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Kinetic equation for dilute, spin-polarized quantum systems

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Résumé. — Nous établissons une équation cinétique pour des systèmes dilués polarisés qui inclut les effets de
dégénérescences par la méthode des fonctions de Green de Kadanoff et Baym. Combinée avec l'approximation
de Born, cette équation se réduit à un résultat dû à Silin. Dans la limite de Boltzmann, notre résultat se réduit à
l'équation de Lhuillier et Laloë, à laquelle s'ajoute un terme de dérive de champ moyen analogue à celui qui
apparaît dans l'équation de Landau-Silin. Nous utilisons notre équation cinétique pour établir une expression
pour le temps de relaxation de la diffusion de spin transverse \( \tau_\perp \) pour un système de Fermi. Dans les limites de
Boltzmann et de polarisation faible, \( \tau_\perp \) se réduit à \( \tau_1 \), le temps de relaxation longitudinal. Touefois, dans un
système dégénéré fortement polarisé, \( \tau_\perp \) peut être beaucoup plus petit que \( \tau_1 \).

Abstract. — A kinetic equation, which includes the effects of degeneracy, is derived for dilute, polarized
systems by the Green's function method of Kadanoff and Baym. When the Born approximation is used for the
self-energy, the equation reduces to a result due to Silin. In the Boltzmann limit our result is equivalent to the
equation of Lhuillier and Laloë, with the addition of a mean-field drift term analogous to that appearing in the
Landau-Silin equation. Our kinetic equation is used to derive an expression for the transverse spin-diffusion
relaxation time, \( \tau_\perp \), for a Fermi system. In the Boltzmann and low-polarization limits \( \tau_\perp \) reduces to
\( \tau_1 \), the longitudinal relaxation time. However, in a highly polarized degenerate system \( \tau_\perp \) can be very much
shorter than \( \tau_1 \).

1. Introduction.

In 1982, Lhuillier and Laloë (LL) [1], published a kinetic equation that provided a generalization of
the ordinary Boltzmann equation for spin-polarized quantum gases. This equation predicted that
polarized Boltzmann gases could have spin waves, enhanced viscosity and thermal conductivity, and
other quantum effects. At the heart of several of these results is the « identical particle spin-rotation »
term, a reactive part of the collision integral in the
LL formulation.

The LL equation is valid for Boltzmann systems. However the spin-rotation term is reminiscent of the
mean-field effect of the Landau-Silin (LS) equation
[2] that leads to the Leggett-Rice [3] effect in
degenerate Fermi systems. Indeed, the possibility of

spin waves in dilute gases had previously been shown by Baskin [4] on the basis of the LS equation.
Although the LS equation was developed for degenerate Fermi systems, as Meyerovich [5], Miyake
et al. (MMS) [6], and Jeon and Mullin [7] have shown, it is also valid in the Boltzmann limit for
dilute systems such as \(^3\)He gas, or dilute solutions of \(^3\)He in \(^4\)He. This point had been also made years ago
by Grossmann [8], who derived the LS equation in this limit. Moreover, MMS demonstrated the identi-
ity of the Leggett-Rice effect and spin-rotation effects by taking the low-temperature limit of the spin-
rotation term and the high temperature limit of the
LS equation.

Spin rotation depends on a unitless « quality »
parameter \( \mu \) [1], which is mainly due to the interference of the transmitted wave and the exchange
portion of the scattered wave when two gas atoms collide. The cross section for this process is \( \xi \sim Aa \)
where $\lambda$ is the de Broglie wave length and $a$ the s-wave scattering length. Then $\mu$ is the ratio of this cross section to the normal one $\sigma \sim a^2$ so $\mu = \zeta / \sigma \sim \lambda / a$. An appropriate measure of the « normal » cross section is the spin diffusion relaxation time $\tau_D$, so that, for example, in the s-wave limit \[ \mu = -4 \pi \hbar \tau_D / m^* \]

where $m^*$ is the particle mass and $n$ the density. The relaxation time, $\tau_D$, that should appear in this formula is $\tau_{\perp}$, the transverse spin-diffusion time, as pointed out by Meyerovich [9]. However, in a Boltzmann system or in a degenerate fluid at low polarization (the usual systems in experiments to date), $\tau_{\perp}$ equals the longitudinal spin-diffusion time $\tau_1$ as we indicate in this paper. The quantity $\tau_1$ has been computed for the degenerate case by Hone [10] and others [7, 11].

For highly polarized, degenerate systems we will demonstrate here that $\tau_{\perp}$ need not equal $\tau_1$, and may indeed be quite different from it in temperature and polarization dependence. An explicit calculation of this quantity has never been carried out to date, in part because the distinction between $\tau_{\perp}$ and $\tau_1$ was not appreciated or because a kinetic equation collision integral of sufficient generality was thought to be unavailable. In fact, however, a collision integral, sufficiently general to include arbitrary polarization, had been derived several years ago by Silin [12]. However, the result has been available only in Russian and is not widely known.

We have undertaken to check Silin’s result, to give it a more rigorous derivation, and to provide a framework for generalizing it beyond the approximations he used. A very powerful Green’s function formalism for deriving the kinetic equation was given some years ago by Kadanoff and Baym [13]. We have generalized this treatment to arbitrary polarization which results in a matrix equation valid for dilute degenerate systems. Our most general result is expressed in terms of proper self-energies. A reduction of this result to a useful form is easily made by use of the Born approximation for the self-energies. At this level our result is found to be equivalent to that of Silin. The equation includes the usual mean-field terms of the Landau-Silin equation as well as Silin’s collision integral.

When we take the Boltzmann limit of our equation, we find that the collision integral reduces properly to the Born approximate version of that of LL [1]. However, the Landau equation mean-field terms persist into this limit, although they are absent in the LL equation and in most standard formulations of the Boltzmann equation. (Such terms can be important in the Boltzmann limit by providing virial-like corrections to any property, including even the gas pressure [5, 8], that is derivable from the transport equation). Laloe [14] has recently shown how the LL equation can be generalized to include these mean-field terms.

It is not difficult to generalize our result to a $T$-matrix formulation. However, the $T$-matrix that appears is a many-body function, the nature of which we cannot specify for practical applications. We can replace this quantity by the vacuum $T$-matrix ($t$-matrix) but this procedure is strictly valid only in the dilute non-degenerate regime. On the other hand, the Born approximation is itself quite useful. A system of prime interest to us is a dilute solution of $^3$He in liquid $^4$He. The effective $^3$He-$^4$He interaction developed for this system can be treated adequately in this limit. (For a large range of temperatures, the momentum dependence of the interaction may also be neglected).

Bashkin [15] pointed out that Silin’s derivation of the collision integral leads to an interesting off-energy shell correction, known as $I_2$, in the spin-rotation term. Levy and Ruckenstein [16] have shown that a corresponding term in the Hartree-Fock self-energy cancels out $I_2$. Laloe [17] has indicated that Silin’s $I_2$ may arises only as an entity in the potential expansion of the $t$-matrix that appears in the LL spin-rotation term. These two views are consistent with one another. We will examine this question briefly and indicate that in degenerate systems the cancellation of $I_2$ is not likely to be complete. However, we will leave to future work a detailed examination of the precise physical implications of a non-vanishing $I_2$.

Most of the small number of experiments done on degenerate polarized Fermi systems that have been sensitive to the spin rotation effect have been carried out at low polarization where $\tau_{\perp} \approx \tau_1$. These include spin-wave detection. The experiments that have had high polarization, that were sensitive to the effects of a nonvanishing $\mu$, and that had at least some degree of degeneracy, have studied dilute solutions of $^3$He in $^4$He. Some of these detected spin waves directly [18, 19] and another investigated the Leggett-Rice effect [20]. The results of the latter experiment still remain somewhat of a mystery. Agreement with theory was found within the Boltzmann regime, but diverged sharply from theory as the degenerate regime was approached. The theory used assumed $\tau_{\perp} = \tau_1$. The results of our present investigation show that such an assumption is incorrect at high polarization in a degenerate system, and that $\tau_{\perp}$ can become much smaller than $\tau_1$. This is in the right direction to explain the data of reference [20].

We present only analytic results in this paper. A numerical calculation of $\tau_{\perp}$, $D_{\perp}$ and $\mu$ will be given in a future publication.

While the present manuscript was in the late
stages of preparation, we learned that Levy and Ruckenstein [21], who had presented an abstract on the subject some time ago [22], had independently completed a paper that discusses a Kadanoff-Baym derivation of the kinetic equation for spin-polarized systems. It is too soon for us to compare this work with that presented here.

