psi^2_\infty(0)}{\beta_A} \frac{T_c(H) - T}{T_0} . \tag{2.6}
\]

b) The free energy difference \( F_s - F_n \) is given by

\[
\Delta F = F_s - F_n = \frac{1}{\beta_A} \Delta F_\infty(0) \left[ \frac{T_c(H) - T}{T_0} \right]^2 + \left[ \frac{h^2}{8 \pi} \right] + \left[ \frac{h^2}{\xi^2_e} \right] . \tag{2.7}
\]

Here \( \Psi^2_\infty(0) = \Psi^2_\infty(T) \big|_{T=0} \) and \( \Psi^2_\infty(T) \) is the usual expression \( \Psi^2_\infty(T) = -\alpha(T)/\beta \) of the uniform bulk solution at which the free energy \( \Delta F = \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 \) assumes its minimum value \( \Delta F_\infty(T) = -\frac{\alpha^2(T)}{2\beta} \) \cite{7}, and \( \beta_A \) is the generalized Abrikosov parameter \cite{8} \( \beta_A = \frac{\langle |\Psi|^4 \rangle}{\langle |\Psi|^2 \rangle^2} \) which is generally field dependent.

c) The configuration of the order parameter and the distribution of supercurrent are obtained from equation (2.3) and reflect the specific vortex structure of the mixed state. For example in a square network the equilibrium configuration consists in a periodic arrangement of basic supercells.

From equations (2.6) and (2.7) it is clear that the specific features of the network topology appears through the field dependence of the critical temperature \( T_c(H) \) and the numerical value of \( \beta_A \). In a type II superconductor, the minima of the free energy are fixed by those of \( \beta_A \) \cite{7} and the reader is directed to reference \cite{6} for further details.

It is important to notice that the above formulation is also valid in a bulk superconductor. For instance at \( H = 0 \), \( \Psi \) becomes uniform and \( \beta_A = 1 \), the order parameter being

\[
\Psi^2_0 = \Psi^2_\infty(0) \frac{T_0 - T}{T_0} = \Psi^2_\infty(T) .
\]

Using equation (2.6), the average supercurrent amplitude \( J_0 \) reads

\[
J_0 = J_n \frac{1}{\beta_A} \frac{\xi(0)}{\xi_e \sin (\ell/\xi_e)} \frac{T_c(H) - T}{T_0} . \tag{2.8a}
\]

In this expression \( \xi_e \) is taken at the band edge (1/\( \xi^2_e \) minimum) and the prefactor \( J_n \) is given by

\[
J_n = \frac{\phi_0}{2 \pi} \frac{c}{4 \pi} \frac{1}{\kappa^2} \frac{\xi(0)}{\xi^2_e} \tag{2.9}
\]

(\( \kappa = \text{GL parameter} \)). As it should be, \( J_0 \) vanishes as \( T_c(H) \) is approached.

The above expressions, valid in the equilibrium state (zero external current) are no longer valid in presence of an applied current since \( \xi_e \) and therefore the critical temperature are expected to be modified. In the following \( T_c(H) \) will be kept as the actual transition temperature in zero current. The approximate expression for \( J_0 \), valid without restriction on the transport current is obtained in terms of \( \xi_e \) using equations (2.4) and (A.16):

\[
J_0 = J_n \frac{1}{\beta_A} \frac{\xi(0)}{\xi_e \sin (\ell/\xi_e)} \times \left( \frac{T_0 - \xi^2_e}{\xi^2_e} \right) . \tag{2.8b}
\]

The influence of an external current on \( \xi_e \) is discussed in the next section.

3. Networks with external currents

When external currents are injected at some nodes of the network, one must solve the Ginzburg-Landau equation with taking into account this additional boundary condition. Within our approximation of the mixed state one has to calculate the new eigenstates \( \xi_e \) of the linearized Ginzburg-Landau equation. \( \xi_e \) now becomes dependent on both the magnetic field and the external current. In this situation equation (2.1) and therefore equation (2.6) must be modified. A convenient procedure for including external currents effects in equation (2.1) has been described in reference \cite{9}. Here we recall briefly this procedure and then describe two simple examples which illustrate this method and its limitations. Critical current calculations are based on the results of this section.

3.1 Network equations with external currents

In order to take into account the presence of an injected current at node \( i \), we separate in equation (2.1) the term associated with strand \((i, n)\):

\[
\sum_{j=1}^{n-1} \left[ -\Psi_i \cot \frac{\ell_{ij}}{\xi_e} + \Psi_j \frac{e^{-\gamma_{ij}}}{\sin (\ell_{ij}/\xi_e)} \right] - \Psi_i \cot \frac{\ell_{in}}{\xi_e} + \Psi_n \frac{e^{-\gamma_{in}}}{\sin (\ell_{in}/\xi_e)} = 0 . \tag{3.1}
\]
A simple calculation shows that for vanishing \( l_{in} \), the expression

\[
T_{in} = \frac{\psi_i^*}{\xi_e} \left[ - \psi_i \cot \frac{\ell_{in}}{\xi_e} + \psi_n \frac{e^{-i\gamma_{in}}}{\sin \left( \frac{\ell_{in}}{\xi_e} \right)} \right]
\]  

(3.2)

becomes a pure imaginary number:

\[
\lim_{l_{in} \to 0} T_{in} = i Q_i |\psi_i|^2
\]  

(3.3)

where we have defined \( Q_i \) by (\( \psi_i = |\psi_i|e^{i\phi_i} \)):

\[
Q_i = \frac{d\phi_i}{ds} - \frac{2\pi}{\Phi_0} A_i.
\]  

