Kinetic equation for dilute, spin-polarized quantum systems
J.W. Jeon, W.J. Mullin

To cite this version:

HAL Id: jpa-00210850
https://hal.archives-ouvertes.fr/jpa-00210850
Submitted on 1 Jan 1988

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Kinetic equation for dilute, spin-polarized quantum systems

J. W. Jeon (1) and W. J. Mullin (1, 2)

(1) Laboratory of Low Temperature Physics, Hasbrouck Lab, University of Massachusetts, Amherst, MA 01003 U.S.A. (*)
(2) Laboratoire de Spectroscopie Hertzienne de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris, France

(Reçu le 20 avril 1988, accepté le 24 juin 1988)

Résumé. — Nous établissons une équation cinétique pour des systèmes dilués polarisés qui inclut les effets de dégénérescences par la méthode des fonctions de Green de Kadanoff et Baym. Combinée avec l'approximation de Born, cette équation se réduit à un résultat dû à Silin. Dans la limite de Boltzmann, notre résultat se réduit à l'équation de Lhuillier et Laloë, à laquelle s'ajoute un terme de dérive de champ moyen analogue à celui qui apparaît dans l'équation de Landau-Silin. Nous utilisons notre équation cinétique pour établir une expression pour le temps de relaxation de la diffusion de spin transverse $\tau_\perp$ pour un système de Fermi. Dans les limites de Boltzmann et de polarisation faible, $\tau_\perp$ se réduit à $\tau_1$, le temps de relaxation longitudinal. Toutefois, dans un système dégénéré fortement polarisé, $\tau_\perp$ peut être beaucoup plus petit que $\tau_1$.

Abstract. — A kinetic equation, which includes the effects of degeneracy, is derived for dilute, polarized systems by the Green's function method of Kadanoff and Baym. When the Born approximation is used for the self-energy, the equation reduces to a result due to Silin. In the Boltzmann limit our result is equivalent to the equation of Lhuillier and Laloë, with the addition of a mean-field drift term analogous to that appearing in the Landau-Silin equation. Our kinetic equation is used to derive an expression for the transverse spin-diffusion relaxation time, $\tau_\perp$, for a Fermi system. In the Boltzmann and low-polarization limits $\tau_\perp$ reduces to $\tau_1$, the longitudinal relaxation time. However, in a highly polarized degenerate system $\tau_\perp$ can be very much shorter than $\tau_1$.

1. Introduction.

In 1982, Lhuillier and Laloë (LL) [1], published a kinetic equation that provided a generalization of the ordinary Boltzmann equation for spin-polarized quantum gases. This equation predicted that polarized Boltzmann gases could have spin waves, enhanced viscosity and thermal conductivity, and other quantum effects. At the heart of several of these results is the « identical particle spin-rotation » term, a reactive part of the collision integral in the LL formulation.

The LL equation is valid for Boltzmann systems. However the spin-rotation term is reminiscent of the mean-field effect of the Landau-Silin (LS) equation [2] that leads to the Leggett-Rice [3] effect in degenerate Fermi systems. Indeed, the possibility of spin waves in dilute gases had previously been shown by Baskin [4] on the basis of the LS equation. Although the LS equation was developed for degenerate Fermi systems, as Meyerovich [5], Miyake et al. (MMS) [6], and Jeon and Mullin [7] have shown, it is also valid in the Boltzmann limit for dilute systems such as $^3$He gas, or dilute solutions of $^3$He in $^4$He. This point had been also made years ago by Grossmann [8], who derived the LS equation in this limit. Moreover, MMS demonstrated the identity of the Leggett-Rice effect and spin-rotation effects by taking the low-temperature limit of the spin-rotation term and the high temperature limit of the LS equation.

Spin rotation depends on a unitless « quality » parameter $\mu$ [1], which is mainly due to the interference of the transmitted wave and the exchange portion of the scattered wave when two gas atoms collide. The cross section for this process is $\xi \sim A_q$.
where $\lambda$ is the de Broglie wave length and $a$ the s-wave scattering length. Then $\mu$ is the ratio of this cross section to the normal one $\sigma \sim a^2$ so $\mu = \zeta / \sigma \sim \lambda / a$. An appropriate measure of the "normal" cross section is the spin diffusion relaxation time $\tau_D$, so that, for example, in the s-wave limit [5]

$$\mu = -4 \pi \hbar \tau_D / m^*$$

where $m^*$ is the particle mass and $n$ the density. The relaxation time, $\tau_D$, that should appear in this formula is $\tau_\perp$, the transverse spin-diffusion time, as pointed out by Meyerovich [9]. However, in a Boltzmann system or in a degenerate fluid at low polarization (the usual systems in experiments to date), $\tau_\perp$ equals the longitudinal spin-diffusion time $\tau_\parallel$ as we indicate in this paper. The quantity $\tau_\parallel$ has been computed for the degenerate case by Hone [10] and others [7, 11].