Our paper is presented as follows: In section 2 we present our results for the generalized kinetic equation. In section 3, we linearize the collision integral and use the kinetic equation to derive an equation governing spin currents for a Fermi system. The solution is presented in terms of \( \tau_1 \) and \( \tau_\perp \). We provide for the first time an explicit expression for \( \tau_\perp \). A physical analysis of the content of these results is given there. The details of the derivation of the kinetic equation are presented in section 4, where a discussion of \( f_2 \) is also given. Section 5 summarizes our conclusions.

2. The kinetic equation.

We will derive the kinetic equation for an arbitrarily polarized system in section 4. More general forms of the result will also be given there. Here we give the Born approximate form of the equation. What is important is not what approximation one uses for the transition probability, but rather the form of the result, that of the drift term and especially of the collision integral. The latter has a form that implies that \( \tau_\perp \) is quite different from \( \tau_1 \) and can provide interesting new physics.

Let \( \bar{q}_p(r, t) \) be the distribution function matrix (a double underlined quantity represents a matrix \( 2 \times 2 \) for spin \( 1/2 \)) having spin trace \( \bar{q}_p(r, t) \) which is the number of particles of any spin component having momentum \( p \) at position \( r \) at time \( t \). For spin \( 1/2 \) the other trace is

\[
\bar{\sigma}_p(r, t) = \text{Tr}(\bar{q}_p(r, t))
\]

which is the magnetization of particles with momentum \( p \) at \( r, t \), where \( \bar{\sigma} \) is the Pauli matrix vector. We find \( \bar{q}_p(r, t) \) satisfies:

\[
\frac{\partial \bar{q}_p}{\partial t} + \frac{1}{2} \left[ \nabla_p \bar{q}_p, \nabla \bar{q}_p \right]_+ - \frac{1}{2} \left[ \nabla \bar{q}_p, \nabla \bar{q}_p \right]_+ + \frac{i}{\hbar} \left[ \bar{q}_p + \bar{q}(p), \bar{q}_p \right]_- = \left( \frac{\partial \bar{q}_p}{\partial t} \right)_{\text{Coll}}.
\]

The energy matrix \( \bar{q}_p(r, t) \) is given by

\[
\bar{q}_p(r, t) = \bar{q}_p \mathbb{L} + \mathbb{U} + \frac{1}{\hbar^2} \int dp \left\{ V(0) \mathbb{L} \text{Tr}(\bar{q}_p(r, t)) + \eta V(|p - p'|) \bar{q}_p', (r, t) \right\}
\]

in which \((^1)\) \( \bar{q}_p = p^2/2m^* \), \( \mathbb{L} \) is a unit matrix, \( \mathbb{U} \) is an external field, \( \eta = +1 \) for Bosons and \(-1\) for Fermions, and the Fourier transform of the potential is given by

\[
V(|p|) = \int dr e^{-ip \cdot r/\hbar} V(|r|).
\]

The brackets \( [ , ] \) and \( [ , ]_+ \) represent a commutator and anticommutator, respectively. We find that in Born approximation the collision integral is given by

\[
\left( \frac{\partial \bar{q}_p}{\partial t} \right)_{\text{Coll}} = \frac{(2 \pi \hbar)^2}{\hbar^3} \int dp_1 dp_2 dp_3 \delta(p_1 + p_2 - p_1 - p_2) \delta(p_1 + p_2 - p_1 - p_2) \times
\]

\[
\times \frac{1}{2} \left[ \frac{1}{2} V(|p_1 - p_1|) \{ \bar{q}_1, \bar{q}_1 \}_+ - \text{Tr}(\bar{q}_2 \bar{q}_2) - \text{Tr}(\bar{q}_2 \bar{q}_2) \right] + \eta V(|p_1 - p_1|) V(|p_1 - p_2|) \{ \bar{q}_1, \bar{q}_2, \bar{q}_1 \}_+ - \{ \bar{q}_1, \bar{q}_2, \bar{q}_1 \}_+ \}
\]

in which

\[
\bar{q}_p = \mathbb{L} + \eta \bar{q}_p.
\]

\((^1)\) We use \( m^* \) for mass to distinguish it from the magnetization \( m \). It is not, of course, a Landau effective mass. In dilute solutions \( m^* \) is the \(^3\)He hydrodynamic mass.
The last term on the left side of equation (2.3) is the spin-rotation commutator and $\xi$ is the so-called $\langle I^2 \rangle$ correction. We give a further discussion of $\xi$ in section 4, but for technical reasons to be discussed in section 4, we ignore it in the remainder of this section and in the physical application discussed in the next section.

The various terms on the left side of equation (2.3) are familiar from Landau-Silin theory [3]. The second and third terms are the matrix forms of Landau’s mean-field terms. The spin-rotation commutator leads to spin waves [1, 2, 4, 18, 19, 23, 24] and to the Leggett-Rice effect [3, 20].

If we consider the Boltzmann limit of our result in equation (2.6), which is obtained by taking the final state factors $\tilde{u}$ equal to the unit matrix, equation (2.6) is easily seen to reduce properly to the Born approximate version of LL [1], (their equation (32b) for example).

When all spins are quantized along the same axis (the longitudinal case) it is easy to see that

$$\left( \frac{\partial p}{\partial t} \right)_{\text{Coll}}$$

reduces to the conventional Uehling-Uhlenbeck diagonal form with components given by

$$\left( \frac{\partial n_{1,\sigma}}{\partial t} \right)_{\text{Coll}} = \frac{(2\pi)^2}{h^2} \int dp_2 dp_1 \left( \delta(p_1 + p_2 - p_1 - p_2) \delta(\bar{E}_1 + \bar{E}_2 - \bar{E}_1' - \bar{E}_2') \right)$$

$$\left[ \left\{ V\left( |p_1 - p_1'| \right) \right\}^2 + \eta V\left( |p_1 - p_1'| \right) V\left( |p_1 - p_2'| \right) \right] \left[ n_{1,\sigma} n_{2,\sigma} \hat{h}_{1,\sigma} \hat{h}_{2,\sigma} - n_{1,\sigma} n_{2,\sigma} \hat{h}_{1,\sigma} \hat{h}_{2,\sigma} + \right]$$

$$+ \left[ V\left( |p_1 - p_1'| \right) \right]^2 \sum_{\sigma} \left[ n_{1,\sigma} n_{2,\sigma} \hat{h}_{1,\sigma} \hat{h}_{2,\sigma} - n_{1,\sigma} n_{2,\sigma} \hat{h}_{1,\sigma} \hat{h}_{2,\sigma} \right]$$

$$\left( \frac{\partial \chi_{p}}{\partial t} \right) = \frac{1}{2} \left( f_p(r, t) + \sigma_p(r, t) \cdot \sigma \right).$$

We can also write

$$\xi_p + e_p(r, t) \xi + h_p(r, t) \cdot \sigma$$

$$\epsilon_p = \bar{V} + \frac{1}{h^2} \int dp' \left[ V(0) + \frac{\eta}{2} V\left( |p - p'| \right) \right] f_p(r, t)$$

and

$$h_p = -\frac{1}{2} h \gamma B + \eta \frac{1}{2} h^2 \int dp' V\left( |p - p'| \right) \sigma_p, (r, t)$$

where $B$ is an external magnetic field and $\gamma$ the gyromagnetic ratio. The resulting equations for $f_p$ and $h_p$ are

$$\frac{Df_p}{Dt} = \frac{\partial f_p}{\partial t} + V_p e_p \cdot V_f + \nabla e_p \cdot V_f \cdot f_p + \sum_i e_i \cdot \nabla \left[ \frac{\partial h_p}{\partial p_i} \cdot \frac{\partial \sigma_p}{\partial r_i} - \frac{\partial h_p}{\partial r_i} \cdot \frac{\partial \sigma_p}{\partial p_i} \right] = \left( \frac{\partial f_p}{\partial t} \right)_{\text{Coll}}$$

$$\frac{D\sigma_p}{Dt} = \frac{\partial \sigma_p}{\partial t} + \sum_i \left[ \frac{\partial e_p}{\partial r_i} \cdot \frac{\partial \sigma_p}{\partial r_i} - \frac{\partial e_p}{\partial p_i} \cdot \frac{\partial \sigma_p}{\partial p_i} + \frac{\partial f_p}{\partial h_p} \cdot \frac{\partial h_p}{\partial r_i} - \frac{\partial f_p}{\partial p_i} \cdot \frac{\partial h_p}{\partial r_i} \right] - \frac{2}{h} (h_p \times \sigma_p) = \left( \frac{\partial \sigma_p}{\partial t} \right)_{\text{Coll}}.$$