(3.4)

Clearly, \( Q_i \) has the meaning of a gauge-invariant velocity field of pairs, related to the external current \( J_{ext} \) at node \( i \) by

\[
J_{ext} = \frac{2 e h}{m^*} |\psi_i|^2 Q_i = J_n \frac{1}{\beta A} \xi_e (0) Q_i |f_i|^2 \times
\]

\[
\left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_e^2} \right] .
\]  

(3.5)

Accordingly the network equation at node \( i \) becomes

\[
\sum_j \left[ - \psi_i \cot \frac{\ell_{ij}}{\xi_e} + \psi_j \frac{e^{-i\gamma_{ij}}}{\sin \left( \frac{\ell_{ij}}{\xi_e} \right)} \right] +
\]

\[
i \xi_e Q_i \psi_i = 0 .
\]  

(3.6)

As we have mentioned, in presence of external current, \( \xi_e \) depends on \( H \) and \( J_{ext} \) according to equations (3.5) and (3.6). For a fixed external current, \( 1/\xi_e^2 \) reaches its local minima at a new equilibrium state, which are the solutions of the new boundary conditions (Eqs. (3.5) and (3.6)), and give the critical temperature in presence of external current.

3.2 Two EXAMPLES. — The physical meaning and the limitations of equation (3.6) become clear when simple cases are considered. This is illustrated here on two examples: single loop and infinite networks.

a) Single loop case (Fig. 2). — The secular equation of the linear system (Eq. (3.6)) corresponding to this geometry reads simply (\( Q = Q_1 = -Q_2 \))

\[
4 \cos^2 \frac{2a}{\xi_e} + \xi_e^2 Q^2 \sin^2 \frac{2a}{\xi_e} - 4 \cos^2 \frac{\gamma}{2} = 0
\]  

(3.7)

where \( a \) refers to the side of the square loop and \( \gamma = 2\pi \Phi/\Phi_0 \) is the reduced flux (\( \Phi = Ha^2 \)). It is found from equations (3.6) and (3.7) that the order parameters at nodes 1 and 2 have the same amplitude: \( |\psi_1|^2 = |\psi_2|^2 \), as expected by symmetry. Solving this equation for \( Q_i \), one obtains the following expression for the current \( J_{ext} \):

\[
J_{ext} = 2 J_n \frac{1}{\beta A} \xi_e (0) \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_e^2} \right] \times
\]

\[
\left[ 1 - \frac{\sin^2 \left( \gamma/2 \right)}{\sin^2 \left( 2a/\xi_e \right)} \right]^{1/2} .
\]  

(3.8)

Note that the factors \( |f_1|^2 = |f_2|^2 = 1 \) have been ignored.

Equation (3.8) can also be solved for \( \xi_e \) under the given external current \( J_{ext} \). This point of view has been adopted in reference [9]. Here we just point out the relation between equation (3.8) and the usual definition of supercurrents [7]. Let \( \alpha \) be the phase difference between the two nodes: \( \psi_2 = \psi_1 e^{i\alpha} \). We have trivial solutions of equation (3.6):

\[
\cos \alpha \cdot \cos \frac{\gamma}{2} = \cos \frac{2a}{\xi_e}
\]

and

\[
2 \sin \alpha \cdot \cos \frac{\gamma}{2} = -\xi_e Q \cdot \sin \frac{2a}{\xi_e}
\]

from which one deduces:

\[
Q = -\frac{4}{\xi_e} \frac{\partial}{\partial \alpha} \left( \frac{a}{\xi_e} \right) = \frac{2a}{\xi_e^2(0)} \frac{\partial}{\partial \alpha} \times
\]

\[
\left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_e^2} \right]^{1/2} .
\]  

(3.9)

therefore

\[
J = J_n \frac{1}{\beta A} \xi_e (0) \frac{\partial}{\partial \alpha} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_e^2} \right].
\]  

(3.10)

Comparing this equation and the \( \alpha \) derivative of free energy \( F \) (see Eq. (A.17)) one has the simple result for the mean current of one single strand \( J/2 \):

\[
J/2 = J_0 \frac{\partial F}{\partial \alpha} \text{ with } J_0 = 4ea/h .
\]  

(3.11)

Clearly equation (3.11) reproduces the usual definition of \( J \) as derivative of the free energy \( F \) with respect to the phase of the order parameter. Actually, equation (3.11) is a very general relation which holds for an arbitrary geometry and can then be used as an alternative to equation (3.5).

b) Square network (Fig. 3). — For an infinite network, it is not convenient to use equation (3.6) and this particularly for massive contacts as shown
Fig. 3. — An infinite square network with horizontal current injected through massive contacts.

on figure 3. For this we adopt a slightly different point of view, by considering first the solutions of equation (2.1) corresponding to the lowest free energy. With obvious notations, the network equations reduce to [2, 4]

\[ f_{m-1} + f_{m+1} + [2 \cos(k - m\gamma) - \varepsilon] f_m = 0, \quad m = \text{integer}, \quad (3.12) \]

where \( \gamma = 2\pi\Phi/\Phi_0, \varepsilon = 4\cos(a/\xi_s) \) and \( f_{m,n} = e^{ikm}f_m \) is a translationally invariant solution. For rational flux \( \gamma = 2\pi p/q \), the general solutions are \( q \)-periodic: \( f_{m+q} = f_m e^{i\alpha} \) with a Floquet factor \( \alpha \). This leads in particular to a determinental equation [10, 11]

\[ P_q(\varepsilon) = 2\cos\alpha + 2\cos qk \]

(3.13)

where \( P_q(\varepsilon) = \varepsilon^q + \cdots \) is a polynomial of degree \( q \) in \( \varepsilon \).