For highly polarized, degenerate systems we will demonstrate here that $\tau_\parallel$ need not equal $\tau_\parallel$, and may indeed be quite different from it in temperature and polarization dependence. An explicit calculation of this quantity has never been carried out to date, in part because the distinction between $\tau_\parallel$ and $\tau_\parallel$ was not appreciated or because a kinetic equation collision integral of sufficient generality was thought to be unavailable. In fact, however, a collision integral, sufficiently general to include arbitrary polarization, had been derived several years ago by Silin [12]. However, the result has been available only in Russian and is not widely known.

We have undertaken to check Silin's result, to give it a more rigorous derivation, and to provide a framework for generalizing it beyond the approximations he used. A very powerful Green's function formalism for deriving the kinetic equation was given some years ago by Kadanoff and Baym [13]. We have generalized this treatment to arbitrary polarization which results in a matrix equation valid for dilute degenerate systems. Our most general result is expressed in terms of proper self-energies. A reduction of this result to a useful form is easily made by use of the Born approximation for the self-energies. At this level our result is found to be equivalent to that of Silin. The equation includes the usual mean-field terms of the Landau-Silin equation as well as Silin's collision integral.

When we take the Boltzmann limit of our equation, we find that the collision integral reduces properly to the Born approximate version of that of LL [1]. However, the Landau equation mean-field terms persist into this limit, although they are absent in the LL equation and in most standard formulations of the Boltzmann equation. (Such terms can be important in the Boltzmann limit by providing virial-like corrections to any property, including even the gas pressure [5, 8], that is derivable from the transport equation). Laloe [14] has recently shown how the LL equation can be generalized to include these mean-field terms.

It is not difficult to generalize our result to a T-matrix formulation. However, the T-matrix that appears is a many-body function, the nature of which we cannot specify for practical applications. We can replace this quantity by the vacuum T-matrix ($t$-matrix) but this procedure is strictly valid only in the dilute non-degenerate regime. On the other hand, the Born approximation is itself quite useful. A system of prime interest to us is a dilute solution of $^3$He in liquid $^4$He. The effective $^3$He-$^4$He interaction developed for this system can be treated adequately in this limit. (For a large range of temperatures, the momentum dependence of the interaction may also be neglected).

Bashkin [15] pointed out that Silin's derivation of the collision integral leads to an interesting off-energy shell correction, known as $I_2$, in the spin-rotation term. Levy and Ruckenstein [16] have shown that a corresponding term in the Hartree-Fock self-energy cancels out $I_2$. Laloe [17] has indicated that Silin's $I_2$ may arises only as an entity in the potential expansion of the $t$-matrix that appears in the LL spin-rotation term. These two views are consistent with one another. We will examine this question briefly and indicate that in degenerate systems the cancellation of $I_2$ is not likely to be complete. However, we will leave to future work a detailed examination of the precise physical implications of a non-vanishing $I_2$.

Most of the small number of experiments done on degenerate polarized Fermi systems that have been sensitive to the spin rotation effect have been carried out at low polarization where $\tau_\perp \approx \tau_\parallel$. These include spin-wave detection. The experiments that have had high polarization, that were sensitive to the effects of a nonvanishing $\mu$, and that had at least some degree of degeneracy, have studied dilute solutions of $^3$He in $^4$He. Some of these detected spin waves directly [18, 19] and another investigated the Leggett-Rice effect [20]. The results of the latter experiment still remain somewhat of a mystery. Agreement with theory was found within the Boltzmann regime, but diverged sharply from theory as the degenerate regime was approached. The theory used assumed $\tau_\perp = \tau_\parallel$. The results of our present investigation show that such an assumption is incorrect at high polarization in a degenerate system, and that $\tau_\perp$ can become much smaller than $\tau_\parallel$. This is in the right direction to explain the data of reference [20].

We present only analytic results in this paper. A numerical calculation of $\tau_\parallel, D_\perp$ and $\mu$ will be given in a future publication.

While the present manuscript was in the late
stages of preparation, we learned that Levy and Ruckenstein [21], who had presented an abstract on the subject some time ago [22], had independently completed a paper that discusses a Kadanoff-Baym derivation of the kinetic equation for spin polarized systems. It is too soon for us to compare this work with that presented here.

