Localized structures generated by subcritical instabilities

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Localized structures generated by subcritical instabilities

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Résumé.– Nous observons l'existence de solutions structurellement stables en forme de pulse, dans un voisinage d'une bifurcation de Hopf inverse. Ces structures localisées correspondent à des gouttes, que l'on sait être instables, pour les transitions de phase du premier ordre. Nous montrons que le mécanisme de stabilisation est un effet non-variationnel, c'est à dire dû à l'absence d'une "énergie libre" à minimiser dans le problème d'instabilité que l'on considère. Nous proposons ce mécanisme comme explication à l'existence de paquets d'ondes localisés dans certains écoulements parallèles ou dans les expériences de convection dans les fluides binaires.

Abstract.– We report the existence of structurally stable pulse-like solutions in the vicinity of an inverted Hopf bifurcation. These localized structures correspond to droplets in first order phase transitions, where they are known to be unstable. We show that the stabilisation mechanism is a non variational effect, i.e. is due to the non existence of a "free-energy" to minimize in the instability problem we consider. We propose this mechanism as an explanation for the existence of localized waves in shear flows or in convection experiments in binary fluid mixtures.

Localized structures are widely observed in systems far from equilibrium. Well known examples are the local regions of turbulent motion surrounded by laminar flow, which develop in many open-flow experiments (e.g. pipe flow, channel flow, boundary layers) [1]. More recently, spatially localized travelling waves have been observed at convection onset in binary fluid mixtures [2-4]. In both cases the possible origin of localized structures lies in the existence of a subcritical instability, which implies that two different homogeneous stable states coexist in an interval range of the control parameter (see Fig. 1). The simplest spatial non-uniformity consists of an interface between the two stable states. A similar situation occurs in first order phase transitions, for instance when droplets of liquid nucleate in a supersaturated vapor. In phase transitions the droplets are always unstable; they either shrink or expand. In the instability problem, a "droplet" consists of a region where the system is in the bifurcated state, surrounded by the basic state. The aim of this letter is to show that non-variational effects, i.e. due to the non-existence of a "free energy" to minimize (a Lyapounov functional), can stabilize the droplet-like structure in the vicinity of a subcritical instability. This provides a possible elementary mechanism to explain the recent observation of stable stationary interfaces between regions of convection and conduction in binary fluid mixtures within a finite interval range of the control parameter [2-4].

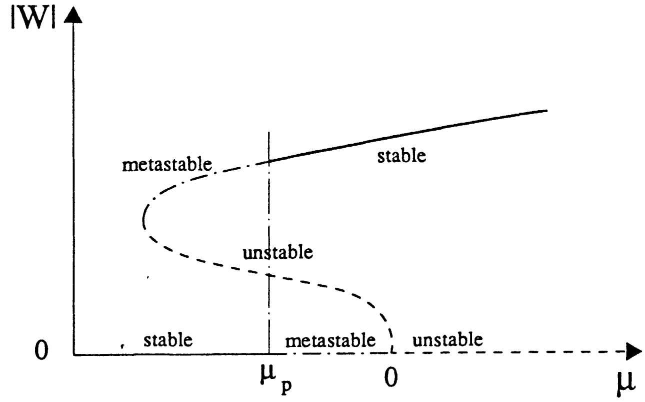

Fig. 1.— Bifurcation diagram of a subcritical Hopf bifurcation. Stationary amplitude $|W|$ as a function of the control parameter μ.

We consider a subcritical Hopf bifurcation, with a one-dimensional complex amplitude $W(x,t)$ governed by the equation,

$$\frac{\partial W}{\partial t} = \mu W + \alpha \frac{\partial^2 W}{\partial x^2} + \beta |W|^2 W + \gamma |W|^4 W, \quad (1)$$

where μ is the distance from criticality and α, β, γ are complex coefficients. Small perturbations are amplified when $\mu > 0$; when the real part of β, β_r is positive, they are not stabilized by the leading order non linearity but only by the quintic term if $\gamma_r < 0$, and the Hopf bifurcation is subcritical (see Fig. 1). Equation (1) can be derived for the two-dimensional disturbances of the plane Poiseuille flow [5]. the travelling waves observed in binary fluid mixtures convection also take place via a subcritical Hopf bifurcation, but the right and left travelling waves, $W_-\exp i(\omega t - kx)$ and $W_+\exp i(\omega t + kx)$ must be both considered, and the amplitude equations for W_- and W_+ are coupled [6]. We have observed numerically localized stationary solutions in both cases, but for simplicity we will describe here only the results obtained with the simplest model (1), and understand the physical mechanism responsible for the stability of these pulse-like solutions.

