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## Statistics of a polymer in a random potential, with implications for a nonlinear interfacial growth model

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**Résumé.** — Nous examinons et généralisons des résultats récents concernant la statistique de chaînes gaussiennes de longueur  $t$  dans un potentiel aléatoire figé  $\mu(r)$ . Ce problème peut être mis en correspondance avec l'équation non linéaire d'évolution  $\dot{h}(r, t) = \nabla^2 h + (\nabla h)^2 - \mu(r)$  d'une surface croissant en présence d'un flux aléatoire  $-\mu(r)$  indépendant du temps mais variant dans l'espace. [Le problème correspondant pour un polymère dirigé, qui équivaut à un flux variant dans le temps et l'espace, a été étudié par M. Kardar, G. Parisi et Y.-C. Zhang, *Phys. Rev. Lett.* **56** (1986) 889]. Nous prédisons l'effondrement d'une chaîne libre avec vrai potentiel de bruit blanc en dimensions  $d = 2, 3$  avec une dimension linéaire  $R \sim v^{1/(d-4)} [\ln V]^{1/(d-4)}$ , où  $v$  est la variance du potentiel (supposée petite) et  $V$ , le volume du système. Ce résultat est en accord avec un calcul récent de S. F. Edwards et M. Muthukumar (*J. Chem. Phys.*, **89** (1988) 2435) par la méthode des répliques, mais avec une dépendance logarithmique en  $V$  supplémentaire. Nous commentons la différence entre le cas du potentiel figé et non figé, faisant référence au phénomène de *localisation* tel qu'il s'applique aux polymères en environnement figé ; nous discutons l'effet d'un potentiel saturant. Au contraire, une chaîne avec une extrémité fixée à une place particulière (soit  $r$ ) est organisée typiquement en *tadpole* consistant en une tête effondrée (occupant un minimum profond de  $\mu$ ) reliée à  $r$  par une queue étendue. La surface d'énergie libre d'une telle chaîne  $F(r, t)$  est composée asymptotiquement d'une série de vallées coniques séparées par des arêtes aiguës. Nous obtenons plusieurs résultats pour la statistique de cette surface. L'évolution du profil dans le modèle de croissance interfaciale équivalent est obtenue par  $h(r, t) = -F(r, t)$  et correspond donc à une série de montagnes coniques.

**Abstract.** — We examine and extend recent results for the statistics of a gaussian polymer chain of length  $t$  in a quenched random potential  $\mu(r)$ . This problem can be mapped onto one involving the nonlinear evolution equation  $\dot{h}(r, t) = \nabla^2 h + (\nabla h)^2 - \mu(r)$  for a growing interface in the presence of a time-independent, but spatially varying, random flux  $-\mu(r)$ . [The corresponding problem for a directed polymer, equivalent to a flux random in both time and space, was studied by M. Kardar, G. Parisi and Y.-C. Zhang, *Phys. Rev. Lett.* **56** (1986) 889.] A free chain in a true white-noise potential, in space dimensions  $d = 2, 3$ , is predicted to collapse to a linear size  $R \sim v^{1/(d-4)} [\ln V]^{1/(d-4)}$ , where  $v$  is the mean-square potential fluctuation (presumed small) and  $V$  the volume of the system. This agrees with a recent replica calculation of S. F. Edwards and M. Muthukumar (*J. Chem. Phys.*, **89** (1988) 2435), but with an extra logarithmic dependence on  $V$ . The difference between this result and that for an annealed potential is commented upon, with reference to the phenomenon of *localization* as it applies to polymers in a quenched environment ; the effect of a saturating potential is discussed. In contrast, a chain with one end fixed at a particular place (say at  $r$ ) is typically arranged as a *tadpole* consisting of a collapsed head region (occupying a deep minimum of  $\mu$ ) connected to  $r$  by an extended tail. The free energy surface of such a chain  $F(r, t)$  consists asymptotically of a series of conical valleys, separated by sharp ridges. Several results for the statistics of this surface are obtained. The evolved profile in the equivalent interfacial growth model is obtained from  $h(r, t) = -F(r, t)$ , and hence corresponds to a series of conical mountains.

### 1. Introduction.

There has been renewed interest recently in the statistical properties of polymer chains in a quenched random environment [1-14]. In this paper we discuss,

using heuristic arguments, the free energy and gyration radius of a chain (without excluded volume) in a quenched white-noise (gaussian, spatially uncorrelated) potential. We pay close attention to the role of system size in determining — through the statistics

of the deepest minima of the potential — the equilibrium gyration radius of a chain, and thereby elucidate the difference between the quenched and annealed average, as exemplified by a recent replica calculation of the gyration radius [5]. The effects of a saturating potential [7] (whose deepest minima do not increase indefinitely with system size) are also discussed.

Then we make some further predictions. These mainly concern the configurations of a chain with one end fixed at a particular spatial location. The statistics of such a chain are, not surprisingly, rather different from those of a chain that is free to move anywhere in the system. While it was shown [5, 7] that in the latter case the chain shrivels up in a deep minimum of the potential (so that its gyration radius becomes independent of chain length in the limit of long chains [7]), this does not generally occur for the fixed-end case. It is argued that instead the chain makes a compromise *tadpole* configuration: part of it is strongly stretched so as to reach a deep minimum of the potential, in which the remainder shrivels up. The gyration radius is controlled by the extended « tail » of the tadpole.

Apart from its relevance to, e.g., the adsorption of polymers on rough surfaces, a major motivation for this study concerns its relation to a problem in interfacial growth. The connection between these two apparently unrelated areas was made by Kardar, Parisi and Zhang (hereafter KPZ) [8], who studied the nonlinear growth equation (in dimensionless form)

$$\dot{h} = \nabla^2 h + (\nabla h)^2 + \eta(r, t) \quad (1.1)$$

where  $h(r, t)$  is the height of an interface at time  $t$  and spatial co-ordinate  $r$  in the basal plane. [For growth in  $d + 1$  spatial dimensions,  $r$  is a  $d$ -dimensional vector.]