Equations in this form were originally derived by Silin [2] and later used by Leggett [3]. However, we now
can provide explicit expressions for the collision integrals. They are

\[ \left( \frac{\partial f}{\partial t} \right)_{\text{Coll}} = \frac{(2\pi\hbar^2)}{\hbar^3} \int dp_2 dp_1 dp_2 \ \delta \left( p_1 + p_2 - p_1 - p_2 \right) \delta \left( \varepsilon_1 + \varepsilon_2 - \varepsilon_1 - \varepsilon_2 \right) \times \]

\[ \times \left[ \left( V \left( |p_1 - p_1| \right) \right)^2 + \frac{\eta}{2} V \left( |p_1 - p_1| \right) V \left( |p_1 - p_2| \right) \right] \left[ \left( f_1 + \frac{\eta}{2} \left( f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \left[ \left( f_1 + \frac{\eta}{2} \left( f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \]

\[ \times \left[ f_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right] \left[ \left( \sigma_1 + \frac{\eta}{2} \left( f_1 + f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \left[ \sigma_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right]

\[ - \left[ \sigma_1 + \frac{\eta}{2} \left( f_1 + f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right] \left[ \sigma_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right] \]

(2.15)

and

\[ \left( \frac{\partial \sigma}{\partial t} \right)_{\text{Coll}} = \frac{(2\pi\hbar^2)}{\hbar^3} \int dp_2 dp_1 dp_2 \ \delta \left( p_1 + p_2 - p_1 - p_2 \right) \delta \left( \varepsilon_1 + \varepsilon_2 - \varepsilon_1 - \varepsilon_2 \right) \times \]

\[ \times \left[ \left( V \left( |p_1 - p_1| \right) \right)^2 + \frac{\eta}{2} V \left( |p_1 - p_1| \right) V \left( |p_1 - p_2| \right) \right] \left[ \left( f_1 + \frac{\eta}{2} \left( f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \left[ \left( f_1 + \frac{\eta}{2} \left( f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \]

\[ \times \left[ f_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right] \left[ \left( \sigma_1 + \frac{\eta}{2} \left( f_1 + f_1 \sigma_1 + \sigma_2 \cdot \sigma_2 \right) \right) \right] \left[ \sigma_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right]

\[ - \left[ f_1 + \frac{\eta}{2} \left( f_1 + f_1 \sigma_1 + \sigma_2 \cdot \sigma_1 \right) \right] \left[ \sigma_2 + \frac{\eta}{2} \left( f_2 \sigma_2 + f_2 \sigma_2 \right) \right]

\[ + \frac{\eta}{2} \left[ (\sigma_1 \cdot \sigma_1) (\sigma_2 \cdot \sigma_2) - (\sigma_2 \cdot \sigma_2) (\sigma_1 \cdot \sigma_1) - (\sigma_1 \cdot \sigma_1) (\sigma_2 \cdot \sigma_2) \right] \right] \]

(2.16)

3. Equation for the spin current.

We want to use our kinetic equation to make contact with spin-diffusion experiments on spin-1/2 Fermi systems. We have in mind principally to make contact with experiments on dilute solutions of $^3$He in $^4$He. We develop an equation for the spin current, analogous to Leggett's equation [3], by which we can describe both longitudinal and transverse spin diffusion.

We follow the usual linearization procedure: a zero-order local-equilibrium distribution function $g^0_p$ causes the collision integral to vanish. The true solution $g_p = g^0_p + \delta g_p$ is inserted into the left side of equation (2.3). The second and third terms are then approximated by dropping the correction $\delta g_p$. We cannot drop $\delta g_p$ from the spin-rotation term since that provides its leading contribution. The collision integral is linearized in $\delta g_p$. We know that the solution for $\delta g_p$ must strongly overlap the drift terms so we choose a variational expression based on their form.

We can show that the appropriate local equilibrium distribution function is given by

\[ g_p^0 = \frac{1}{2} \left( f_p^0 L + \sigma_p^0 \hat{\epsilon} \cdot \sigma \right) \]

(3.1)

where

\[ f_p^0 = n_p^0 + n_p^0 \]

(3.2a)

\[ \sigma_p^0 = n_p^0 - n_p^0 \]

(3.2b)

\[ \hat{\epsilon} \] is the local direction of the magnetism, and

\[ n_{pp}^0 = [e^{\beta (\mu_p - \mu_p)} - 1]^{-1} \]

(3.3)

Here $\mu_p$ is a chemical potential for spin species $\sigma$ that may depend on temperature and position. Only the kinetic energy appears in equation (3.3) because of the form of the energy conservation $\delta$-function in equation (2.6), (see section 4). At this point we simplify the mathematics by making the assumption that a contact interaction (which can be made equivalent to an s-wave approximation) is sufficient.
and take $V(|p|) = V(0) = 2V_0$. This is a reasonable approximation for the dilute-solution problem. Then the zero-order expressions for the energy terms are

$$\epsilon_p^0 = \epsilon_p + \frac{V(0)}{2\hbar^2} \int dp' f_p = \epsilon_p + V_0 n$$  \hspace{1cm} (3.4)

with $n(r, t)$ the particle density and

$$\eta_p^0 = -\frac{\hbar^2}{2} B - \frac{V(0)}{2\hbar^2} \epsilon \int dp' \sigma_p^0 = -\frac{\hbar^2}{2} B - V_0 m$$  \hspace{1cm} (3.5)

where the magnetization at position $r$ and time $t$ is

$$m(r, t) = \frac{1}{\hbar^3} \int dp (n_p^0 - n_p^0) \epsilon.$$

With the substitution of the local equilibrium functions, the $r$ and $p$ gradient terms on the left side of equation (2.14) become

$$\sum_i [\ldots] = \sum_i v_p \left[ -\sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} \frac{\partial \epsilon_{\sigma}}{\partial r_i} \epsilon + \right.
\left. + \sum_{\sigma} (\sigma n_{p\sigma}) \frac{\partial \epsilon}{\partial r_i} - V_0 \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} \frac{\partial \epsilon}{\partial r_i} \right]
+ V_0 \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} m$$  \hspace{1cm} (3.7)

where $v_p = p/m^*$ and we have dropped the superscript 0 from the local equilibrium functions to simplify the notation.

The use of the Gibbs-Duhem relation (which can be derived from the kinetic equation), given by

$$\sum_\sigma n_\sigma \nabla \mu_\sigma = 0,$$  \hspace{1cm} (3.8)

and the relation

$$n_\sigma = \frac{1}{2} (\sigma m + n)$$  \hspace{1cm} (3.9)

allow us to show easily that

$$\frac{\partial n}{\partial r_i} = -d \frac{\partial m}{\partial r_i}$$  \hspace{1cm} (3.10)

and

$$\frac{\partial \mu_\sigma}{\partial r_i} = \sigma \langle G \rangle \frac{\partial m}{\partial r_i}$$  \hspace{1cm} (3.11)

where

$$d = \sum_\sigma \sigma n_\sigma G_{-\sigma}/\langle G \rangle,$$

$$G_\sigma = \frac{1}{\hbar^3} \int dp \left( -\frac{\partial n_{p\sigma}}{\partial \epsilon_p} \right),$$

and

$$\langle G \rangle = \sum_\sigma n_\sigma G_{-\sigma}.$$  \hspace{1cm} (3.14)

The quantities $G_\sigma$ can be evaluated for high degeneracy or for a Boltzmann system; however, we keep the formulas valid for an arbitrary temperature.

If we define the correction $\delta g_p$ to the local equilibrium function as

$$\delta g_p = \frac{1}{2} (\delta f_p \epsilon + \delta \sigma_p \cdot \epsilon)$$  \hspace{1cm} (3.15)

then the left side of equation (2.14) becomes

$$\sum_i v_p \left[ \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} \frac{\partial \epsilon_{\sigma}}{\partial r_i} \epsilon + \right.$$
$$\left. + \sum_{\sigma} (\sigma n_{p\sigma}) \frac{\partial \epsilon}{\partial r_i} - V_0 \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} \frac{\partial \epsilon}{\partial r_i} \right]
+ V_0 \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} m$$

$$\sum_i v_p \left[ \sum_{\sigma} \sigma n_{p\sigma} + V_0 m \sum_{\sigma} \frac{\partial n_{p\sigma}}{\partial \epsilon_p} \right] - 2 \frac{\hbar}{m} \delta \sigma_p \times \left[ \frac{\hbar^2}{2} B + V_0 m \right]$$  \hspace{1cm} (3.16)

where

$$t_\sigma = \frac{1}{\langle G \rangle} [n_{-\sigma} - 2V_0 n_\sigma G_{-\sigma}].$$  \hspace{1cm} (3.17)

The quantities in this drift term that have the factors $\partial n_{p\sigma}/\partial \epsilon_p$ are nonvanishing only at the Fermi surfaces in a degenerate system. All longitudinal terms (proportional to $\epsilon$) are of this nature. On the other hand, there is one term, which is transverse because it depends on $\frac{\partial \epsilon}{\partial r_i}$, and is not confined just to the Fermi surface. This is the term containing $\sum_\sigma \sigma n_{p\sigma}$.