We have numerically intergrated equation (1) with a pseudo-spectral method involving 512 complex modes and periodic boundary conditions on the interval $[0, L]$. A typical pulse-like solution is shown in figure 2. For the convection problem, it corresponds to a small convective region surrounded by the conduction state. Notice that the amplitude of the pulse is strongly localized while its phase varies almost linearly in space. Solutions with a similar shape have been observed on large interval ranges of the constants α, β, $(\alpha_r = 1,\ \alpha_i \in [0,10], \beta_r = 1, \beta_i \in [0,4])$. Their typical size does not depend on the box length L which has been varied from 4π to 30π. The pulses exist for values of μ within a finite band (see below). They are obtained with a great variety of initial conditions. For instance, a phase-unstable homogeneous state $|W| \neq 0$, often evolves to a pulse-like solution [7]. Stationary localized pulses are thus structurally stable solutions of equation (1).

Let us first consider the case $\alpha_i = \beta_i = \gamma_i = 0$ for which equation (1) has a Lyapounov functional $\mathcal{L}\{W\}$:

$$\frac{d}{dt}\mathcal{L}\{W\} = -\int_0^L \left|\frac{\partial W}{\partial t}\right|^2 dx \leq 0, \quad (2)$$

where

$$\mathcal{L}\{W\} = \int_0^L \left[\frac{1}{2}\left|\frac{\partial W}{\partial x}\right|^2 - V(|W|)\right] dx, \quad (3)$$

with

$$V(|W|) = \frac{1}{2}\mu |W|^2 + \frac{1}{4}\beta_r |W|^4 + \frac{1}{6}\gamma_r |W|^6. \quad (4)$$

$\mathcal{L}$ decreases in time and is minimum for uniform solutions which maximize $V(|W|)$. There exists a particular value $\mu_p = 3\beta_r^2/16\gamma_r$ of the control parameter μ, for which the $W = 0$ and $W \neq 0$ uniform solutions have the same "energy" $-V$. For $\mu = \mu_p$, an isolated interface between the two uniform states remains at rest; this corresponds to the Maxwell plateau in first order phase transitions [8]. Pulse-like solutions are always unstable; they either shrink or expand in such a way that the lowest "energy" state increases in size (see Fig. 3). The "variational pulse" is even unstable for $\mu = \mu_p$ because of the interaction between its limiting interfaces. Therefore, the stability of pulse-like solutions can be explained only with a non-variational effect.

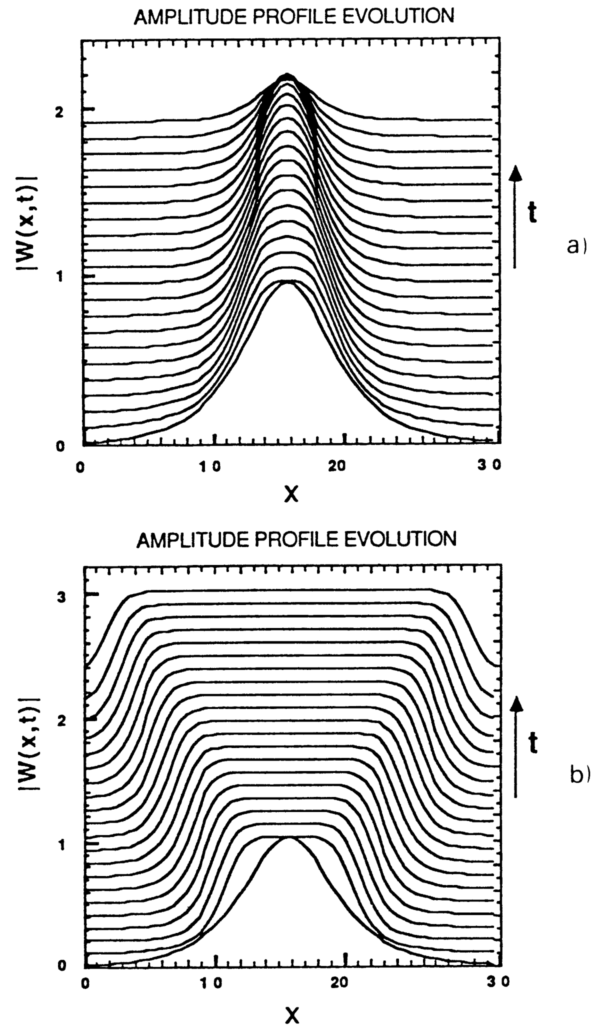

Fig. 2.— 1D pulse-like solution in the case $\alpha_i = 0$. Other parameters are $\mu = -0.1$, $\alpha_r = 1$, $\beta = 3 + i$, $\gamma = -2.75 + i$, interval length : $L = 30$; a) amplitude profile $|W(x)|$; b) phase profile $\phi(x) = \arg W(x)$.