In what follows we consider mainly the normalized average current per elementary strand \( j_\alpha = J_\alpha/J_0 \) (horizontal) and \( j_t = J_t/J_0 \) (vertical). Due to the \( q \)-periodicity, we have simply

\[ j_\alpha = \frac{1}{q} \sum_{m=0}^{q-1} |f_m|^2 \sin(k - m\gamma). \]

These expressions can be simplified further by using the following identities [11]

\[ \text{Im}(f_m^* f_{m+1}) = \frac{1}{2} \sum_{m=0}^{q-1} |f_m|^2 \frac{\partial \varepsilon}{\partial \alpha} \]

(3.14)

and

\[ \frac{1}{2} \sum_{m=0}^{q-1} |f_m|^2 \sin(k - m\gamma) = \frac{1}{2} \sum_{m=0}^{q-1} |f_m|^2 \frac{\partial \varepsilon}{\partial k} \]

(3.15)

where \( P'(\varepsilon) = dP_q(\varepsilon)/d\varepsilon \). Therefore one obtains finally

\[ j_\alpha = -\frac{q}{2} \frac{\partial \varepsilon}{\partial \alpha} \frac{\varepsilon}{P'(\varepsilon)}, \quad j_t = -\frac{q}{2} \frac{\partial \varepsilon}{\partial qk} \frac{\varepsilon}{P'(\varepsilon)} \]

(3.16)

where the normalization \( \sum_{m=0}^{q-1} |f_m|^2 = q \) has been used. Note that equation (3.16) have been derived here for particular wave functions \( f_{m,n} \). However it is not difficult to check the validity of this result for an arbitrary solution

\[ f_{m,n} = \sum_{\ell=0}^{q-1} c_{\ell} e^{i(k + nq)\gamma} \quad \text{providing} \quad \sum_{\ell=0}^{q-1} |c_{\ell}|^2 = 1. \]

Therefore, the final expression for the current densities become

\[ j_\alpha = J_\alpha \frac{\xi(0)}{\beta \Lambda \xi_s \sin(a/\xi_s)} \frac{q \sin \alpha}{P'(\varepsilon)} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_s^2} \right] \]

(3.17)

\[ j_t = J_t \frac{\xi(0)}{\beta \Lambda \xi_s \sin(a/\xi_s)} \frac{q \sin qk}{P'(\varepsilon)} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_s^2} \right]. \]

Clearly, \( J_\alpha \) and \( J_t \) involve in addition to \( T \) and \( H \), the energy \( \varepsilon \) of the corresponding solution as well as the phase factors \( \alpha \) and \( qk \). For example at the spectrum edge of equation (3.12) (ground state) one has \( P(\varepsilon) = 4, \alpha = qk = 0 \) and then \( J_\alpha = J_t = 0 \). In general equation (3.17) will be used to calculate the allowed solutions \( \varepsilon \) giving a set of external currents \( J_\alpha \) and \( J_t \). In cases where more than one solution is allowed, the lowest in energy will be considered as corresponding to the real state. Making further restriction \( qk = 0 \) or \( \pi \), only the horizontal current remains and \( \alpha \) is fixed by \( \alpha = \text{Arc cos}[P(\varepsilon)/2 \mp 1] \). In this case, one gets

\[ J_\alpha = \pm q \sqrt{\frac{|P(\varepsilon)| - P^2(\varepsilon)/4}{P'(\varepsilon)}} \]

(3.18)

and

\[ J_t = \pm J_t \frac{\xi(0)}{\beta \Lambda \xi_s \sin(a/\xi_s)} \times \]

\[ \times q \sqrt{|P(\varepsilon)| - P^2(\varepsilon)/4} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_s^2} \right]. \]

(3.19)

This complicated expression for the horizontal current can actually be written in terms of other properties of the spectrum of equation (3.12). Indeed, in the appendix C, the following relations are derived:

\[ j_\alpha = \frac{1}{4\pi q(\varepsilon)} \]

(3.20)

\[ J_\alpha = J_\alpha \frac{a\xi(0)}{\beta \Lambda \xi_s^2} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi_s^2} \right] \frac{1}{g(E)} \]

(3.21)
where \( g(\epsilon) \) is the density of states of equation (3.12), \( E = 1/\beta e \) and \( F \) refers to the free energy (Eq. (A.17)).

More generally, the obtained expression for the currents can be cast in a more transparent form. Using equations (3.15) and (A.17), one gets

\[
J_\perp = J_0 \frac{\partial F}{\partial \alpha} \\
J_\parallel = J_0 \frac{\partial F}{\partial \alpha_q}
\]

with \( J_0 = 4 e a / \hbar \). Note that equation (3.22) is identical to equation (3.11) and this is a non surprising common feature to finite and infinite networks.

Finally, on figure 4 is shown a diagonal configuration of the same square network. For such a diagonal injection of currents, we have \( J_\perp = J_\parallel \) and this gives \( k_q = \alpha \) and \( P_q(\epsilon) = 4 \cos \alpha \) respectively. Similar calculation leads in this case to the following expressions of the current:

\[
\begin{align*}
J = \frac{q}{P'(\epsilon)} \sqrt{1 - P^2(\epsilon)/16} \\
J = J_n \frac{\xi(0)}{\beta \xi_s} \sin (\alpha/\xi_s) \times \\
\times \left[ 1 - \frac{T}{T_0} \frac{\epsilon^2(0)}{\xi_s^2} \right]
\end{align*}
\]

which are the counterparts of equations (3.18) and (3.19). This configuration is relevant for the experimental results of section 5.