Our paper is presented as follows: In section 2 we present our results for the generalized kinetic equation. In section 3, we linearize the collision integral and use the kinetic equation to derive an equation governing spin currents for a Fermi system. The solution is presented in terms of \( \tau_1 \) and \( \tau_\perp \). We provide for the first time an explicit expression for \( \tau_\perp \). A physical analysis of the content of these results is given there. The details of the derivation of the kinetic equation are presented in section 4, where a discussion of \( I_2 \) is also given. Section 5 summarizes our conclusions.

2. The kinetic equation.

We will derive the kinetic equation for an arbitrarily polarized system in section 4. More general forms of the result will also be given there. Here we give the Born approximate form of the equation. What is important is not what approximation one uses for the transition probability, but rather the form of the result, that of the drift term and especially of the collision integral. The latter has a form that implies that \( \tau_\perp \) is quite different from \( \tau_1 \) and can provide interesting new physics.

Let \( \varrho_p(r, t) \) be the distribution function matrix (a double underlined quantity represents a matrix \(-2 \times 2\) for spin 1/2) having spin trace which is the number of particles of any spin component having momentum \( p \) at position \( r \) at time \( t \). For spin 1/2 the other trace is which is the magnetization of particles with momentum \( p \) at \( r, t \), where \( \sigma \) is the Pauli matrix vector. We find \( \varrho_p(r, t) \) satisfies:

\[
\frac{\partial \varrho_p}{\partial t} + \frac{1}{2} \left[ \nabla \varphi_p, \nabla \varrho_p \right]_+ = \frac{1}{2} \left[ \nabla \varphi_p, \nabla \varrho_p \right]_+ + \frac{i}{\hbar} \left[ \varphi_p + \varphi_p(p), \varrho_p \right]_+ = \left( \frac{\partial \varrho_p}{\partial t} \right)_{\text{Coll}}.
\]

The energy matrix \( \varphi_p(r, t) \) is given by

\[
\varphi_p(r, t) = \varphi_p \mathbb{L} + U + \frac{1}{\hbar^2} \int dp' \{ V(0) \mathbb{L} \text{Tr}(\varrho_p)(r, t) \} + \eta V(|p - p'|) \varrho_p(r, t)
\]

in which \((^1)\) \( \varphi_p = p^2/2 m^* \), \( \mathbb{L} \) is a unit matrix, \( U \) is an external field, \( \eta = +1 \) for Bosons and \(-1\) for Fermions, and the Fourier transform of the potential is given by

\[
V(|p|) = \int dr e^{-i p \cdot r/\hbar} V(|r|).
\]

The brackets \([, ]\) and \([, ]^+\) represent a commutator and anticommutator, respectively. We find that in Born approximation the collision integral is given by

\[
\left( \frac{\partial \varrho_p}{\partial t} \right) = \frac{(2 \pi)^2}{\hbar^2} \int dp_1 dp_2 dp_3 dp_4 \delta(p_1 + p_2 - p_3 - p_4) \delta(\varphi_p, \varrho_p) \times
\]

\[
\times \frac{1}{2} \left[ V(|p_1 - p_4|) \right]^2 \left[ \{q_1, q_4\}, \{q_2, q_3\}, \text{Tr}(\varrho_2 \varrho_2) - \{q_1, q_4\}, \text{Tr}(\varrho_1 \varrho_1) \} \right]
\]

\[
+ \eta V(|p_1 - p_4|) V(|p_1 - p_3|) \left[ \{q_1, q_2, q_3, q_4\} \{q_2, q_3\}, \varrho_1, \text{Tr}(\varrho_2 \varrho_2) \right] \left[ \{q_1, q_2, q_3, q_4\}, \varrho_1, \text{Tr}(\varrho_2 \varrho_2) \right]
\]

in which

\[
\tilde{\varrho}_p = \mathbb{L} + \eta \varrho_p.
\]

\((^1)\) We use \( m^* \) for mass to distinguish it from the magnetization \( m \). It is not, of course, a Landau effective mass. In dilute solutions \( m^* \) is the \(^3\)He hydrodynamic mass.
The last term on the left side of equation (2.3) is the spin-rotation commutator and $\xi$ is the so-called «$I_2$» correction. We give a further discussion of $\xi$ in section 4, but for technical reasons to be discussed in section 4, we ignore it in the remainder of this section and in the physical application discussed in the next section.