Let us consider for simplicity equation (1) with $\alpha_i = 0$, and try to understand the shape of the pulse-like solution shown in figure 2. In the outer region the pulse amplitude is very small and we can neglect non linear terms. We look for a solution under the form $W(x,t) = R_0(x) \exp i\,[\Omega t + \theta_0(x)]$ and find the asymptotic behavior for large $|x|$ ($\mu < 0$) :

$$R_0 \approx \exp\left(-\frac{1}{\sqrt{2}}\sqrt{-\mu+\sqrt{\mu^2+\Omega^2}}\right)|x|\,, \quad \text{(5a)}$$

$$\theta_0 \approx \left(-\frac{1}{\sqrt{2}}\operatorname{sgn}(\Omega)\sqrt{\mu+\sqrt{\mu^2+\Omega^2}}\right)|x|\,. \quad \text{(5b)}$$

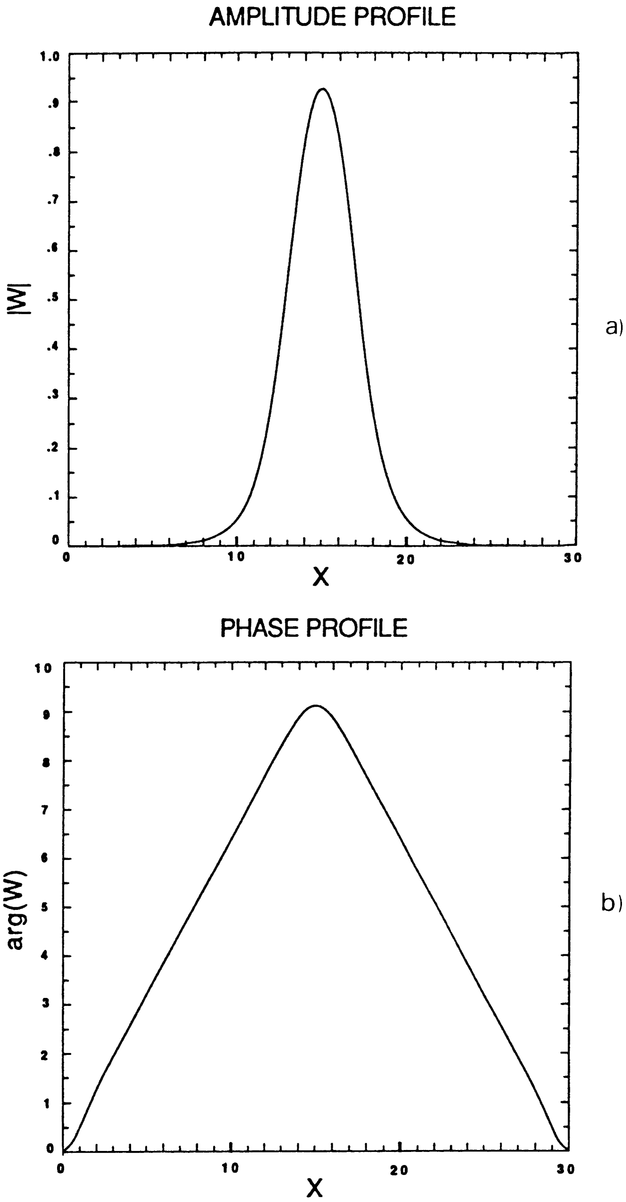

Fig. 3.— Time evolution of a pulse in the variationel case : $\alpha = 1 + 4i$, $\beta = 13.33 + 4i$, $\gamma = -10+4i$, interval length : $L = 5\pi$. For these parameter $\mu_p = -3.33$; a) case $\mu = \mu_p - 0.05$: the "droplet" shrinks; b) case $\mu = \mu_p + 0.10$: the "droplet" grows.

We thus easily explain the shape of the solution in the outer region.

Ω traces back to the dependence of the frequency of the oscillatory field (the travelling waves) with respect to its amplitude, i.e. $R_0(x)$; indeed, multiplying (1) by the complex conjugates of W and integrating over the interval [0, L] gives after equating real and imaginary parts :

$$\Omega = \beta_i \frac{\int R_0^4(x)\,\mathrm{d}x}{\int R_0^2(x)\,\mathrm{d}x} + \gamma_i \frac{\int R_0^6(x)\,\mathrm{d}x}{\int R_0^2(x)\,\mathrm{d}x}. \quad (6)$$

If $\alpha_i = \beta_i = \gamma_i = 0$, we find $\Omega = 0$ and thus $\theta_0 = 0$ as expected. When $\beta_i \neq 0$ or $\gamma_i \neq 0$, the frequency of the travelling waves depends on their amplitude, and this implies from equation (5.b) that the phase $\theta_0(x)$ varies linearly outside the pulse, with a slope which depends on the pulse shape $R_0(x)$. Therefore the dependence of the wave frequency on its amplitude generates the following feedback mechanism : the frequency correction Ω depends on the pulse shape (Eq. (6)), and in turn affects it (Eq. (5)).