4. Critical current of a superconducting network

Besides the thermodynamical properties, critical currents \( J_c \) in superconducting networks are of basic interest. Already in bulk superconductors, \( J_c \) is a very important topic both because of its origin (flux jump, thermal instabilities, pair-breaking, etc.) and for its dependence on various parameters (temperature, magnetic field, voltage, ...). In the case worked here, we have a slight simplification, due to the fact that magnetic fluxes are fixed and the inductive fields are smaller than \( H_{cl} \) around wires.

4.1 Definition and simple example. — The critical current \( J_c \) is defined usually [7] as the value of the external current beyond which the normal state is recovered. Let us illustrate this definition on the example of a single wire in zero magnetic field as described in reference [7]. Assuming a small width \( d < \xi(T) \), one can write \( \Psi(s) = \Psi_0 e^{i\psi(s)} \) for the order parameter along the wire, \( \Psi_0 \) being independent of the coordinate \( s \). In this limit, the free energy assumes the following form:

\[
F_s = F_n + \alpha |\Psi|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{1}{2} m^* v_s^2 |\Psi|^2 + \frac{\hbar^2}{8 \pi} (4.1)
\]

where \( v_s \) denotes the velocity of pairs

\[
v_s = \frac{\hbar}{m^*} \frac{\partial \psi}{\partial s}. \quad (4.2)
\]

The conjugate variables \( J \) and \( v_s \) are related by

\[
J = 2 e |\Psi|^2 v_s \quad (4.3)
\]

and the explicit expression of \( J_c \) is obtained in two steps. First, the free energy is minimized with respect to \( |\Psi|^2 \). In the present case this yields

\[
|\Psi|^2 = \Psi_0^2 \left[ 1 - \left( \frac{\xi(T) m^* v_s}{\hbar} \right)^2 \right]. \quad (4.4)
\]

Next, \( J \) as given by equation (4.3) is maximized under the constraint (Eq. (4.4)). Eliminating \( |\Psi|^2 \), \( J \) can be expressed as a cubic function of \( v_s \):

\[
J = 2 e \Psi_0^2 \left[ 1 - \left( \frac{\xi(T) m^* v_s}{\hbar} \right)^2 \right] v_s \quad (4.5)
\]

\( J_c \) is given by the maximum of equation (4.5), and this leads to the 3/2 law

\[
J_c = \frac{2}{3} \frac{\hbar}{\sqrt{3}} \frac{2}{m^*} \frac{\Psi_0^2}{\xi(T)} = J_n \frac{2}{3} \frac{1 - T}{T_0}^{3/2} \quad (4.6)
\]
In order to make contact with the procedure used below, we give another derivation of this result. For this we notice that \((m^* v_\ell /h)^2\) corresponds to the eigenvalue \(1/\xi_e^2\) of the linearized GL equation:

\[
\left( -i \nabla - \frac{2 \pi}{\Phi_0} A \right)^2 \Psi = \frac{1}{\xi_e^2} \Psi
\]  
(4.7)

which for a single wire reads simply

\[
\frac{1}{2} m^* v_i^2 |\Psi|^2 = \frac{\hbar^2}{2 m^*} \frac{1}{\xi_e^2} |\Psi|^2.
\]  
(4.8)

The eigenvalue is trivially

\[
\frac{1}{\xi_e^2} = \frac{m^* v_i^2}{\hbar^2}
\]

and this provides a one to one correspondence between \(\xi_e\) and \(v_i\). With this change of variables, equations (4.4) and (4.5) become

\[
|\Psi|^2 = \Psi^2 \left[ 1 - \left( \frac{\xi(T)}{\xi_e} \right)^2 \right]
\]  
(4.9)

and

\[
J = J_n \frac{\xi^3(0)}{\xi^2(T) \xi_e} \left[ 1 - \left( \frac{\xi(T)}{\xi_e} \right)^2 \right].
\]  
(4.10)

Maximizing equation (4.10) with respect to \(\xi_e\) reproduces the above expression (Eq. (4.6)) of \(J_c\).

4.2 SINGLE LOOP GEOMETRY. — Consider the configuration depicted on figure 2 and let calculate the critical current \(J_c\). The first step of minimization of the free energy leads to the network equation considered in section 2. The resulting expression of the current \(J\) (Eq. (3.8)) is

\[
J = 2 J_n \frac{1}{\beta_A} \left[ \frac{\xi(0)}{\xi_e} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi^2_e} \right] \times \right.
\]

\[
\times \left[ 1 - \frac{\sin^2(\gamma/2)}{\sin^2(2a/\xi_e)} \right] \right]^{1/2}.  \]  
(4.11)

The critical current \(J_c\) is obtained, as for the single wire, by taking the maximum of \(J\) with respect to \(\xi_e\). In general \(J_c\) is a decreasing function of temperature \(T\). At zero field, one gets:

\[
J_c = J_n \frac{4}{3 \sqrt{3}} \left[ 1 - \frac{T}{T_0} \right]^{3/2}.  \]  
(4.12)

As expected, \(J_c\) is twice that of the single wire equation (4.6). For finite magnetic fields, \(J_c\) as obtained numerically from equation (4.11) is still given by a similar expression

\[
J_c = J_n C (\Phi/\Phi_0) \left[ 1 - \frac{T}{T_c(H)} \right]^{3/2}
\]  
(4.13)

where \(C (\Phi/\Phi_0)\) is a numerical constant shown in figure 5. At half quantum flux \(\Phi/\Phi_0 = 1/2\), \(C (\Phi/\Phi_0)\) vanishes and this is due to the symmetric location of nodes 1 and 2 which leads to \(2a/\xi_e = \pi/2\), i.e. \(Q = 0\) at this value of the magnetic field. Such an accidental degeneracy is no longer present for a generic disposition of nodes.