In equation (1.1) the term in  $\nabla^2 h$  represents the effects of surface tension [15] (presuming some local rearrangement of the growing surface is possible) whereas the nonlinear term  $[(\nabla h)^2]$  is the first correction due, for example, to the departure of the local growth direction (normal to the surface) from the normal to the underlying basal plane. Finally  $\eta(r, t)$  is a source term representing a fluctuating flux onto the surface (by taking a moving frame,  $\eta$  can be made to have zero mean) whose statistical properties are prescribed. Thus the KPZ equation (1.1) provides a canonical model for the study of nonlinear effects in interfacial growth, describing, for example, the time evolution of a solid surface bombarded by a random « rain » of sticky particles [16]. The same mathematics may also describe simple nonlinear erosion processes so long as the replacement  $h \rightarrow -h$  is made, thus reversing the sign of the nonlinear term [8]. The growth equation

is also closely related to Burgers' equation

$$\dot{u} + (u \cdot \nabla) u = \nabla^2 u - \nabla \eta$$

for the velocity  $u$  of a curl-free fluid in the presence of a stirring term  $\sim \nabla \eta$  [17]; this may be derived by setting  $u = -\nabla h$  in (1.1).

Fortunately (1.1) can be easily linearized by a Cole-Hopf transformation,  $W = \exp(h)$  to give

$$\dot{W}(r, t) = \nabla^2 W - \mu(r, t) W \quad (1.2a)$$

$$\mu(r, t) \equiv -\eta(r, t). \quad (1.2b)$$

If one were to impose the boundary condition

$$W(r, 0) = \delta^d(r - r'), \quad (1.3)$$

this would reduce to the standard equation [18] for the partition function  $W(r, t)$  of a polymer chain of length  $t$ , whose zeroth monomer is at spatial position  $r'$  and whose last ( $t$ -th) is at position  $r$ ; the chain experiences a potential  $\mu(r, t)$  which depends on both the spatial coordinates, and, in general, the position ( $t$ ) of a monomer in the chemical sequence. [Note: Eq. (1.2) is *not* the diffusion equation for a particle in the potential  $\mu$ .]

For the interfacial growth problem, the relevant initial condition is usually not (1.3) but rather

$$h(r, 0) = 0 \quad (1.4)$$

corresponding to a flat interface at time zero. Since (1.2) is linear, an appropriate solution may be constructed as

$$Z(r, t) = \int d^d r' W_{r'}(r, t) \quad (1.5)$$

with  $W_{r'}(r, t)$  obeying (1.2) and (1.3). Equation (1.5) is the partition function of a chain of length  $t$  whose last (or, equivalently, first) monomer is held at  $r$ , but whose other end occupies an unrestricted position. Writing as usual the free energy  $F = -\ln Z$ , we see that the solution  $h(r, t)$  to equation (1.1) for initial condition (1.5) obeys

$$h(r, t) = -F(r, t). \quad (1.6)$$

That is, the height profile  $h(r, t)$  of the growing interface at time  $t$  is simply the upside-down free energy landscape for a chain of length  $t$ , with one end fixed at  $r$ , in the potential  $\mu(r, t)$ .

Having established this, KPZ [8], and also Kardar and Zhang [9] focussed attention on the case when  $\eta \equiv -\mu(x, t)$  is fully random in both space and time. In that case it is convenient to think of  $t$  as an extra spatial co-ordinate, in which the chain is fully directed. These authors found various scaling results describing the asymptotic roughness of the interface and/or the free energy landscape of the

directed polymer in this case, including the exact values of the relevant scaling exponents for  $d = 1$ . [See also Ref. 10.] Specifically the width of the interface  $w(R, t)$  measured over a lateral length  $L$  after time  $t$  was predicted to obey  $w(L, t) \sim L^\chi w_0(t/L^z)$  with  $z = 3/2$  and  $\chi = 1/2$ . However, the overall picture remains far from complete, insofar as the detailed structure of the evolved surface is only partially characterized by the two critical indices  $\chi$  and  $z$ .

In the present work we consider the case when  $\eta(r, t) \equiv -\mu(r)$  is a time-independent flux that varies randomly in space :

$$\dot{h}(r, t) = \nabla^2 h + (\nabla h)^2 - \mu(r). \quad (1.7)$$

This is equivalent to the problem of an *undirected* polymer in a  $d$ -dimensional random environment, as was introduced at the start of this section. Equation (1.7) has not been studied in detail, although it clearly relates to a realizable, and perhaps common, type of interfacial growth.

In one sense the model is less interesting than the case studied by KPZ — for example, we shall see that the scaling indices characterizing the evolution of the surface are comparatively trivial. On the other hand, by simple arguments applied to the equivalent polymer problem, we can predict in some detail the statistics of the surface, rather than just its exponents. This is rather remarkable, considering the simultaneous presence of nonlinearity (albeit removable by Cole-Hopf) and noise ( $\mu(r)$ ) in (Eq. (1.7)). The insight thus gained could be valuable in a number of related nonlinear noisy problems, for which there is no hope of any comparable simplification.