The drift terms constitute the driving forces for the various currents and the deviations from local equilibrium will be determined by the form of these terms. Thus $\delta \sigma_p$ will contain a transverse term not confined to the Fermi surfaces. This simply results from the
fact that spins tipped away from local equilibrium (while maintaining constant $|\mathbf{m}|$), will constitute a disturbance everywhere in momentum space that there is a net magnetization including between the two Fermi spheres.

If we carry out a similar local-equilibrium substitution in equation (2.13) we find

$$\frac{Df_p}{Dt} = \frac{\delta f_p}{\delta t} - \sum_{i} v_{pi} \frac{\partial m}{\partial r_i} \times \sum_{\sigma} \left( \sigma \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} \right)$$  \hspace{1cm} (3.18)

Particle and spin currents may be defined in terms of $f_p$ and $\sigma_p$, respectively, according to

$$J_j = \frac{1}{\hbar^3} \int dp v_{pj} f_p = \frac{1}{\hbar^3} \int dp v_{pj} \delta f_p,$$  \hspace{1cm} (3.19)

$$J_j(\mathbf{m}) = \frac{1}{\hbar^3} \int dp v_{pj} \sigma_p = \frac{1}{\hbar^3} \int dp v_{pj} \delta \sigma_p.$$  \hspace{1cm} (3.20)

Examination of the drift terms leads us to take the following variational form for $\delta f_p$:

$$\delta f_p = q \sum_{i} v_{pi} \frac{\partial m}{\partial r_i} \sum_{\sigma} \left( \sigma \frac{\partial n_{p\sigma}}{n_{\sigma}} \varepsilon_p \right)$$  \hspace{1cm} (3.21)

where $q$ is a variational constant. The form chosen automatically guarantees a vanishing particle current as required in a spin diffusion experiment. The magnetization increment is taken to be

$$\delta \sigma_p = \delta \sigma_p^I + \delta \sigma_p^\perp$$  \hspace{1cm} (3.22)

where

$$\delta \sigma_p^I = q \sum_{i} v_{pi} \frac{\partial m}{\partial r_i} \hat{\varepsilon} \left( \sum_{\sigma} \frac{1}{n_{\sigma}} \frac{\partial n_{p\sigma}}{\varepsilon_p} \right)$$  \hspace{1cm} (3.23)

is longitudinal, and

$$\delta \sigma_p^\perp = A \sum_{i} v_{pi} \hat{\mathbf{g}}_i \left( \sum_{\sigma} \sigma n_{p\sigma} \right)$$  \hspace{1cm} (3.24)

is the transverse part. Here $A$ is a variational constant. The form for $\delta \sigma_p^\perp$ corresponds to that used in our previous work [7] and will lead to the same relaxation time $\tau_1$ and the same (7) diffusion constant $D_1$. The unit vectors $\hat{g}_i$ in the transverse part are perpendicular to the magnetization direction $\hat{\varepsilon}$. To be explicit we can write

$$\hat{g}_i = x \frac{\partial \hat{\varepsilon}}{\partial r_i} + y \hat{\varepsilon} \times \frac{\partial \hat{\varepsilon}}{\partial r_i}$$  \hspace{1cm} (3.25)

where $x$ and $y$ are constants.

We now multiply equation (3.16) by $v_{pj}$ and integrate over $p$ to find an equation for the spin current $J_j(\mathbf{m})$. We find

$$\frac{\partial J_j(\mathbf{m})}{\partial t} + \alpha_1 \frac{\partial m}{\partial r_j} \hat{\varepsilon} + \alpha_\perp m \frac{\partial \hat{\varepsilon}}{\partial r_j} - \gamma J_j(\mathbf{m}) \times \mathbf{B} + \frac{2}{\hbar} V_0 \mathbf{m} \times J_j(\mathbf{m})$$

\hspace{1cm} $= \frac{1}{\hbar^3} \int dp v_{pj} \left( \frac{\partial \sigma_p}{\partial t} \right)_{Coll}$  \hspace{1cm} (3.26)

whith

$$\alpha_1 = \frac{1}{m^*} \sum_{\sigma} \sigma n_{\sigma}$$  \hspace{1cm} (3.27a)

$$\alpha_\perp = \frac{2}{3 m^*} \sum_{\sigma} \sigma k_{\sigma} - V_0 \frac{n}{m^*}$$  \hspace{1cm} (3.27b)

and

$$k_{\sigma} = \frac{1}{\hbar^3} \int dp \bar{\varepsilon}_p n_{p\sigma}.$$  \hspace{1cm} (3.28)

The left side of equation (3.26) has roughly the same general form as that found by Leggett [3] and LL [1]. However, there is the important difference that $\alpha_\perp$ and $\alpha_1$ are not equal. Further the present equation is not limited to low polarization as is Leggett’s. When the polarization is small and the system degenerate then we find $\alpha_\perp = \alpha_1$ and the result becomes equivalent to that given in MMS [5] (which equals Leggett’s equation with a constant interaction). Furthermore, when the system is in the Boltzmann regime of temperature, then we again have $\alpha_1 = \alpha_\perp$ for arbitrary polarization, and the result of LL [1] is reproduced. By finding differing polarization dependences of $\alpha_1$ and $\alpha_\perp$ and their effect on the longitudinal and transverse diffusion constants we are repeating results noted previously by Meyerovich [9].

We will shortly linearize the right side of equation (3.26). However, we anticipate our answer and write down the form that results from our analysis of the collision term. Examination of $\left( \frac{\partial \sigma_p}{\partial t} \right)_{Coll}$ in equation (2.16) tells us what to expect. Linearization will lead to terms in $\delta f_p$ and $\delta \sigma_p$, $\sigma_p$. These will necessarily be longitudinal. However, there are other possible forms having factors like $\delta \sigma_p f_p$, $\delta \sigma_p f_p f_p'$, and $\delta \sigma_p(\sigma_p', \sigma_p')$ which can produce transverse terms. The longitudinal factors are localized on the Fermi surfaces at low temperatures and so the collision integral ultimately produces factors [2b, 7] like $n_{1+} n_{2-} (1 - n_{1+}) (1 - n_{2-})$ which, together with energy and momentum conservation, restrict all collisions to the Fermi surfaces and lead to the usual $\tau_1 \sim T^{-2}$ temperature dependence. On the other hand, the transverse quantity $\delta \sigma_p^\perp$ is not restricted to the Fermi surface and

(7) Equation (3.16) contains a previously neglected mean-field term that will change $D_1$ a bit.
different distribution function factors appear in $\tau_\perp$. Hence we anticipate $\tau_\perp \neq \tau_1$. However, when the polarization is small or in the non-degenerate limit it will turn out that $\tau_\perp = \tau_1$.

Meyerovich [9] has previously noted what physics results from having differing values of $\tau_1$ and $\tau_\perp$. We repeat that argument here in a slightly different form for the sake of completeness.

We will show below that the collision integral gives

$$\frac{1}{h^3} \int dp \, v_{pj} \left( \frac{\partial \sigma_p^j}{\partial t} \right)_{\text{coll}} = -\frac{1}{\tau_1} \langle J_j(m) \cdot \hat{e} \rangle \hat{e} - \frac{1}{\tau_\perp} \langle J_j(m) \cdot \hat{g}_j \rangle \hat{g}_j \quad (3.29)$$

The solution to equation (3.26), with equation (3.29) is found easily to be (Cf. Ref. [1, 3, 9])

$$J_j(m) = -D_\perp \frac{\partial m}{\partial r_j} \frac{D^{(0)}_\perp}{[1 + (\mu m/n)^2]} \times \left[ \frac{m \frac{\partial \epsilon}{\partial r_j}}{n} + \frac{\mu m^2}{n} \frac{\epsilon \frac{\partial \epsilon}{\partial r_j}}{n} \right] \quad (3.30)$$

where $\mu$ is the spin-rotation parameter given here by

$$\mu = -\frac{2nV_0}{h} \tau_\perp \quad (3.31)$$

and the diffusion constants are

$$D_1 = \alpha_1 \tau_1 \quad (3.32)$$

$$D^{(0)}_\perp = \alpha_\perp \tau_\perp \quad (3.33)$$

The complete transverse diffusion constant is

$$D_\perp = \frac{D^{(0)}_\perp}{[1 + (\mu m/n)^2]} \quad (3.34)$$

In Leggett's theory [3], $\tau_\perp$ was assumed to be equal to $\tau_1$ (which is quite correct for the low polarization case he considered) and $\alpha_\perp$ was equal to $\alpha_1$. It is $D_\perp$ that is measured in a spin-echo experiment. Simpler theories that assume $D^{(0)} = D_1$ at high polarization (as done in Ref. [7]) are incorrect.