The result shown in figure 5 has to be compared with the recent calculation reported by Fink et al. (Ref. [5]). Agreement is found for \(\Phi/\Phi_0 = 0\), \(1/2\) and 1. However, our approach is valid close to the critical temperature and this allows us to consider various geometries. In this respect, the work reported in reference [5] is an exact treatment for the single ring case with arms and then the difference with our results appears far away from \(T_c\).

4.3 INFINITE SQUARE NETWORK. — As for the single loop case the maximum of \(J_c\) with respect to \(\epsilon\) (or \(\xi_e\)) will be identified as the critical current. The simple limit \(H = 0\) and \(T = T_0\) can be worked out explicitly since \(J_c\) (Eq. (3.19)) takes a simple form \((p = 0, q = 1\) and \(P_1(e) = e\)):

\[
J_c = 2 J_n \frac{1}{\beta_A} \left[ \frac{\cos (a/\xi_e)}{\cos (a/\xi_e) + 1} \right]^{1/2} \times \right.
\]

\[
\times \frac{\xi(0)}{\xi_e} \left[ 1 - \frac{T}{T_0} - \frac{\xi^2(0)}{\xi^2_e} \right]^{1/2}
\]  
(4.14)

and \(\xi^2(0)/\xi^2_e \leq 1 - T/T_0\) is vanishing when \(T \to T_0\). One gets the 3/2 law for \(J_c\):

\[
J_c = J_n \frac{4}{3 \sqrt{6}} \left[ 1 - \frac{T}{T_0} \right]^{3/2}.  \]  
(4.15)

When compared with the single wire result, the only difference between equations (4.15) and (4.6) is
only a geometrical factor of $\sqrt{2}$. Furthermore, equation (4.15) reproduces exactly the main result of reference [5] where the complete GL equations have been considered.

The same calculation of $J_c$ has been repeated for a diagonal configuration (Fig. 4). The final result reads

$$J'_c = J_n \frac{2}{3} \sqrt{3} \left[ 1 - \frac{T}{T_0} \right]^{3/2} \quad (4.16)$$

that is $J'_c = J_c / \sqrt{2}$. This difference is purely geometrical and can be understood as follows. The current through the whole sample reads $I = nwdJ$ where $w$ and $d$ are respectively the width and the thickness of wires, $n$ being the number of strands across the cross-section normal to the current direction. Equation (4.16) results immediately from the ratio of $n$'s in the two considered configurations.

Below $T_0$, and for finite magnetic fields we have calculated $J_c$ numerically. Here we describe some of these results for the horizontal critical current $J_c$.

i) In figure 6, a numerical solution $J_c/J_n$ vs. $\Delta t = (T_0 - T)/T_0$ is shown for $\xi(0)/a = 0.78$ and $H = 0$. As can be seen, the 3/2 law is well obeyed close to $T_c$ in perfect agreement with equation (4.15), but a systematic deviation appears for large $\Delta t$. For this value of $\xi(0)/a$, the maximum of $J_\infty$ with respect to $1/\xi_e$ (unbounded in comparison with $\varepsilon$) is reached in the first quadrant $a/\xi_e \in [0, \pi/2]$. However, the optimal value of $a/\xi_e$ can belong to other quadrants and this depends on the value of $\xi(0)/a$. Such a situation occurs for very small values of $\xi(0)/a$ where the optimal $a/\xi_e$ can assume values larger than $\pi/2$. For instance, if $\pi/2 < a/\xi(0) < 3 \pi/2$, the second quadrant $[\pi/2, 3 \pi/2]$ becomes accessible to $a/\xi_e$, and the possible maximum of $J_\infty$ in this quadrant must be considered. Similarly, for $a/\xi(0) > 3 \pi/2$, the third quadrant $[3 \pi/2, 5 \pi/2]$ becomes accessible and so on. In figure 7 the current $J_\infty$ is shown as a function of $a/\xi_e$ for $\xi(0)/a = 0.063$. For $\Delta t = 0.5$ the maximum of $J_\infty$ takes place in the third quadrant, whereas for $\Delta t = 0.8$, this occurs in the fourth quadrant.

![Fig. 6.](image_url)

![Fig. 7.](image_url)

We have calculated $J_c$ at $H = 0$ and some results are shown on figures 8 and 9. We notice that the maxima of $J_\infty$ occur always at $a/\xi_e = n\pi$, $n = \text{integer}$ (see Fig. 8), where the factor

$$2 \left[ \frac{\cos (a/\xi_e)}{\cos (a/\xi_e) \pm 1} \right]^{1/2}$$

in equation (4.14) assumes a value $\sim \sqrt{2}$. This
ii) The same study has been extended to rational magnetic fields. For simplicity we ignore, in this section, the small field variation of $\beta_n$. In figure 10 we show the temperature variations of $J_c$ at different fields. Here again, the broken line shape is obtained, its origin is the same as for zero field. Furthermore, the 3/2 law is always a good representation and this whatever the value of the magnetic field. The only difference with the zero field case is the value of the slope $C(\Phi/\Phi_0)$ in the 3/2 law:

$$J_c = J_n C(\Phi/\Phi_0) \left[ 1 - \frac{T}{T_c(H)} \right]^{\frac{3}{2}}, \quad T \leq T_c(H).$$

(4.17)

In figure 11 we show the numerical values of $C(\Phi/\Phi_0)$ for different rational fields. As can be seen $C(\Phi/\Phi_0)$ exhibits oscillations as function of $\Phi/\Phi_0 = \frac{p}{q}$. Similar oscillations have been obtained for the generalized Abrikosov parameter.
The main conclusion of this section is certainly the persistence of the 3/2 law on superconducting networks and this whatever the value of the magnetic flux. This law seems to hold in various situations, the net effect of the magnetic field is simply a non trivial prefactor $C(\Phi/\Phi_0)$.