The various terms on the left side of equation (2.3) are familiar from Landau-Silin theory [3]. The second and third terms are the matrix forms of Landau’s mean-field terms. The spin-rotation commutator leads to spin waves [1, 2, 4, 18, 19, 23, 24] and to the Leggett-Rice effect [3, 20].

If we consider the Boltzmann limit of our result in equation (2.6), which is obtained by taking the final state factors $\xi$ equal to the unit matrix, equation (2.6) is easily seen to reduce properly to the Born approximate version of LL [1], (their equation (32b) for example).

When all spins are quantized along the same axis (the longitudinal case) it is easy to see that

$$\left(\frac{\partial \eta_1}{\partial t}\right)_\text{Coll}$$

reduces to the conventional Uehling-Uhlenbeck diagonal form with components given by

$$\left(\frac{\partial \eta_1}{\partial t}\right)_\text{Coll} = \frac{2 \pi}{h^3} \int dp_2 \ dp_1 \ \delta (p_1 + p_2 - p_2 - p_2) \ \delta (\bar{v}_1 + \bar{v}_2 - \bar{v}_1 - \bar{v}_2)$$

$$\left[ (\nabla (|p_1 - p_2|)^2 + \eta \nabla (|p_1 - p_2|) \nabla (|p_1 - p_2|)) [n_{1,0} n_{2,0} \bar{n}_{1,0} \bar{n}_{2,0} - n_{1,0} n_{2,0} \bar{n}_{1,0} \bar{n}_{2,0}] + \nabla (|p_1 - p_2|)^2 \sum_{\sigma' \sigma} [n_{1,0} n_{2,0} \bar{n}_{1,0} \bar{n}_{2,0} - n_{1,0} n_{2,0} \bar{n}_{1,0} \bar{n}_{2,0}] \right]$$

(2.8)

in which the $n_{p\sigma}$ are the components of the diagonal matrix $\eta_p$. For spin 1/2, $\sigma = \pm \frac{1}{2}$; for other spin values, $\sigma \rightarrow m$, the magnetic quantum number. This standard expression was used in reference [7] (in the s-wave approximation) to compute $\tau_1$ at all temperatures through both degenerate and Boltzmann regimes.

Equations (2.3-2.6) can be shown to be equivalent to the kinetic equation originally derived by Silin [12] (if we note our differing definitions of the distribution function, expand Silin’s drift term and keep only up to first order, and correct several minor errors in Silin’s quoted result).

For the spin-1/2 case we can write

$$\eta_p (r, t) = \frac{1}{2} \left[f_p (r, t) + \sigma_p (r, t) \cdot \sigma \right]$$

(2.9)

We can also write

$$\xi_p + \epsilon_p (r, t) + h_p (r, t) \cdot \sigma$$

(10.10)

where

$$\epsilon_p = \bar{v}_p + \frac{1}{\hbar^2} \int dp' \left\{ V(0) + \frac{\eta}{2} V(|p' - p'|) \right\} f_p (r, t)$$

(11.11)

and

$$h_p = -\frac{1}{2} \hbar \gamma B + \eta \frac{1}{2 \hbar^2} \int dp' \left\{ V(|p' - p'|) \sigma_p (r, t) \right\}$$

(12.12)

where $B$ is an external magnetic field and $\gamma$ the gyromagnetic ratio. The resulting equations for $f_p$ and $h_p$ are

$$\frac{Df_p}{Dt} = \frac{\partial f_p}{\partial t} + \nabla_p \epsilon_p \cdot \nabla f_p - \nabla_p \epsilon_p \cdot \nabla f_p + \sum_{i=\text{spin}} \left[ \frac{\partial h_i}{\partial p_i} \cdot \frac{\partial \sigma_p}{\partial r_i} - \frac{\partial h_i}{\partial r_i} \cdot \frac{\partial \sigma_p}{\partial p_i} \right] = \left(\frac{\partial f_p}{\partial t}\right)_\text{Coll}$$

(13.13)

$$\frac{D\sigma_p}{Dt} = \frac{\partial \sigma_p}{\partial t} \left[ \frac{\partial \epsilon_p}{\partial p_i} \cdot \frac{\partial \sigma_p}{\partial r_i} - \frac{\partial \epsilon_p}{\partial r_i} \cdot \frac{\partial \sigma_p}{\partial p_i} + \frac{\partial f_p}{\partial p_i} \frac{\partial \sigma_p}{\partial r_i} - \frac{\partial f_p}{\partial r_i} \frac{\partial \sigma_p}{\partial p_i} \right] - \frac{2}{\hbar} (h_p \times \sigma_p) = \left(\frac{\partial \sigma_p}{\partial t}\right)_\text{Coll}$$