We are now able to give an explicit expression for $\tau_\perp$ that will show how it can be considerably smaller than $\tau_1$ in the degenerate limit. Then $\mu$, which also depends on $\tau_\perp$, will also be smaller. This result may explain the anomalous results of the experiment of reference [20].

To derive $\tau_1$ and $\tau_\perp$ we must explicitly linearize the collision integral. Rather than working with the form given in equation (2.16), we find it more convenient to return to the general form, equation (2.6). Linearization results in the following lengthy expression

$$\left( \frac{\partial \delta \bar{a}_1}{\partial t} \right)_{\text{coll}} = \frac{(2 \pi)^2}{h^2} \int dp_1 dp_2 dp_2 \delta (p_1 + p_2 - p_2) \delta (\bar{e}_1 + \bar{e}_2 - \bar{e}_1 - \bar{e}_2) \times$$

$$\times \frac{1}{2} \left[ V(|p_1 - p_2|) \langle \{ \delta \bar{a}_1 + \delta \bar{a}_2 \} \{ \delta \bar{a}_1 + \delta \bar{a}_2 \} \rangle - \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle - \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle - \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle \langle \delta \bar{a}_1 + \delta \bar{a}_2 \rangle \right]$$

where all $\bar{a}_p$'s are local equilibrium quantities. In the limit of low polarization this reduces to the form given by Pal and Bhattachari [25]. Longitudinal effects can be gotten by substituting

$$\delta \bar{a}_p^\perp = \frac{1}{2} [\delta \bar{a}_p L + \delta \sigma_p^j \cdot \hat{g}] \quad (3.36)$$

with $\delta \bar{a}_p$ and $\delta \sigma_p^j$ given in equations (3.21) and (3.23). All matrices are diagonal along the direction of the magnetization and the results are the same as those derived for $\tau_1$ in reference [7]. We examine here only the more interesting transverse case in which

$$\delta \bar{a}_p^\perp = \frac{1}{2} \delta \sigma_p^j \cdot \hat{g} \quad (3.37)$$

and $\delta \sigma_p^j$ is given by equations (3.24). We work in the reference frame whose $z$ axis is along $\hat{e}$, the local
magnetization direction. In that frame all the $n_{pq}$ (local equilibrium distribution functions) are diagonal with matrix elements $n_{pq} (\sigma = \pm 1)$. The increment $\delta n_{pq}^\pm$, on the other hand, is completely off-diagonal and has the form

$$\delta n_{pq}^\pm = \begin{bmatrix} 0 & s_p^\pm \sum_\sigma \sigma n_{pq} \\ s_p \sum_\sigma \sigma n_{pq} & 0 \end{bmatrix}$$  \hspace{1cm} (3.38)

where $s_p = s_{px} + i s_{py}$ and $s_p = \frac{1}{2} A \sum_i v_{pi} \hat{g}_i$. We now simply carry out the matrix multiplications indicated in equation (3.35). A typical term is, say,

$$s_p = \frac{1}{2} A \sum_i v_{pi} \hat{g}_i$$  \hspace{1cm} (3.40)

We have used the fact that $\delta n = - \delta n$ and $n_{pq} = - n_{qp}$. The cross product in equation (3.41), if present, would actually contribute to the spin-rotation term (last term on the left of Eq. (3.26)); however, all such terms can be shown to cancel out (just as no cross terms occur in Eq. (2.16)). The advantage of working with forms as given by the first term in equation (3.41) is that they preserve products of four distribution functions, such as, $n_{1+} n_{2-} n_{1+} n_{2-}$. Then the standard energy conservation equality, in which this latter product becomes $n_{1+} n_{2-} n_{1+} n_{2-}$, can be used.

We carry out all the matrix multiplications, then multiply the resulting $(\delta s_{pq}^\pm d \tau)_\text{Coll}$ by $\alpha$ and take the spin trace to give $(\delta s_{pq}^\pm d \tau)_\text{Coll}$. For the sake of simplicity we use the constant potential approximation $V(p) = V(0) = 2 V_0$. We find then

$$\left( \frac{\delta \sigma_{p_i}}{\delta \tau} \right)_\text{Coll} = \sum_i \hat{g}_i A S_i(p_i)$$  \hspace{1cm} (3.42)

where

$$S_i(p_i) = \left( \frac{2 \pi}{h^2} \right)^2 V_0^2 \int dp_2 dp_3 dp_4 \times$$

$$\times \delta(p_1 + p_2 - p_1 - p_2) \delta(\bar{v}_1 + \bar{v}_2 - \bar{v}_1 - \bar{v}_2) \times 2(v_{11} - v_{12} \pm v_{21}) \left( + + + + \right) \left( + + + - \right) \left( + - + - \right) \left( + - - + \right) \left( + - - - \right)$$  \hspace{1cm} (3.43)

where the notation is, for example,

$$(- + + +) = n_{1-} n_{2+} \hat{n}_{1+} \hat{n}_{2+} \hat{n}_{1+} n_{2+} +$$

$$\hat{n}_{1+} \hat{n}_{2+} n_{1+} n_{2+} -$$  \hspace{1cm} (3.44)

To get this in the desired form of equation (3.29) we note that substitution of equation (3.24) into the definition equation (3.20) gives a transverse current

$$J_{\perp}(m) = \frac{1}{h^2} \int dp v_{pj} \delta \sigma_{p} = A \hat{g}_j \alpha'$$  \hspace{1cm} (3.45)

with

$$\alpha' = \frac{2}{3 m^2} \sum_\sigma \sigma k_{\sigma}$$  \hspace{1cm} (3.46)

where $k_{\sigma}$ is defined in equation (3.28). If we then multiply equation (3.42) by $v_{1j}$ and integrate over $p_i$ we find that

$$\frac{1}{\tau_{\perp}} = - \frac{1}{\alpha' h^2} \int dp_1 v_{1j} S_j(p_1)$$  \hspace{1cm} (3.47)

It is easy to test this result in the non-degenerate or low polarization limits. We find in each case that $\tau_{\perp}$ reduces as expected to $\tau_1$.

The unusual features in $\tau_{\perp}$ can be seen from the forms appearing in equations (3.43) and (3.44). Energy conservation causes quantities such as $(+++)$ or $(- - -)$ to vanish. However, each of the forms written in equation (3.43) has only one different spin component as if spin were not conserved in the collision. Energy conservation does not make the quantity in equation (3.44) vanish. The source of this unusual form is simply that in a transverse process a spin from a neighbouring region enters a given regime, where the local magnetization direction is $\hat{e}$, with a spin component perpendicular to $\hat{e}$. It, in a sense, has a foot in each local Fermi sphere. This mixing in $\delta \sigma_{p}$ is the source of the mixed spin configurations that occur in $S_i(p_i)$. Factors as occur in equation (3.44) do not confine the collision to the Fermi surfaces. Suppose for example, that the
up-spin Fermi sphere is larger than the down-spin sphere. Then a factor like
\[ n_{1+} n_{2-} \tilde{n}_{1-} \tilde{n}_{2-} = n_{1+} n_{2-} (1 - n_{1-})(1 - n_{2-}) \]  
(3.48)
requires an up spin 1 from anywhere in the up-sphere colliding with a down spin 2 to have both spins end up anywhere outside the smaller down-spin sphere. It is easy to show that energy and momentum conservation put no severe restriction on the collision. Processes far from the Fermi surfaces are easily possible. (Processes in which both final state factors are down-spin, as given in equation (3.48), can be shown easily to dominate all other terms). Because of the large phase space for these processes we expect \( \tau_\perp \ll \tau_\parallel \) as we have mentioned above.

We have not yet made further explicit analytic or numerical reductions of \( \tau_\perp \). We plan to treat this matter in a future publication.

We now go back to seek a justification for the kinetic equation used in the preceding analysis.

4. Derivation of the kinetic equation.

Our procedure is a generalization of the Green's function method of Kadanoff and Baym and we need not give all the details here. The main complication is that the Green's functions, which become matrices here, are not diagonal in spin space.