The remainder of this paper is organized as follows. In section 2 we consider the case of a free chain in a true white-noise potential. In section 3 we discuss how the results must be modified in the presence of a saturating (bounded) potential. In section 4 we return to the true white noise limit, and predict the statistics of a chain with one end fixed at a prescribed position. (There is a close relation with the work of Zhang [13] and Engel and Ebeling [14] who studied the strong-disorder limit for this case.) In section 5 we study the implications for the nonlinear growth problem. Section 6 contains a discussion of our approximations, and a comparison of our results for the growth model, which pertain to the case of a time-independent random flux, with what is known about other cases. (These are a sudden burst of random flux at time zero, and a flux that is random in both space and time, both of which were studied by KPZ [8].)

## 2. A free polymer in a white-noise potential.

Consider a gaussian polymer chain. As a function of its size (either end-to-end distance, or radius of

gyration)  $R$ , the free energy of such a chain may be written as [18]

$$F \sim R^2/tl^2, \quad R^2 \gg tl^2 \quad (2.1a)$$

$$F \sim tl^2/R^2, \quad R^2 \ll tl^2. \quad (2.1b)$$

Here  $t$  is the number of monomers in the chain and  $l$  a monomeric length, so that the gyration radius in equilibrium  $\sim t^{1/2}l$ . The limit of large  $t$  is always the one of interest.

Now let us define our random potential. We imagine space to be divided into monomersized regions, and choose a potential  $\mu$  independently in each place. We take this to be governed by gaussian statistics of zero mean so that

$$P(\mu(r)) \propto v^{-1/2} e^{-\mu^2/2v}. \quad (2.2)$$

The continuum limit for the polymer and the potential [the latter yielding white noise for which  $\langle \mu(r)\mu(r') \rangle = v\delta^d(r-r')$ ] may be taken simultaneously by letting  $l \rightarrow 0$  at fixed  $tl^2$ . In this work we will retain a cutoff at short distance scales, and measure all lengths in multiples of this cutoff : this involves setting  $l = 1$ . Note also that  $v$  must be small for the description of the polymer by use of equation (1.2) to be valid.

Next, consider the coarse-grained (average) potential  $\bar{\mu}$  in some region of volume  $a$ . This is the average of  $a$  independent gaussian terms and therefore is also gaussian distributed with  $\langle \bar{\mu}^2 \rangle = v/a$  :

$$P_a(\bar{\mu}) \propto (v/a)^{-1/2} e^{-a\bar{\mu}^2/2v}. \quad (2.3)$$

Note that for  $a$  large, the same applies even if  $\mu$  itself is not gaussian, so long as one is not interested in the extreme tails of the  $\bar{\mu}$  distribution. For example if  $\mu = \pm v^{1/2}$  with equal probability,  $P_a(\bar{\mu})$  obeys (2.3) for  $\bar{\mu}^2 \ll v$  whereas for  $\bar{\mu}^2 > v$ ,  $P_a(\bar{\mu})$  is zero.

Combining results (2.1a) and (2.3) we can start to discuss the statistics of a polymer in a random potential. We presume the chain to shrink into a volume  $a$  corresponding to a place where the mean potential  $\bar{\mu}$  takes on a lower value than usual. In this case the total free energy of the chain may crudely (neglecting all numerical factors) be estimated as

$$F(a, \bar{\mu}) = t/R^2 + t\bar{\mu} + a\bar{\mu}^2/v \quad (2.4)$$

where  $R = a^{1/d}$ . The last term allows for the number of suitable places available to the polymer, i.e., it is the entropy  $\ln P_a(\bar{\mu})$  associated with a fluctuation in the potential of mean  $\bar{\mu}$  over a region of volume  $a$ . [Below we will argue that this term must be modified, at least for finite system sizes, but first we follow through to see the consequences of equation (2.4).] Minimizing this free energy, over both  $\bar{\mu}$  and  $a$ , shows that for any  $d \geq 2$ ,  $F \rightarrow -\infty$  as

$R \rightarrow 0$ . Thus the mean size of the chain is zero, or in the presence of the cut-off as described above, one monomer length :

$$R \sim 1, \quad d \geq 2. \quad (2.5a)$$

For  $d < 2$ , equation (2.4) has a minimum at

$$R \sim (tv)^{1/(d-2)} \quad d < 2, \quad (2.5b)$$

which, since it is a decreasing function of  $t$ , is also cut off at  $R \sim 1$  in the long-chain limit ( $t \geq 1/v$ ).

These results are the same as in the self-attracting polymer problem. The equivalence arises because the above treatment of the entropy takes no account of the fact that the random potential is quenched ; equation (2.4) is no different from what one would write down for an *annealed* potential that is able to adjust itself locally to lower the total free energy of the system. [Such a potential can easily be integrated out to give a self-attracting polymer.] The result (2.5a) apparently contradicts a recent prediction of Edwards and Muthukumar [5] that for  $v$  small,

$$R \sim v^{-1/(4-d)} \quad d = 2, 3. \quad (2.6)$$

This prediction was found with the replica formalism which deals directly in quenched averages and should therefore be more reliable than the simple argument behind equation (2.4). On the other hand, in an infinite system containing a finite (albeit long) chain, one might expect the quenched and annealed averages to coincide. The infinite system can be divided into subregions (much larger than the chain) each containing a different realization of the potential ; the chain can sample all of these, including configurations arbitrarily similar to those which it would induce around itself in the annealed case.