5. Preliminary experiments.

We describe in this section some preliminary measurements on square networks of aluminium and indium filaments. For this purpose we designed a pattern appropriate for critical current measurements. The input current is injected along the diagonal of the cells in order to achieve the same current density in every strand of the network (Fig. 12). The elementary filaments have the following dimensions: length 2.82 $\mu$m, thickness 0.1 $\mu$m and width 0.3 $\mu$m (nominal). The distance between voltage probes is 100 $\mu$m (25 cells) and the total width is 40 $\mu$m (10 cells). The network structure was realized by direct writing of the pattern on PMMA resist by use of an electron beam micro-fabrication (CNET Meylan). Subsequently, the indium film was deposited by thermal evaporation followed by a lift-off of the resist. Optimal uniformity of the film was achieved by cooling the silicon substrate to liquid nitrogen during the evaporation cycle.

Fig. 12. — Micrograph of the grid used for critical currents measurements. Currents are applied along the horizontal axis. Voltage probes are shown.

The measurements were performed close to the critical temperature $T_c(H)$ where the calculations of this paper are expected to be valid. Typical current bias range from $10^{-9}$ to $10^{-5}$ A. The dc current was supplied by a programmable current source (Time Electronics 9818) and voltages induced across the network were measured using a low temperature chopper (resolution $\sim 10$ nV). Data acquisition was controlled by a microcomputer in either mode: $I$-$V$ characteristic and critical current measurement. The sample was shielded from non desirable external fields by a double cylinder of demagnetized $\mu$ metal.

Within a few milliKelvin from $T_c$, $I$-$V$ characteristic curves are reversible in all measured samples. However, the detailed shape of the $I$-$V$ curve depends strongly on the network material. For example, in disordered granular Al networks ($T_c = 1.95$ K, $R = 25$ $\Omega$/square) we observed a characteristic $I$-$V$ shape with a well defined voltage jump at the critical current $I_c$. In this sample $I_c$ is likely controlled by Josephson tunneling between Al-grains. In contrast, the indium network shows, in its $I$-$V$ characteristic, the existence of a resistive state with voltage steps above $T_c$. These steps are due to flux bundles crossing the sample, as confirmed by recent imaging experiments performed on these samples [13]. In this sample the actual critical current can be measured with accuracy provided the voltage threshold $V_s$, for defining $I_c$ is taken at the foot of the resistive state (typically $V_s = 0.2$-0.5 $\mu$V). Measurements at $V_s = 0.5$ $\mu$V are shown in figure 13 in zero

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12.png}
\caption{Micrograph of the grid used for critical currents measurements. Currents are applied along the horizontal axis. Voltage probes are shown.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig13.png}
\caption{Critical current (\mu A) vs. temperature (mK) at zero field. Voltage threshold $V_s = 0.5$ $\mu$V, sample resistance in the normal state $R_N = 0.2588$ $\Omega$. (a) $I_c$ vs. $T$, (b) $I_c^{2/3}$ vs. $T$.}
\end{figure}
external field \((H < 10^{-3} \text{ gauss})\). As can be seen, \(I_c\) follows the expected 3/2 law

\[
I_c = I_0 (1 - T/T_0)^{3/2}
\]

with \(T_0 = 3.405 \text{ K}\) and \(I_0 = 1.247 \text{ A}\). This value of \(I_0\) corresponds to a current density \(J_0 = I_0/nwd = 2 \times 10^8 \text{ A/cm}^2\) deduced from the following sample parameters: \(n = 20\) (number of parallel filaments), \(w = 0.3 \mu \text{m}\) (width of the filaments) and \(d = 0.1 \mu \text{m}\) (thickness). This value has to be compared with equations (4.16) and (2.9). Reasonable agreement is found by using: \(\xi(0) = 0.178 \mu \text{m}\) extracted from the \(T_c(H)\) oscillations and \(\kappa = 0.093\) for \(In\).

Preliminary experiments in a magnetic field (Fig. 14) show that the 3/2 law is also observed at half quantum flux \(\Phi/\Phi_0 = 0, 1,\) and \(1/2\). However it has not yet been possible to test the details predicted in section 4. Far from \(T_c\) one gets a \((T_c - T)^{1/2}\) law which can be attributed to heating effects. Measurements are in progress now and detailed an account will be published elsewhere.

\[
\begin{align*}
\Phi/\Phi_0 &= 0.5 \quad \Phi/\Phi_0 = 1 \quad \Phi/\Phi_0 = 0
\end{align*}
\]

Fig. 14. — Critical current vs. temperature at rational fluxes \(\Phi/\Phi_0 = 0, 1,\) and \(1/2\).

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Appendix A.