We suppose that our system (either Fermi or Bose) was in thermodynamic equilibrium in the distant past and we have then turned on a perturbation given by

\[ H_{\text{ext}} = \sum_{ab} \int \! d\mathbf{r} \, \psi^+_a(r, t) U_{a\beta} \psi_{\beta}(r, t) \]  
(4.1)

where \( \psi^+_a(r, t) \) creates a particle having spin \( \alpha \) at \( r \) at time \( t \). For spin 1/2 we can take (3)

\[ U_{a\beta} = -\frac{\gamma}{2} \mathbf{B}(r, t) \cdot \sigma_{a\beta} \]

with \( \mathbf{B} \) a magnetic field and \( \sigma_{a\beta} \) a component of a Pauli matrix. The response of the system to the perturbation \( H_{\text{ext}} \) is given by the non-equilibrium, real-time, Green's functions defined by

\[ \varrho_{a\beta}(1, 1'; \mathcal{U}) = \frac{1}{i} \langle T(\psi_{\alpha}(1; \mathcal{U}) \psi^+_{\beta}(1'; \mathcal{U})) \rangle \]  
(4.2a)

\[ \varrho^\circ_{a\beta}(1, 1'; \mathcal{U}) = \frac{1}{i} \langle \psi_{\alpha}(1; \mathcal{U}) \psi^+_{\beta}(1'; \mathcal{U}) \rangle \]  
(4.2b)

\[ \varrho_{a\beta}(1, 1'; \mathcal{U}) = \frac{i}{\hbar} \langle \psi^+_a(1'; \mathcal{U}) \psi_{\alpha}(1; \mathcal{U}) \rangle \]  
(4.2c)

in which (1) stands for \((r_1, t_1)\), etc, and

\[ \psi_{\alpha}(1; \mathcal{U}) = v^+(t_1) \psi_{\alpha}(1) v(t_1) \]  
(4.3)

with

\[ v(t) = T \exp \left(-i \int_{-\infty}^{t} d\tilde{t} \, H_{\text{ext}}(\tilde{t}) \right) \]  
(4.4)

and \( T \) is the time-ordering operator. The average \( \langle \ldots \rangle \) in equation (4.2) denotes a grand canonical ensemble average

\[ \langle \mathcal{O}(\mathcal{U}) \rangle = \text{Tr} \left\{ e^{-\beta(H - \mu N)} \mathcal{O}(\mathcal{U}) \right\} / \text{Tr} \left\{ e^{-\beta(H - \mu N)} \right\} \]  
(4.5)

Here \( H \) is the second quantized Hamiltonian of the system, not including \( H_{\text{ext}} \), and \( N \) is the number operator. The trace with a small \( \ll t \), \( \text{Tr} \), is a sum over eigenstates as opposed to the spin trace, \( \text{Tr} \), introduced previously. We take \( H \) to be

\[ H = \sum_a \int \! d\mathbf{r} \, \psi^+_a(r) \left( -\frac{\nabla^2}{2 m^*} \right) \psi_a(r) + \frac{1}{2} \sum_{a\beta} \int \! d\mathbf{r} \, \int \! d\mathbf{r}' \, \psi^+_a(r_1) \psi^+_\beta(r_2) V(|\mathbf{r}_1 - \mathbf{r}_2|) \psi_\beta(r_2) \psi_a(r_1) \]  
(4.6)

with \( V(|\mathbf{r}|) \) a spin-independent potential.

Following reference [13], we get the equations of motion for \( g^\alpha(1, 1'; \mathcal{U}) \) (with \( \mathcal{U} \) dropped from the \( g \) notation)

\[ \left( i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2 m^*} \right) g^\alpha(1, 1'; \mathcal{U}) - \mathcal{U}(1) g^\alpha(1, 1') - \int \! d\tilde{t} \, \Sigma_{\text{HF}}(1, \tilde{1}) \, g^\alpha(\tilde{1}, 1') = \right. \]

\[ = \left. \int_{-\infty}^{t_1} \! d\tilde{t} \left[ \Sigma^\alpha(1, \tilde{1}) - \Sigma^\alpha(1, \tilde{1}) \right] g^\alpha(\tilde{1}, 1') - \int_{-\infty}^{t_1} \! d\tilde{t} \, \Sigma^\alpha(1, \tilde{1}) \left[ g^\alpha(\tilde{1}, 1') - g^\alpha(\tilde{1}, 1') \right] \right) \]  
(4.7a)

and

\[ \left( -i \frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2 m^*} \right) g^\alpha(1, 1') - g^\alpha(1, 1') \mathcal{U}(1') - \int \! d\tilde{t} \, g^\alpha(1, \tilde{1}) \, \Sigma_{\text{HF}}(1', \tilde{1}) = \right. \]

\[ = \left. \int_{-\infty}^{t_1} \! d\tilde{t} \left[ g^\alpha(1, \tilde{1}) - g^\alpha(1, \tilde{1}) \right] \Sigma^\alpha(1, \tilde{1}) - \int_{-\infty}^{t_1} \! d\tilde{t} \, g^\alpha(1, \tilde{1}) \left[ \Sigma^\alpha(1, \tilde{1}) - \Sigma^\alpha(1, \tilde{1}) \right] \right) \]  
(4.7b)

(3) In this section we take \( \hbar = 1 \).
The proper self-energy $\Sigma$ has been split into a Hartree-Fock part and a collisional part according to

$$\Sigma(1, 1') = \Sigma_{\text{HF}}(1, 1') + \Sigma_c(1, 1').$$  \hspace{1cm} (4.8)

$\Sigma_{\text{HF}}$ contains a time $\delta$-function, $\delta(t_1 - t_{1'})$:

$$\Sigma_{\text{HF}}(1, 1') = \delta(t_1 - t_{1'}) \Sigma_{\text{HF}}(r_1, r_{1'}, t_1)$$  \hspace{1cm} (4.9)

and $\Sigma_c$ is composed of two analytic functions of the time variables in the form

$$\Sigma_c(1, 1') = \begin{cases} \Sigma^+(1, 1') & \text{for} \ t_1 > t_{1'} \\ \Sigma^-(1, 1') & \text{for} \ t_1 < t_{1'} \end{cases}.$$  \hspace{1cm} (4.10)

If we subtract (4.7b) from (4.7a) we find

$$\left( i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_{1'}} + \frac{v_1^2}{2 m^*} - \frac{v_{1'}^2}{2 m^*} \right) g^\pm(1, 1') - [\mathcal{U}(1) g^\pm(1, 1') - g^\pm(1, 1') \mathcal{U}(1')] -$$

$$- \int d\bar{T} \left[ \Sigma_{\text{HF}}(1, \bar{T}) g^\pm(\bar{T}, 1') - g^\pm(1, \bar{T}) \Sigma_{\text{HF}}(1', \bar{T}) \right]$$

$$= \int_{-\infty}^{t_1} d\bar{T} \left[ [\Sigma^+(1, \bar{T}) - \Sigma^-(1, \bar{T})] g^\pm(\bar{T}, 1') - \left[ g^\pm(1, \bar{T}) - g^\pm(1, \bar{T}) \right] \Sigma^+(1', \bar{T}) \right]$$

$$- \int_{-\infty}^{t_1} d\bar{T} \left[ \Sigma^+(1, \bar{T}) g^\pm(\bar{T}, 1') - g^\pm(\bar{T}, 1') \right] - g^\pm(1, \bar{T}) [\Sigma^+(1', \bar{T}) - \Sigma^-(1', \bar{T})] \right].$$

(4.11)

We change variables to

$$R = \frac{1}{2} (r_1 + r_{1'}) \quad ; \quad T = \frac{1}{2} (t_1 + t_{1'})$$

$$r = r_1 - r_{1'} \quad ; \quad t = t_1 - t_{1'}$$  \hspace{1cm} (4.12)

and write $g^\pm(1, 1')$ as $g^\pm(r, t, R, T)$.