To study this issue in more detail, and reconcile the predictions (2.5a) and (2.6), consider placing the chain in a finite box of volume  $V$ . Since the box is finite, arbitrarily deep potential minima (which dominate the annealed average) are not present. Instead, if we consider a subdivision into regions of volume  $a$  (for which the probability distribution of  $\bar{\mu}$  still obeys (2.3)) we can estimate that the most negative  $\bar{\mu}$  (call this  $-u$ ) of any of these regions obeys

$$\int_{-\infty}^{-u} P_a(\bar{\mu}) \simeq a/V \quad (2.7)$$

or

$$au^2/v \sim \ln V + O(\ln a) + O(\ln u). \quad (2.8)$$

Retaining only the leading behaviour at large volumes (which amounts to keeping only the first term on the right), as will systematically be done below without comment, we now write instead of (2.4)

$$F(a) = t/R^2 - t[va^{-1} \ln V]^{1/2}. \quad (2.9)$$

Thus we presume the polymer to have a certain size  $R \sim a^{1/d}$ , and place it in the deepest well in the system of that size. Minimization of  $F$  over  $a$  now yields, for  $d < 4$ , a finite radius  $R$  obeying

$$R \sim [v \ln V]^{-1/(4-d)} \quad (2.10a)$$

where we have again retained the leading behaviour for large  $V$ . Note that the corresponding potential  $\bar{\mu} = -u$  obeys

$$u \sim [v \ln V]^{2/(4-d)}; \quad (2.10b)$$

the chain is bound to its well by a net attractive energy of order  $tu$ .

The above calculation should be correct for large  $V$  if (and only if) the configurations of the polymer are dominated by those in which it is confined to a single small region, namely, the deepest well in the system of linear size  $R$ . [In principle a finite number of such regions could be involved ; however, it is easily shown that the few deepest wells in the system have differences in  $u$  which remain of order  $\delta u \equiv v^{2/(4-d)}$  as  $V \rightarrow \infty$ . This means that a long enough chain will almost always be found in the single, deepest well.] Roughly speaking, this holds when the binding energy of the chain in the well,  $tu$ , exceeds the translational entropy it would have if free to move throughout the entire system ( $\sim \ln V$ ). Using (2.10a) this gives

$$\ln V < t[v \ln V]^{2/(4-d)} \quad (2.11)$$

as the criterion under which the result (2.10b) can be applied. In space dimensions  $2 < d < 4$ , this is always satisfied for large enough  $V$  ; in  $d = 2$  it also holds so long as  $t$  is large enough. Correspondingly for  $d \geq 2$  the polymer is *localized* in the sense that, even as  $V \rightarrow \infty$ , its configurations are dominated by those in which it is trapped in a single small neighbourhood. For the same reason, if two long chains are placed in the system and the volume  $V$  is allowed to increase indefinitely, the chains will nonetheless be found in the same place with finite probability. [Indeed, with probability 1 as  $t \rightarrow \infty$ .] In the replica calculation, the corresponding phenomenon is the formation of bound-states among different replicas. The word *localization* [5, 6] seems quite appropriate in this context ; note that a chain can be collapsed ( $R \sim t^0$ ) without being localized in this sense. The localization phenomenon appears to be closely related to the « freezing » of a spin glass at low temperatures, as has been discussed by Derrida and Spohn [12] in their study of the statistics of directed polymers on a disordered tree. Indeed, the above calculation has several features in common with Derrida's analysis of the random energy model (REM) for spin glasses [19] ; but since in the polymer problem the relevant energy scale

( $tu$ ) increases linearly with chain length, one is always in the frozen (*i.e.*, localized) phase for large  $t$ .

In contrast, for  $d < 2$ , equation (2.10) holds only for system volumes  $V$  smaller than a critical value  $V'$  at which equality occurs in (2.11). A simple manipulation shows that the corresponding radius  $R = R'$  from (2.10) coincides with that predicted by the annealed average (2.5b). For systems larger than  $V'$ , the chain samples a large number of similar minima, so that the reasoning behind equation (2.9) breaks down; the annealed result (2.5b) is valid instead.

These arguments allow us to reconcile, to a large extent, the replica and annealed approaches. In  $d = 3$ , for any noninfinite large volume  $V$ , the dependence in (2.10) on the strength of the potential ( $v$ ) is that given by Edwards and Muthukumar, (Eq. (2.6)). The same holds in  $d = 2$  so long as  $t$  is large. Nonetheless, in the infinite volume limit the chain does collapse to zero size (as a negative power of  $\ln V$ ). For  $d = 1$ , however, the chain is not localized in the infinite volume limit so that the annealed result, (2.5b) is valid. But in practice, equation (2.10b) still applies for systems that are not large compared to  $V' (d = 1) \sim \exp[t^3 v^2]$ , a quantity which can be enormous for large  $t$ .

Throughout the above, we assumed that the collapsed chain can be modelled as roughly spherical in shape. This assumption is justified in slightly more detail in section 6.

### 3. Effects of saturation.

The conclusions just reached must be modified for a potential which is bounded (unlike true white noise). In the saturating case, the binding energy of a chain to its coarse-grained potential well does not increase logarithmically with system size  $V$ , but remains finite as  $V \rightarrow \infty$ . Since the translational entropy of an unbound chain does increase with volume, a chain cannot be localized, in the sense defined above, for a large enough system. In terms of replicas, the binding energy between different replicas reaches a saturating value as  $V \rightarrow \infty$ , so the replica bound state must always dissociate when placed in a large enough volume. On the other hand, the volume required for dissociation is exponentially large in the binding energy, which is a power of  $t$ , so that a quenched average may still be appropriate for reasonable-sized systems.

It is interesting to study the saturation effect directly in a simple case: we take the local potential to be discretized as  $\mu = \pm v^{1/2}$ . (As usual, we take  $v$  small so that the potential is weak on a monomeric scale.) As pointed out in connection with equation (2.3), the statistics of  $\bar{\mu}$ , the coarse-grained potential in a region of volume  $a$ , remain gaussian for  $\bar{\mu}^2 \ll v$ . Thus the calculations of the previous

section remain valid so long as the depth  $\bar{\mu}$  of the well in which the chain is typically found obeys this criterion. Inspection of the quenched results (2.9) and (2.10) reveal that the gaussian approximation fails when  $R$  falls below a critical value  $R_{\text{sat}} \sim v^{-1/4}$ .