The Ginzburg-Landau free energy is written as

\[
F_s = F_n + \int \text{d}^3r \left\{ \frac{\hbar^2}{2m^*} \left[ -\frac{1}{\xi^2(T)} \right] |\Psi|^2 + \left( -i \nabla - \frac{2\pi}{\Phi_0} A \right) \Psi \right|^2 + \frac{\beta}{2} |\Psi|^4 + \frac{h^2}{8\pi} \right\}.
\]  

(A.1)

Near the critical temperature \(T_c(H)\), the linearized GL equation holds and the order parameter \(\Psi(r)\) is an eigenfunction of the following operator with eigenvalue \(E\) (here \(E\) is identical to \(1/\xi^2\)):

\[
\left( -i \nabla - \frac{2\pi}{\Phi_0} A \right)^2 \Psi = E \Psi.
\]  

(A.2)

As the temperature is lowered below the critical line \(T_c(H)\) the order parameter increases progressively in the superconducting phase. The linearized GL equation is no longer sufficient to describe the order parameter distribution. However, as pointed out by Abrikosov, close to \(T_c\) the exact solution must (by continuity) have a strong resemblance to certain solutions of the linearized equation.

Let us introduce the induced field \(h_s\) defined as the difference between the microscopic field \(h\) and the external field \(H\):

\[
h_s = h - H.
\]  

(A.3)

Choosing the external field \(H\) as free variable, the functional to be minimized is the Gibbs free energy \(G\), which is given by the Legendre transformation:

\[
G = F - \hbar H/4\pi.
\]  

(A.4)

Although \(h_s\) is of order \(\Psi^2\) and therefore is assumed small, it cannot be neglected below \(T_c\). By accounting for the variation due to \(h_s\), the equation (A.2) becomes

\[
\Psi \left( -i \nabla - \frac{2\pi}{\Phi_0} A \right)^2 \Psi = \left[ E(H) + h_s \frac{dE}{dH} \right] |\Psi|^2.
\]  

(A.5)

The gradient term in the free energy (A.1) can be replaced by the right hand side of (A.5) when neglecting the contribution of the surface integral. It is useful to set

\[
\Psi(r) = \Psi_0 f(r)
\]  

(A.6)

where \(f(r)\) is a normalized eigenfunction and \(\Psi_0\) is the corresponding amplitude. By regrouping (A.1) and (A.3)-(A.6), one obtains for the Gibbs free energy

\[
G_s = G_n + \int \text{d}^3r \left\{ \frac{\hbar^2}{2m^*} \left[ E(H) + h_s \frac{dE}{dH} \frac{1}{\xi^2(T)} \right] \times \Psi_0^2 |f|^2 + \frac{1}{2} \frac{\beta}{\Psi_0^4} |f|^4 + \frac{h^2}{8\pi} \right\}.
\]  

(A.7)

An error may be made in this substitution. If one wants to keep the higher order term in using the nonlinear GL equation, and replace the \(r \cdot h \cdot s\) by \(E(h)|\Psi|^2 + 2m^*\beta h^2/\hbar^2 |\Psi|^4\), one will find an equation
similar to (A.7) but with the minus sign in front of \( \beta \). The explanation of this paradox is based in the fact that a function \( f(x) = ax + \frac{\beta}{2} x^2 \) may be expressed as \( f(x) = -\frac{\beta}{2} x^2 \) after minimizing \( w.r.t. x \). Because the complete GL equation is obtained from minimization of equation (A.1), the eigenvalue \( E \) may not be anything but \( 1/\xi^2(T) \). Therefore the coefficient of \( |\Psi|^2 \) vanishes and \( \Delta G = -\frac{\beta}{2} |\Psi|^4 + \cdots \). What we assumed up to now is only that \( \Psi \) is eigenfunction of the operator (A.2). The minimization procedure will be done below to find the amplitude of \( \Psi \).

Varying \( h_\perp \) by \( \delta h_\perp \), \( \Psi_0 \) by \( \delta \Psi_0 \), and stating that \( G \) must be stationary, one obtains the two basic equations:

\[
\begin{align*}
\hbar^2 & \frac{\partial^2}{\partial m^2} \frac{dE}{dH} f^2 \Psi_0^2 = 4\pi \hbar^2 \left( 2 m^* \right)^2 \frac{dE}{dH} f^2 \Psi_0^2 \\
\frac{\hbar^2}{2 m^*} \left( \frac{E - \frac{\xi^2(T)}{2}}{\xi^2(T)} \right) |f|^2 & + \beta - 4\pi \left( \frac{2 m^*}{2 m^*} \right)^2 |f|^4 = 0.
\end{align*}
\]

(A.8)

(A.9)

Noting

\[
\beta = 2\pi \left( \frac{2 e h \kappa}{m^* c} \right)^2 = 8\pi \left( \frac{m^*}{2 m^*} \right)^2 \frac{2\pi}{\Phi_0} \kappa^2
\]

(A.10)

where \( \kappa \) is the Ginzburg-Landau parameter, equation (A.9) becomes

\[
E - \frac{1}{\xi^2(T)} + \frac{8\pi e^2}{m^* c^2} (2\kappa^2 - \mu^2) \beta \Lambda(H) \Psi_0^2 = 0
\]

(A.11)

with the parameter \( \mu \) and \( \beta \Lambda \) defined as:

\[
\mu = \frac{\Phi_0}{2\pi} \frac{dE}{dH}
\]

(A.12)

\[
\beta \Lambda(H) = \frac{|\Psi|^4}{|\Psi|^4} = |f|^4.
\]

(A.13)

The latter parameter is simply the generalization of Abrikosov's parameter \( \beta \Lambda \), which was first introduced in the original theory of type II superconductors [8, 14]. In the case of bulk superconductors, \( h_{d2} = \Phi_0/2\pi \xi^2(T) \) so \( \mu = 1 \), and equations (A.8) and (A.11) reduce to equations (6.87) and (6.88) of reference [14].