The $r, t$ Fourier transforms of these functions are

$$g^+(p, \omega, R, T) = \int dr \int dt e^{-i p \cdot r + i \omega t} \eta g^+(r, t, R, T)$$  \hspace{1cm} (4.13a)

and

$$g^-(p, \omega, R, T) = \int dr \int dt e^{-i p \cdot r + i \omega t} i g^-(r, t, R, T).$$  \hspace{1cm} (4.13b)

For a slowly varying external field, $g^\pm(p, \omega, R, T)$ is a slowly varying function of $R$ and $T$. If we expand equation (4.11) and keep terms up to first order in $\nabla_R$ and $\nabla_T$, we get,

$$\frac{1}{2} \left[ \left( \omega - \varepsilon_p \right) \mathcal{L} \mathcal{U}(R, T) - \Sigma_{\text{HF}}(p, R, T) - \frac{\xi(p, \omega, R, T)}{e} \mathcal{L} \mathcal{U}(R, T) \right] +$$

$$+ \left[ \mathcal{D}(p, \omega, R, T), \frac{\xi(p, \omega, R, T)}{e} \mathcal{L} \mathcal{U}(R, T) \right] +$$

$$+ i \left[ \mathcal{U}(R, T) + \Sigma_{\text{HF}}(p, R, T) + \frac{\xi(p, \omega, R, T)}{e} \mathcal{L} \mathcal{U}(R, T) \right] - \left[ \mathcal{D}(p, \omega, R, T), \frac{\xi(p, \omega, R, T)}{e} \mathcal{L} \mathcal{U}(R, T) \right]$$

$$= \frac{1}{2} \left[ \left[ \Sigma^+(p, \omega, R, T), g^+(p, \omega, R, T) \right] - \left[ \Sigma^-(p, \omega, R, T), g^-(p, \omega, R, T) \right] \right] +$$

$$+ i \left[ \left[ \Sigma^+(p, \omega, R, T), g^+(p, \omega, R, T) \right] \right] +$$

$$+ \frac{i}{4} \left[ \left[ \Sigma^+(p, \omega, R, T), g^+(p, \omega, R, T) \right] \right] - \left[ \Sigma^-(p, \omega, R, T), g^-(p, \omega, R, T) \right] \right].$$  \hspace{1cm} (4.14a)
and a similar equation for $g'_{\omega}$:

$$\eta \frac{1}{2} \left\{ (\omega - \mathcal{E}_p) \mathcal{L} - \mathcal{U} - \mathcal{E}_{\text{hf}} - \xi, g_{\omega} \mathcal{P} \right\} + [\xi, g_{\omega} \mathcal{P}_r] + i \eta \{ [\mathcal{U} + \mathcal{E}_{\text{hf}} + \xi, g_{\omega} \mathcal{P}_r] - [\xi, g_{\omega} \mathcal{P}_r] \} =$$

$$= \frac{1}{2} \left\{ [\mathcal{E}_{\omega}, g_{\omega}] - [\mathcal{E}_{\omega}, g_{\omega}] - [\xi, g_{\omega}] \right\} + \frac{i}{4} \left\{ [\mathcal{E}_{\omega}, g_{\omega} \mathcal{P}_r] - [\mathcal{E}_{\omega}, g_{\omega} \mathcal{P}_r] \right\} \tag{4.14b}$$

where $[\ ,\ ]_c$ represents a commutator (anticommutator) and we define a "Poisson commutator" as

$$\langle A, B \rangle_c = \langle A, B \rangle^p \pm \langle B, A \rangle^p \tag{4.15}$$

with the generalized Poisson bracket given by

$$\langle A, B \rangle^p = \nabla_k A \cdot \nabla_k B - \nabla_k B \cdot \nabla_k A + \frac{\partial A}{\partial \omega} \frac{\partial B}{\partial \omega} - \frac{\partial A}{\partial T} \frac{\partial B}{\partial T}. \tag{4.16}$$

In equation (4.14) we have introduced the quantities

$$\mathcal{E}_{\text{hf}}(p, \mathcal{R}, T) = \int \text{d}r e^{-ip \cdot r} \mathcal{S}_{\text{hf}}(r, 0, \mathcal{R}, T), \tag{4.17}$$

$$\xi(p, \omega, \mathcal{R}, T) = \int \text{d}r \int \text{d}t \frac{1}{2} \left[ \frac{t}{|t|} \right] \left[ \mathcal{E}_{\omega}(r, t, \mathcal{R}, T) - \mathcal{E}_{\omega}(r, t, \mathcal{R}, T) \right] =$$

$$= P \int \frac{d\omega'}{2\pi} \frac{\mathcal{L}(p, \omega', \mathcal{R}, T)}{\omega - \omega'}, \tag{4.18}$$

where

$$\mathcal{L}(p, \omega, \mathcal{R}, T) = \mathcal{E}_{\omega}(p, \omega, \mathcal{R}, T) - \eta \mathcal{E}_{\omega}(p, \omega, \mathcal{R}, T), \tag{4.19}$$

$$\mathcal{E}_{\omega}(p, \omega, \mathcal{R}, T) = \int \text{d}r \int \text{d}t e^{-ip \cdot r + i\omega t} \xi \mathcal{E}_{\omega}(r, t, \mathcal{R}, T), \tag{4.20a}$$

$$\mathcal{E}_{\omega}(p, \omega, \mathcal{R}, T) = \int \text{d}r \int \text{d}t e^{-ip \cdot r + i\omega t} \eta \xi \mathcal{E}_{\omega}(r, t, \mathcal{R}, T), \tag{4.20b}$$

and

$$\mathcal{B}(p, \omega, \mathcal{R}, T) = \int \text{d}r \int \text{d}t \frac{1}{2} \left[ \frac{t}{|t|} \right] \left[ g_{\omega}(r, t, \mathcal{R}, T) - g_{\omega}(r, t, \mathcal{R}, T) \right] = P \int \frac{d\omega'}{2\pi} \frac{g_{\omega}(p, \omega', \mathcal{R}, T)}{\omega - \omega'}, \tag{4.21}$$

The spectral density is defined as

$$g(p, \omega, \mathcal{R}, T) = g_{\omega}(p, \omega, \mathcal{R}, T) - \eta g_{\omega}(p, \omega, \mathcal{R}, T). \tag{4.22}$$

By subtracting equation (4.14b) from (4.14a) we get an equation for the spectral density $g(p, \omega, \mathcal{R}, T)$:

$$\frac{1}{2} \left\{ (\omega - \mathcal{E}_p) \mathcal{L} - \mathcal{U} - \mathcal{E}_{\text{hf}}, g_{\omega} \mathcal{P} \right\} + i \{ \mathcal{U} + \mathcal{E}_{\text{hf}}, g_{\omega} \mathcal{P} \} =$$

$$= \frac{1}{2} \left\{ \{ \xi, g_{\omega} \mathcal{P} \mathcal{P}_r \} - [\xi, \mathcal{E}_{\omega} \mathcal{P}_r] \right\} - i \left\{ \{ \xi, g_{\omega} \} - [\xi, g_{\omega}]_c \right\} \tag{4.23}$$

Were we able to solve this equation, then an equation for the distribution function

would follow from equation (4.14a). However, equation (4.23) is too complicated for us to find a general solution easily. As is discussed in reference [13] the determination of a Boltzmann equation involves a further simplification, namely, the neglect of the terms of order $\frac{1}{T}$ or $V_{\mathcal{R}}$ that originate from the "collisional" parts of the Green's functions. (In Eq. (4.23), these are on the right-hand side). Then instead of equations (4.14a) and (4.23) we have

$$\frac{1}{2} \left\{ (\omega - \mathcal{E}_p) \mathcal{L} - \mathcal{U} - \mathcal{E}_{\text{hf}}, g_{\omega} \mathcal{P} \right\} + i \left\{ \{ \mathcal{U} + \mathcal{E}_{\text{hf}}, g_{\omega} \mathcal{P} \} - [\xi, g_{\omega}]_c \right\} \tag{4.25}$$
From its definition in equation (4.22) we can show that \( a \) satisfies

\[
\frac{1}{2} \left[ (\omega - \bar{\varepsilon}_p) \mathbb{L} - \mathcal{U} - \mathbf{S}_{hf}, a \right]^p + \\
+ i \left\{ \mathcal{U} + \mathbf{S}_{hf} + \xi, a \right\}_- - \left[ b, \mathbb{L} \right]_+ = 0 \quad (4.26)
\]

results. The \( \varepsilon_{\pm} \) are values of energy that would appear in the local equilibrium distribution function and can be specified from equation (4.26). We should then substitute this result into equation (4.28) and integrate over \( \omega \) to determine the equation for \( \bar{g}_p \). This gives us a rather complicated form. The main role of the \( \delta \)-functions in \( \bar{g} \) is to determine the energies in the overall energy conservation \( \delta \)-function in the collision integral (as in Eq. (2.8)). The Hartree corrections to the energy are not very important in this context. (Note that the usual Boltzmann equation contains only kinetic energies in the energy conservation \( \delta \)-function). So we replace \( \varepsilon_{\pm} \) with the kinetic energy by the simplifying assumption (4).

\[
\tilde{g}_p (\p, \omega, R, T) = 2 \pi \delta (\omega - \bar{\varepsilon}_p) \mathbb{L} . \quad (4.30)
\]

Upon substituting equation (4.30) into equation (4.25) and integrating over \( \omega \), we derive the following equation for the distribution function:

\[
\frac{\partial \tilde{g}_p (R,T)}{\partial t} + \frac{1}{2} \left[ V_p \mathbb{L}_p (R,T), \mathbf{V}_R \tilde{g}_p (R,T) \right]_+ - \frac{1}{2} \left[ V_R \mathbb{L}_p (R,T), \mathbf{V}_p \tilde{g}_p (R,T) \right]_+ + \\
+ i \left[ \mathbb{L}_p (R,T) + \xi (\p, \omega = \bar{\varepsilon}_p, R, T), \tilde{g}_p (R,T) \right]_- = \left( \frac{\partial \tilde{g}_p}{\partial t} \right)_{coll} \quad (4.31)
\]

Equation (4.33) has the form of the Landau-Silin equation of Fermi-liquid theory with the addition of the off-energy-shell term \( i [\xi, \tilde{g}_p]_- \). We discuss this term below.