This occurs for  $V > V_{\text{sat}}$  where  $\ln V_{\text{sat}} \sim v^{-d/4}$ , as found by substituting  $R = R_{\text{sat}}$  in (2.10a). In this regime, the depth of the deepest well cannot increase further beyond the saturated value  $u \sim v^{1/2}$ . Instead, as the system volume increases, the size of the *largest* such well increases: this we estimate from

$$V \exp[-a] \sim 1. \quad (3.1)$$

The exponential factor is the probability that  $\mu$  takes the value  $-v^{1/2}$  in (a finite fraction of) the sites in a region whose volume is of order  $a$ . The corresponding chain radius (assuming  $R \sim a^{1/d}$ ) obeys

$$R \sim [\ln V]^{1/d}. \quad (3.2)$$

On the other hand, an annealed average similar to (2.4) may be estimated in this saturating case by writing

$$F(R) \sim t/R^2 - tv^{1/2} + R^d \quad (3.3)$$

where the first term is the energy to compress a chain into a region of linear size  $R$ ; the second corresponds to the (saturated) potential; and the third is the entropy associated with a saturated well of linear size  $R$  (as in Eq. (3.1)). The annealed prediction is therefore

$$R \sim t^{1/(d+2)} \quad (3.4)$$

which is, in contrast to equations (2.5) for the nonsaturating case, an increasing function of chain length. As stated at the beginning of this section, the annealed behaviour must take over for large enough  $V$ ; we argue that this happens when the quenched prediction for the radius (which is for a saturating potential a slowly increasing function of  $V$ ) exceeds the annealed prediction. This criterion identifies the disappearance of localization, in the saturating case, at system sizes of order  $V \sim \exp[t^{d/(d+2)}]$ .

To summarize, for the saturating potential  $\mu = \pm v^{1/2}$ , the quenched-average white noise calculation given in the previous section remains valid so long as it predicts a chain radius  $R > R_{\text{sat}} \sim v^{-1/4}$ . This holds for systems of volume  $V < V_{\text{sat}}$  obeying  $\ln V_{\text{sat}} \sim v^{-d/4}$ ; for  $V_{\text{sat}} < V < \exp[t^{d/(d+2)}]$  the chain remains localized in a region of volume  $a \sim \ln V$ , whereas for very large volumes the chain is no longer localized. In practice, some details of these results, especially in the intermediate regime, may be altered by fluctuations in the shape of the saturated « well » in which the chain resides. (For a discussion, see Sect. 6.)

#### 4. A chain with one end fixed in space.

The above discussion is for a chain which is free to move throughout the system. We now extend it to the case of a polymer with one end fixed ; to avoid excessive elaboration, we restrict ourselves to the true white-noise potential. The results will be useful in discussing the long-time properties of the non-linear interfacial growth equation (1.7). The arguments given in this section are quite closely related to those given by Zhang [13] and discussed by Engel and Ebeling [14], in the context of diffusion and localization for Schrödinger's equation in a random potential (that is Eq. (1.2a) with imaginary time). Note, however, that in Zhang's variational approach, all polymer configurations other than the one of lowest energy (*not* free energy) are neglected entirely : the chain entropy term (Eqs. (2.1)) is not taken into account. Such an approach is valid for strong disorder ( $v \gg 1$ ), whereas in this paper we are concerned with the opposite limit of small  $v$ . Correspondingly, our variational calculation (which includes entropic contributions to the free energy) is rather more complicated, and our results are somewhat different.

First we observe that with a fixed end, the chain can only be influenced, even in principle, by the potential in a neighbourhood of volume  $V \sim t^d$  (since  $t$  is the length of the chain when fully extended). It is clear physically that a quenched average is always needed in this case, even for  $d < 2$ . [This may be confirmed by noting that this explorable volume is, for large  $t$ , much less than  $V'$  as defined in section 2.] A free chain, enclosed in the explorable volume, but not constrained by its end, would always be confined to a single well. Rather than do a replica calculation, which is quite complicated in the case of a fixed end, we predict the configuration of the chain by a simple argument, as follows.

Suppose we confine a free chain to a region of linear dimension  $X$ . Within this region there will be an optimal well of radius  $R \ll X$  in which the chain will sit with binding energy per monomer (cf. Eqs. (2.10))

$$R \sim [v \ln X]^{-1/(4-d)} \quad (4.1a)$$

$$u(X) \sim [v \ln X]^{2/(4-d)}. \quad (4.1b)$$

In these expressions we have retained only the leading nontrivial behaviour at large  $X$ . Now imagine pulling one end of the chain out of the well to a distance  $Y \gg R$ . Since one is pulling against a fixed binding energy per monomer, the free energy should increase linearly with distance. The corresponding chain configuration is a *tadpole* with a collapsed head (in the deep well) and an extended tail of length  $Y$ . Indeed, we may write

$$F \sim Y^2/m - (t - m) u(X) \quad (4.2)$$

where the first term is the elastic energy to pull a section of chain containing  $m$  monomers out to distance  $Y$ , and the second reflects the lost binding energy that this entails. There is also a term of order  $m^{1/2}$  from the random potential felt by the extended section, but this is small (as may be checked afterwards). Note that  $u(X)$  is itself a random variable ; its fluctuations are of order  $\delta u \sim u(2X) - u(X) \sim v^{2/(4-d)}$ .