The averaged value of the order parameter and therefore the free energy follow immediately from equation (A.11):

\[
\Psi_0^2 = \frac{m^* e^2}{8\pi e^2 \beta \Lambda 2 \kappa^2 - \mu^2} \left[ \frac{1}{\xi^2(T)} - E \right]
\]

(A.14)

\[
F_s = F_n - \frac{1}{4\pi \beta \Lambda} \left[ \frac{\kappa \Phi_0}{2\pi} \frac{1}{\beta^2 - \mu^2} \left( \frac{1}{\xi^2(T)} - E \right) \right]^2
\]

(A.15)

\[
+ \left( \frac{\hbar^2}{8\pi} \right)
\]

As indicate in reference [6], for a real 2D sample, the demagnetization effect must be accounted for by replacing the factor \( (2\kappa^2 - \mu^2) \) in equations (A.14) and (A.15) by \( (2\kappa^2 - \mu^2 + \mu^2 N_s/\beta \Lambda) \). Since the demagnetization factor \( N_s \) is near 1 for 2D sample and \( \beta \Lambda \approx 1 \), the remaining factor is simply \( 2\kappa^2 \) and is always positive. Physically, this means that a 2D superconductor behaves as a type II even when \( \kappa = \sqrt{2}/2 \) [15]. By using the usual expression

\[
\Psi_\infty^2(T) = -\alpha(\beta) \frac{1}{\alpha^2/2\beta}
\]

and \( \Delta F_\infty(T) = -\alpha^2/2\beta \), equations (A.14) and (A.15) can be written as

\[
\Psi_\infty^2 = \frac{1}{\beta \Lambda} \Psi_0^2(0) \left[ \frac{1}{T - T_0} - \frac{\xi^2(0)}{\xi^2_\infty} \right]
\]

(A.16)

\[
F_s - F_n = \frac{1}{\beta \Lambda} \Delta F_\infty(0) \left[ \frac{1}{T - T_0} - \frac{\xi^2(0)}{\xi^2_\infty} \right]^2
\]

(A.17)

Appendix B.

For an equilateral network of lattice spacing \( a \), if \( N \) is total the number of nodes and \( z \) the coordination (assumed constant), according to equation (2.3a), the average of order parameter reads

\[
\langle |\Psi|^2 \rangle = \frac{1}{Nza} \sum_{(i,j)} \int d\xi_j \langle \Psi(s_{ij}) \rangle^2
\]

\[
= \frac{2\pi c}{Nza} \sum_{(i,j)} \left[ \langle |\Psi|^2 \rangle + \langle |\Psi|^2 \rangle \right] S_{21}
\]

\[
+ 2 \text{Re} \left( \langle \Psi^* \Psi \rangle e^{-i\gamma_{ij}} \right) S_{22}
\]

where

\[
S_{21} = a - \frac{1}{2} \sin^2 \frac{2a}{\xi_\infty}
\]

\[
S_{22} = \sin \frac{a}{\xi_\infty} \cos \frac{a}{\xi_\infty} \xi_\infty
\]

\[
\sum_{(i,j)} \text{represents the sum over all the pairs of adjacent nodes and can be written as } \sum_{(i,j)} = \sum_i \sum_j \text{, where } j \text{ runs over all the nearest neighbours of } i \text{. According to equation (2.1b),}
\]

\[
\sum_j \Psi_j e^{-i\gamma_{ij}} = z \cos \frac{a}{\xi_\infty} \Psi_i
\]
Noting the equality $S_{21} + \cos \frac{a}{\xi_e} S_{22} = \sin \frac{a}{\xi_e}$, we find
\[
\langle |\Psi|^2 \rangle = \frac{1}{N} \sum_i |\Psi_i|^2.
\]

Appendix C.

In order to prove the result of equations (3.20) and (3.21), we define the density of states $g(E)$ of the eigenvalues of equation (3.12) as follows
\[
g(E) = \frac{1}{4 \pi q} \sum_{m=0}^{q-1} \int_0^{2\pi} \sum_{k=0}^{1} \delta \left( \epsilon_m(\alpha, k \pi) - \epsilon \right).
\]
(C.1)

Here $\epsilon_m$ refers to the $m$-th subband of the spectrum. As shown in reference [12], $g(E)$ can be expressed in term of the polynomial $P(\epsilon)$:
\[
g(E) = \frac{1}{4 \pi q} \frac{|P'(\epsilon)|}{\sqrt{|P(\epsilon)| - P^2(\epsilon)/4}}.
\]
(C.2)

Comparison with equation (3.18) leads immediately to the result equation (3.20).

Using $E = 1/\xi_e^2$ as variable and the relation $e = 4 \cos (a/\xi_e)$ one deduces from equation (C.2) the density of states for $E$:
\[
g(E) = \frac{dE}{d\epsilon} = \frac{a \xi_e}{2 \pi q} \left| \sin \frac{a}{\xi_e} \right| \times
\frac{x}{\sqrt{|P| - P^2/4}}.
\]
(C.3)

Equation (3.21) follows from (C.3). Finally, the expression (A.17) for the free energy allows to deduce
\[
J_0 = \frac{4ae}{\hbar} \frac{1}{g(F)}
\]
(C.4)

where $g(F)$ denotes the density of states of the free energy.

References