In Born collision approximation it is easy to write explicit expressions for the quantities in these equations. We find

\[
\mathbf{S}_{hf} (\p, R, T) = \int \frac{dp'}{(2 \pi)^3} \int \frac{d\omega'}{2 \pi} \left\{ V(0) \mathbb{L} \text{Tr} [\tilde{g}_p (\p', \omega', R, T)]_+ + \eta V (|p - p'|) g^c (\p', \omega', R, T) \right\} = \\
= \int \frac{dp'}{(2 \pi)^3} \left\{ V(0) \mathbb{L} \text{Tr} \tilde{g}_p (R, T) + \eta V (|p - p'|) \tilde{g}_p (R, T) \right\} , \quad (4.34)
\]

(\( \dagger \) One result of the approximation of equation (4.30) is that \( b \) (Eq. (4.21)) is then proportional to \( \mathbb{L} \) and so the last term on the left of equation (4.25) drops out. It might be interesting to investigate the physical significance of this term when a less restrictive approximation is made.)
and

\[ \langle \bar{\Sigma}_{n} | P, \omega_1 = \bar{\epsilon}_1, R, T \rangle = \frac{1}{(2 \pi)^6} \int dp_2 dp_1 dp_x \int d\omega_2 d\omega_1 d\omega_2 \delta(p_1 + p_2 - p_1 - p_2) \times \]

\[ \times \delta(\omega_1 + \omega_2 - \omega_1 - \omega_2) \left\{ [V(|p_1 - p_1|)]^2 \tilde{g}^\alpha (p_1, \omega_1, R, T) \text{Tr}[\tilde{g}^\alpha (p_2, \omega_2, R, T) \tilde{g}^\alpha (p_2, \omega_2, R, T)] \right\} \]

\[ + \eta V(|p_1 - p_1|) \left\{ [V(|p_1 - p_2|)]^2 \tilde{g}^\alpha (p_1, \omega_1, R, T) \tilde{g}^\alpha (p_2, \omega_2, R, T) \tilde{g}^\alpha (p_2, \omega_2, R, T) \right\} \]

\[ = \frac{1}{(2 \pi)^6} \int dp_2 dp_1 dp_x \delta(p_1 + p_2 - p_1 - p_2) \delta(\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2) \]

\[ \times \left\{ [V(|p_1 - p_1|)]^2 \tilde{g}^\alpha \text{Tr} [\tilde{g}^\alpha \tilde{g}^\alpha] + \eta V(|p_1 - p_1|) V(|p_1 - p_2|) \tilde{g}^\alpha \tilde{g}^\alpha \tilde{g}^\alpha \right\} \]

(4.35)

where

\[ \tilde{g}^\alpha = g^\alpha \]

(4.36)

From the definition of \( \xi \) in equation (4.18) we find, in Born approximation,

\[ \xi(p_1, \omega_1 = \bar{\epsilon}_1, R, T) = \frac{1}{(2 \pi)^6} \int dp_2 dp_1 dp_x \delta(p_1 + p_2 - p_1 - p_2) P \left[ \frac{1}{\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2} \right] \]

\[ \times \left\{ [V(|p_1 - p_1|)]^2 \tilde{g}^\alpha \text{Tr}(\tilde{g}^\alpha \tilde{g}^\alpha) - \eta \tilde{g}^\alpha \text{Tr}(\tilde{g}^\alpha \tilde{g}^\alpha) \right\} \]

\[ + \eta V(|p_1 - p_1|) V(|p_1 - p_2|) \tilde{g}^\alpha \tilde{g}^\alpha \tilde{g}^\alpha \eta \tilde{g}^\alpha \tilde{g}^\alpha \tilde{g}^\alpha \right\} . \]

(4.37)

With these results we arrive at the kinetic equation quoted in equations (2.3) and (2.6) and in agreement with the equation of Silin.

The off-energy-shell term \( i \xi \eta_{\eta} \) has been discussed before [15-17]. The previous discussions were not meant to apply to the degenerate limit. Levy and Ruckenstein [16] claimed that \( \eta \) was cancelled out by a term second order in the t-matrix in \( \Sigma_{nh} \). This argument is valid only in the non-degenerate limit. In such a Boltzmann limit, we have

\[ \langle \xi(1), \xi_1 \rangle = \frac{1}{(2 \pi)^6} \int dp_2 dp_1 dp_x \delta(p_1 + p_2 - p_1 - p_2) \times \]

\[ \times P \left[ \frac{1}{\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2} \right] V(|p_1 - p_1|) V(|p_1 - p_2|) \xi_2 \xi_2 \xi_2 \]

(4.38)

so that

\[ \Sigma_{BH}(1) + \xi(1), \xi_1 \rangle = \frac{\eta}{(2 \pi)^3} \int dp_2 \xi_2 \xi_2 \xi_2 \langle V(|p_1 - p_2|) + \]

\[ + \int \frac{dq}{(2 \pi)^3} P \left[ \frac{m^*}{(p_1 - p_2)^2 - q^2} \right] V \left( \left| \frac{p_1 - p_2}{2} - q \right| \right) V \left( \left| q - \frac{p_2 - p_1}{2} \right| \right) \right]. \]

(4.39)

The T-matrix for two particles scattering in vacuum is given by

\[ \langle k_t | t | k_i \rangle = V(|k_t - k_i|) + m^* \int \frac{dq}{(2 \pi)^3} \frac{V(|k_t - q|)|k_t - q|^2 + i \varepsilon}{k_i^2}. \]

(4.40)

Laloë has argued [17] that if all higher order terms were considered equation (4.39) would sum simply to

\[ \Sigma_{BH}(1) + \xi(1), \xi_1 \rangle = \eta \int \frac{dp_2}{(2 \pi)^3} Re \left\{ \frac{p_1 - p_2}{2} | t | \frac{p_2 - p_1}{2} \right\} \xi_2 \xi_2 \xi_2 \]

(4.41)
so that $\xi$ (or $I_2$) is gathered harmlessly into the spin-rotation term.

However, the degenerate final state factors in equation (4.37) invalidate such an argument and, moreover, the generalization beyond Born approximation in the degenerate limit would replace $V'$ s by many-body $T$ matrices which depend on the non-equilibrium Green's functions and thereby the distribution functions, in contrast to the vacuum $t$-matrix of equation (4.40). We recall that we have neglected first order derivative terms in deriving equations (4.25) and (4.26) and the kinetic equation (4.31). We are unable to say what effect the inclusion of these terms would have on $I_2$. So we feel the question of $I_2$ or $\xi$ is unresolved. We should point out that Miyake [26] has carried out an analysis of a dilute Fermi system, based on the renormalization procedure of Abrikosov and Dzaloshinskii [27], that finds a non-zero off-energy-shell term in spin-rotation.

5. Conclusion.

We have derived a kinetic equation valid in the degenerate regime and equivalent to that of Silin. Our Born approximation result should be valid for dilute solutions of $^3\text{He}$ in liquid $^4\text{He}$. For other systems it is possible to think of $V(p)$ as an effective potential rather than a bare one or as an approximation to replace $V(p)$ by a vacuum $t$-matrix to treat systems interacting via a potential having no Fourier transform. We believe the form of our result will not change fundamentally.

The solution of the kinetic equation by a variational method, valid at arbitrary polarization leads to both longitudinal and transverse relaxation times and diffusion constants. In highly polarized and degenerate Fermi systems these quantities will not be the same for the longitudinal and transverse cases. We have shown how the transverse relaxation time can be much smaller than the longitudinal one resulting in a smaller diffusion constant and a smaller spin-rotation parameter. These results may explain previously anomalous spin-echo experiments.

Several items remain for future work. We must reduce our analytic formulas, for $\tau_L$, for example, to numerical values to provide comparison with experiment. Several aspects of the derivation of the kinetic equation, including questions of the use of a more accurate spectral density and the nature of $I_2$, remain to be investigated more fully.

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