Minimizing (4.2) on  $m$  one finds

$$m \sim Yu(X)^{-1/2} \quad (4.3a)$$

$$F \sim -tu(X) + Yu(X)^{1/2}. \quad (4.3b)$$

The second equation demonstrates a linear dependence on  $Y$ , corresponding to a V-shaped potential (conical or hyperconical in  $d = 2, 3$ ), of slope  $\partial F/\partial Y \sim u(X)^{1/2}$ . The first gives the number of monomers  $m$  in the extended tail of the tadpole. If we continue to pull on the tail of the tadpole, the conical free energy surface will eventually cross that centered on a different well nearby. When this happens, the chain will suddenly jump to a new configuration in which the head of the tadpole is in this other well. To estimate the typical length  $Y^*(t)$  of the tadpole at which this occurs (which is comparable to the mean length of a tadpole whose end point  $r$  is chosen at random) we minimize

$$F(Y) \sim -tY + Yu(Y)^{1/2} \quad (4.4)$$

with respect to  $Y$ . This expression estimates the free energy of a typical tadpole of length  $Y$  ; for self-consistency, the confinement length  $X$ , introduced above, is also set equal to  $Y$ . The minimization gives (for  $Y$  large) a value

$$Y^*(t) \sim tv^{1/(4-d)} [\ln Y^*]^{1/(4-d)-1} \\ \sim tv^{1/(4-d)} [\ln t]^{1/(4-d)-1}. \quad (4.5)$$

The corresponding number of monomers in the tail is

$$m^*(t) \sim Y^* u(Y^*)^{-1/2} \sim t/\ln t. \quad (4.6)$$

The mean binding energy per monomer,  $u^*(t)$ , and well size  $R^*(t)$  obey

$$u^*(t) \equiv u(Y^*) \sim [v \ln t]^{2/(4-d)} \quad (4.7a)$$

$$R^*(t) \sim [v \ln t]^{-1/(4-d)}. \quad (4.7b)$$

To check that our use of a gaussian elasticity term in equation (4.2) is selfconsistent, we can estimate the number of monomers  $m_b$  in a « blob » of the extended tail section, defined as a unit for which the gaussian stretching energy (which is in turn roughly equal to the lost binding energy of the same unit) is of order unity :  $m_b \sim 1/u^*$ . One can ask what size of blob ( $\bar{m}_b$ , say) is required before the random poten-

tial induces significant departures from gaussian statistics. The r.m.s. fluctuation in the coarse-grained potential on the scale of such a blob,  $(v/\tilde{m}_b^{d/2})^{1/2}$ , multiplied by the number of monomers in it, should be of order unity. This gives  $\tilde{m}_b \sim v^{-2/(4-d)}$ . Since, for large  $t$ ,  $m_b \ll \tilde{m}_b$ , our treatment of the extended tail using a gaussian elasticity expression should be valid.

More seriously, the estimate (4.4) neglects the fact that the depth  $u(Y)$  of the deepest well within a linear range  $Y$  is itself a fluctuating quantity with r.m.s. deviation of order

$$\delta u \sim u(Y) - u(2Y) \sim v^{2/(4-d)}. \quad (4.8)$$

The neglect of this fluctuation is consistent so long as the predicted mean tadpole length  $Y^*$  corresponds to an energy of the tail  $\Delta F \sim Y^* u(Y^*)^{1/2}$  that exceeds the mean fluctuation  $t \delta u$  in the binding energy of the head :

$$\begin{aligned} \Delta F(t) \sim m^*(t) u^*(t) \sim t v^{2/(4-d)} [\ln t]^{2/(4-d)-1} \\ \gg t \delta u \sim t v^{2/(4-d)}. \end{aligned} \quad (4.9)$$

If this holds (which it does asymptotically for  $d > 2$ ),  $\Delta F$  obeying (4.9) is, to within a numerical factor, the r.m.s. fluctuation in the height of the free energy surface. The minima of the surface are at the tips of conical valleys (depth  $\sim tu^*(t)$ ) whereas the maxima lie at the tops of hyperbolic ridges along which the conical surfaces associated with neighbouring wells cross one another ; these are elevated from the minima by  $\sim \Delta F(t)$ . Note that the ridges are asymptotically sharp in the limit of long chains : only in a region of negligible width compared to  $Y^*$  (the spacing between minima) do tadpole configurations involving two different wells simultaneously contribute to  $F(r, t)$ .

These considerations almost completely determine the asymptotic (large  $t$ ) statistics of the free energy surface for  $2 < d < 4$ . This surface consists of conical sections of mean depth  $\sim tu^*$  and fluctuation  $t \delta u$ . The slope of each cone is proportional to its depth ; the mean slope is  $\sim u^{*1/2}$ . (The distributions of both depth and slope become narrow compared to the mean at large  $t$ .) The minima are placed randomly with average spacing  $\sim Y^*(t)$  ; the hyperbolic ridges that divide neighbouring conical valleys lie at height  $\sim \Delta F$  (Eq. (4.9)) above the minima, with fluctuations in height also of order  $\Delta F$ . Since  $\Delta F(t) \gg t \delta u$ , the ridge loci in  $r$ -space are determined by a Voronoi construction about the randomly placed minima. On top of all this structure is noise of order  $(vt/\ln t)^{1/2}$  arising from the random environment seen by the extended tails. The noise is largest at the ridges and smallest at the valleys, but for long chains is small everywhere compared to the main features of the landscape just described. There are

various other small corrections, such as rounding of the conical tips on a length  $E^*(t)$  (see Eq. (4.7b)), and small wiggles in the Voronoi construction, all of which become less and less noticeable as  $t$  becomes large.