To illustrate this feedback mechanism on the pulse stability, let us write the equations for $R(x,t)$ and $\phi(x,t)$, where $W(x,t) = R(x)\exp i\phi(x)$, $(\alpha = 1)$

$$\frac{\partial R}{\partial t} = \left[\mu - \left(\frac{\partial \phi}{\partial x}\right)^2\right] R + \beta_r R^3 + \gamma_r R^5 + \frac{\partial^2 R}{\partial^2 x}, \tag{7a}$$

$$R\frac{\partial \phi}{\partial t} = \beta_i R^3 + \gamma_i R^5 + 2\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial \phi}{\partial x}\right) + R\frac{\partial^2 \phi}{\partial^2 x}, \tag{7b}$$

and notice that a linear variation in space of the phase ϕ simply renormalizes μ in the equation for R. As said above, we have for pulse-like solutions $\phi = \Omega t + \theta_0(x)$, where $\left(\frac{\partial \theta_0}{\partial x}\right)^2$ is constant, say p^2, almost on all the interval $[0, L]$. We thus define $\mu_{\text{eff}} = \mu - p^2$. We plot below μ_{eff} as a function of μ, for the values of μ within the interval range where the pulses are stable (the numerical values of the parameters are $\alpha = 1$, $\beta = 3+i$, $\gamma = -2.75+i$, $L = 30$).

μ	−0.450	−0.400	−0.350	−0.300	−0.250	−0.200	−0.150	−0.100
μ_{eff}	−0.666	−0.693	−0.703	−0.708	−0.708	−0.707	−0.703	−0.700

We can notice that μ_{eff} is kept approximatively constant by the phase gradient, which shows its effect on the stabilizating mechanism. Note that the value of μ_{eff} is close to $\mu_p = -0.641$, the value corresponding to the Maxwell plateau in the potential case $(\alpha_i = \beta_i = \gamma_i = 0)$.

This mechanism also works for spatially two-dimensional fields $W(x, y, t)$, solutions of equation (1) where the diffusion term is the two-dimensional Laplacian. Figure 4 displays a stable 2D pulse.

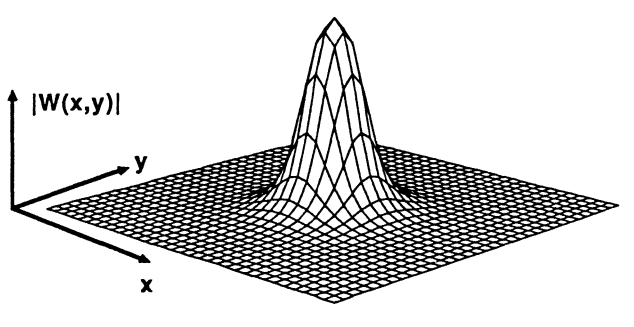

Fig. 4.— 2D pulse-like solution in the $\alpha_i = 0$ case. Same parameters as for figure 3. Amplitude profile $|W(x)|$.

In conclusion, let us discuss the implications of our model with respect to the experimental results on binary fluid mixture convection. Confined states have been first observed in parallelepipedic containers [2, 3], and related to the reflection of the waves from the lateral boundaries [9]. This mechanism cannot explain the similar localized structures observed in an annulus [4]. Another mechanism have been proposed, that consider exponentially small effects due to the locking of the wave envelope to the underlying structure [10, 11]. In our model, stable localized structures result from a non-variational effect; we predict that the wave amplitudes in the confined states are comparable to the one of the homogeneous bifurcated state, in agreement with the experimental results [4]. We also explain the different frequencies of these two regimes; indeed, it follows from equation (6) that the non-linear frequency correction Ω depends on the amplitude profile $R(x)$ of the wave envelope. Obviously, the model that couples left and right travelling waves with non zero group velocities should be used for a more realistic study. The main result of our paper is that localized structures in dissipative systems far from equilibrium, can result from a general and simple non-linear effect : the amplitude-dependence of a wave frequency. This basic mechanism traces back to the one involved for solitary waves in conservative systems.

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