The above arguments specify the free energy landscape for  $d > 2$ . For  $d \leq 2$ , the consistency criterion (4.9) is not obeyed for long chains, and another approach is therefore required. In this case the average fluctuation in the height of the free energy surface is controlled by the intrinsic fluctuation  $t \delta u$  in the depths of the conical minima themselves. The heights of the ridges between neighbouring valleys must also be of this order, which means that the typical tail energy must obey

$$\Delta F(t) \sim m^* u(Y^*) \sim Y^* u(Y^*)^{1/2} \sim t \delta u. \quad (4.10)$$

This fixes the mean spacing between minima as

$$\begin{aligned} Y^*(t) \sim t v^{1/(4-d)} [\ln Y^*]^{-1/(4-d)} \\ \sim t v^{1/(4-d)} [\ln t]^{-1/(4-d)} \end{aligned} \quad (4.11a)$$

and the corresponding mean depth of the conical valleys as

$$tu^*(t) = tu(Y^*) \sim t [v \ln t]^{2/(4-d)}. \quad (4.11b)$$

The fluctuation in the depths of the conical tips is  $t \delta u$ . The slope of each cone is again proportional to its depth (so that the fluctuation in the slope becomes small compared to the mean at large  $t$ ). The loci of the minima of the cones are completely random ; however, the ridges no longer correspond to a Voronoi (equidistant) construction, since the difference in depth of two adjacent minima is comparable to the ridge height between them. [Note that the ridges are nonetheless sharp : only in a narrow region of spatial width  $\sim Y^*/(t \delta u) \ll Y^*$ , near the top of a ridge, do configurations involving two distinct wells differ by energies comparable to  $k_B T$  and thus simultaneously contribute to the free energy.] Instead the ridges form a random polygonal network of a slightly different specification, for which we have been unable to find a simple closed form. However, it is fully specified, implicitly, by the construction of the free energy surface as the lower envelope of all conical pieces centered on local minima. As in the case of  $d > 2$ , there is additional ( $O([vt]^{1/2} [\ln t]^{-1/(4-d)})$ ) noise whose effects are minor in the limit of long chains. The resulting structure in one dimension is sketched in figure 1.

## 5. Implications for the nonlinear growth model.

Consider now the surface structure after a long time  $t$ , resulting from equation (1.7), for an interface growing in the presence of a time-independent white-noise flux  $-\mu(r)$  (starting from a flat surface at time  $t = 0$ ). This is found from the free energy

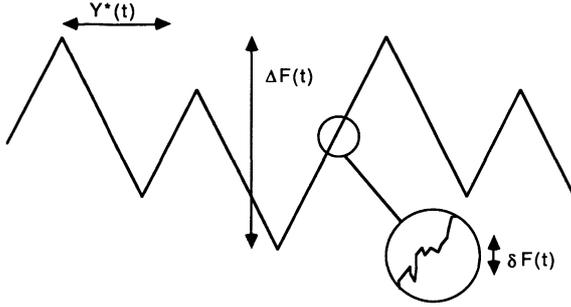


Fig. 1. — Sketch of the free energy surface  $F(r, t)$  for  $d = 1$  at large  $t$ . Here  $Y^*(t) \sim tv^{1/3} [\ln t]^{-1/3}$ , and  $\Delta F(t) \sim tv^{2/3}$ ; these characterize respectively the horizontal and the vertical scale of roughness. There is in addition noise of order  $\delta F(t) \sim (tv)^{1/2} [\ln t]^{-1/3}$  (inset).

landscape  $F(r, t)$  for a chain with one end fixed at  $r$ , simply by identifying chain length  $t$  with the time, and free energy  $F$  with  $-h$ . Thus the free energy landscape, described above for the polymer, must merely be turned upside-down. Note that for the *erosion* version of the model, in which the sign of the nonlinear term in (1.7) is reversed, even this inversion is unnecessary: one has a pattern of conical pits separated by ridges.

It should be observed that, although  $\mu$  was assumed to have mean zero, the average height of the interface increases with time as  $tu^*(t)$  (obeying (4.7) in  $d > 2$  and (4.10) in  $d < 2$ ). This is expected because the nonlinear term  $(\nabla h)^2$  in (1.7) leads to nonconservation of flux. For modelling, it may be more realistic to restore global conservation by a trivial shift of the mean height (so that this is zero in a moving frame for which the incident flux vanishes). The remaining structure consists of conical mountains of mean spacing  $Y^*(t)$  and peak-to-valley separation  $\Delta F(t)$ . We recall from equations (4.5) and (4.11) that  $Y^*(t) \sim tv^{1/(4-d)} [\ln t]^{1/(4-d)-1}$  in  $d > 2$ , and  $Y^*(t) \sim tv^{1/(4-d)} [\ln t]^{-1/(4-d)}$  in  $d \leq 2$ ; hence  $Y^*(t)$  is linear in time in  $d = 3$  and weakly sublinear in  $d = 1, 2$ . In contrast,  $\Delta F$  is of order  $tv^{2/(4-d)} [\ln t]^{2/(4-d)-1}$  in  $d > 2$ , and of order  $tv^{2/(4-d)}$  in  $d \leq 2$ . In  $d = 1, 2, 3$ , the average slope of the mountains is of order  $\Delta F/Y^* \sim u^*(t)^{1/2} \sim [v \ln t]^{1/(4-d)}$ , which is a (very) slowly increasing function of time.

It is quite simple to qualitatively describe the manner in which the evolved structure « coarsens » as time goes by. The velocities of the peaks are simply proportional to their heights; this is obvious in the polymer language since the free energy of a chain of minimal free energy (which lies entirely at the tip of a deep well) scales as  $t$ . As time evolves, then, the taller peaks move further ahead, and the lower peaks are occluded. Very gradually, the angle of the surviving peaks sharpens up.

## 6. Discussion.

The work described above uses simple ideas to predict the configuration and free energy landscape of a polymer chain in a random potential, which, for the case of a chain with one end fixed, may simply be turned upside down to give the evolved profile for the nonlinear growth model of equation (1.7).

Throughout, the mathematics was at the level of Flory-type [18] energy balance arguments; however, these were more sophisticated than usual, since they accounted for the fact that the potential was quenched. This was done by estimating directly the depth of the best potential well in an accessible region, whereas a conventional (annealed) approach would simply add on an entropy penalty for *creating* a well of a certain type. Since, as was seen, the statistics of the polymer are controlled entirely by *extremely unlikely* potential configurations, this distinction is crucial. Our results for a chain in a true white noise potential helped us to clarify the concept of *localization* of the polymer, which is not to be confused with mere collapse ( $R \sim t^0$ ). We predicted localization in  $d = 2, 3$  (in agreement with the replica calculation of Ref. [5]) but not in  $d < 2$ . Localization was also not predicted for a saturating potential of the form  $\mu = \pm v^{1/2}$ . In practice, the distinction may frequently be an academic one since the system volumes required to see delocalized behaviour can be enormously large — as are the kinetic barriers involved in exploring different regions of a finite system. For these reasons, Monte Carlo simulation data [3, 7] must probably be treated with some caution.

It is likely that our new results for the statistics of a chain with one end fixed could be reproduced with a more sophisticated replica calculation. Probably a variational treatment would still be required, in which case the occurrence of « tadpole » configurations might have to be explicitly allowed for in the choice of trial states. This remark raises an interesting issue as to whether our own, simplified, variational treatment captures all the physical ingredients that determine the chain configuration — and hence the structure of the free energy landscape — for long chains. In particular one might worry about the assumption that the collapsed part of the chain is a roughly spherical droplet. After all, given a volume  $a \sim R^d$  in which the average potential  $\bar{\mu}$  has some anomalously low value ( $\bar{\mu}$ , say), it is more likely that this volume is arranged as a random « animal »-like structure (or indeed as a polymer) than as a sphere. This argument is misleading, however; for if we ascribe a structural length  $r$  (a « blob size ») to this random shape, the free energy penalty of confining a chain to it becomes of order  $t/r^2$  (compare (Eq. (2.1))) rather than  $t/R^2$ . Because the energy recovered by exploiting the low

potential is still only  $\bar{\mu}$ , the chain may as well collapse into a single « blob » of size  $r$  (especially as, if  $r \ll R$ , it will typically be able to find one for which  $\bar{\mu} \ll \mu$ ). It is less clear to what extent this remains true in the case of a saturating potential. For the annealed (very large  $V$ ) regime, a similar argument can be made, whereas in the intermediate regime (Eqs. (3.1) and (3.2)) the shape fluctuations may perhaps become important.

The above reasoning suggests that shape fluctuations are not too large, and that the spherical droplet approximation does not lead to substantial error, at least in our results for the nonsaturating potential at large  $t$  (Sect. 2, 4, 5). It is possible, in principle, that other kinds of fluctuations lead to significant corrections to these results ; however, it is not obvious what these fluctuations would be like. For this reason it seems reasonable to conjecture that our various results for the asymptotic statistics of the free energy surface and/or interfacial growth profile are correct in their essential features, for dimensions  $d = 1, 2, 3$ .

It is interesting to try to understand these features more directly in terms of the growth model. While certain aspects (such as cuspidity) are familiar consequences of nonlinearity, and others (such as the Voronoi cells, and the  $t^{1/2}$  noise) those of randomness, these do seem to have interfered in a nontrivial way. For example, the positions in the maxima in the profile at time  $t$  correspond to those places where the incident flux is greatest, not at a point, but averaged over a region of linear size  $R^*(t) \sim [v \ln t]^{-1/(4-d)}$ . This length scale, which also determines the rounding at the conical tips, is not obvious simply from inspection of (1.7). Nor is it trivial that any such scale should be a decreasing function of time.

Note that the predicted structure is much coarser than would arise in the absence of the nonlinear term  $(\nabla h)^2$  in the equation of motion. Without that term, the equation reads

$$\dot{h}(r, t) = \nabla^2 h - \mu(r) \quad (7.1)$$

which is readily analysed by Fourier transforms (following Edwards and Wilkinson [15]). One finds a

characteristic transverse length scale  $L \sim t^{1/2}$  and roughness with r.m.s. height

$$h \sim v^{1/2} t^{(4-d)/4}. \quad (7.2)$$

The much enhanced coarsening in the present case may be crudely explained by noting that the nonlinear term encourages configurations with large local gradients.

We should mention that the evolved surface has several features in common with one calculated by KPZ [8] for the case of a random burst of flux at time zero [i.e.,  $\mu(r, t) = \mu(r) \delta(t)$ ]. In that case the surface is made up of paraboloid maxima, divided by sharp valleys. The results of this calculation are readily understood in the polymer context as they concern the free energy surface of a chain whose first monomer (only) feels the random potential  $\mu$ , while the other end is held at position  $r$ . The resulting free energy minima are parabolic, since the chain obeys the gaussian elasticity law (Eq. (2.1a)) ; in contrast, for our case of a time-independent flux they are conical, because the chain is being pulled out of a reservoir of stored length at fixed potential. Other features, such as the sudden jump from one well to another when the chain end is pulled too far, are similar in the two cases.

Finally, we compare our predictions for the surface structure, which apply in the case of a time independent random flux, with those for a flux random in both space and time (as also studied by KPZ [8]). In our case both the scale of vertical roughness, and that of horizontal coarsening, increase (to within logarithms) linearly with time. The former scaling, at least, is not surprising, as there is a finite difference in flux between different places. (But note that quite different behaviour arises in the linearized model, Eq. (7.2)). In contrast, the surface resulting from a flux random in both space and time exhibits fractional power law exponents (as mentioned in Sect. 1) [8]. Regrettably, very little else is known about the structure in that case [8-10], and so a more detailed comparison must await future developments.

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