(14.14)

Equations in this form were originally derived by Silin [2] and later used by Leggett [3]. However, we now
can provide explicit expressions for the collision integrals. They are
\[
\left( \frac{\partial f_1}{\partial t} \right)_{\text{Coll}} = \frac{(2\pi)^2}{h^7} \int dp_x dp_y dp_z \delta(p_1 + p_2 - p_y - p_z) \delta(\epsilon_1 + \epsilon_2 - \epsilon_y - \epsilon_z) \times
\]
\[
\left[ \left\{ V(|p_1 - p_1|) \right\}^2 + \frac{n}{2} V(|p_1 - p_1|) V(|p_1 - p_2|) \right] \left\{ \left[ f_1 + \frac{n}{2} (f_1 f_1 + \sigma_1 \cdot \sigma_1) \right] \left[ f_2 + \frac{n}{2} (f_2 f_2 + \sigma_2 \cdot \sigma_2) \right] \right\}
\]
\[
\times \left[ f_y + \frac{n}{2} (f_y f_y + \sigma_y \cdot \sigma_y) \right] - \left[ f_1 + \frac{n}{2} (f_1 f_1 + \sigma_1 \cdot \sigma_1) \right] \left[ f_2 + \frac{n}{2} (f_2 f_2 + \sigma_2 \cdot \sigma_2) \right]
\]
\[
+ \frac{n}{2} V(|p_1 - p_1|) V(|p_1 - p_2|) \left\{ \left[ \sigma_x + \frac{n}{2} f_x \sigma_1 + f_1 \sigma_y \right] \cdot \left[ \sigma_z + \frac{n}{2} f_z \sigma_2 + f_2 \sigma_y \right] \right\}
\]
\[
- \left[ \sigma_1 + \frac{n}{2} (f_1 \sigma_1 + f_1 \sigma_y) \right] \cdot \left[ \sigma_2 + \frac{n}{2} (f_z \sigma_2 + f_z \sigma_y) \right].
\]
(2.15)

and
\[
\left( \frac{\partial \sigma_1}{\partial t} \right)_{\text{Coll}} = \frac{(2\pi)^2}{h^7} \int dp_x dp_y dp_z \delta(p_1 + p_2 - p_y - p_z) \delta(\epsilon_1 + \epsilon_2 - \epsilon_y - \epsilon_z) \times
\]
\[
\left[ \left\{ V(|p_1 - p_1|) \right\}^2 + \frac{n}{2} V(|p_1 - p_1|) V(|p_1 - p_2|) \right] \left\{ \left[ f_y + \frac{n}{2} (f_y f_y + \sigma_y \cdot \sigma_y) \right] \left[ f_z + \frac{n}{2} (f_z f_z + \sigma_z \cdot \sigma_z) \right] \right\}
\]
\[
\times \left[ f_1 + \frac{n}{2} (f_1 f_1 + \sigma_1 \cdot \sigma_1) \right] - \left[ f_2 + \frac{n}{2} (f_2 f_2 + \sigma_2 \cdot \sigma_2) \right] \left[ f_1 + \frac{n}{2} (f_1 f_1 + \sigma_1 \cdot \sigma_1) \right]
\]
\[
+ \frac{n}{2} V(|p_1 - p_1|) V(|p_1 - p_2|) \left\{ \left[ f_z + \frac{n}{2} (f_z f_z + \sigma_z \cdot \sigma_z) \right] \left[ f_y + \frac{n}{2} (f_y f_y + \sigma_y \cdot \sigma_y) \right] \right\}
\]
\[
- \left[ \sigma_1 + \frac{n}{2} (f_1 \sigma_1 + f_1 \sigma_y) \right] \cdot \left[ \sigma_2 + \frac{n}{2} (f_z \sigma_2 + f_z \sigma_y) \right]
\]
\[
+ \frac{n}{2} \left[ (\sigma_1 \cdot \sigma_1)(\sigma_2 - \sigma_y) - (\sigma_2 \cdot \sigma_2)(\sigma_1 - \sigma_y) + (\sigma_y \cdot \sigma_2)(\sigma_1 - \sigma_y) \right].
\]
(2.16)

3. Equation for the spin current.

We want to use our kinetic equation to make contact with spin-diffusion experiments on spin-$1/2$ Fermi systems. We have in mind principally to make contact with experiments on dilute solutions of $^3$He in $^4$He. We develop an equation for the spin current, analogous to Leggett's equation [3], by which we can describe both longitudinal and transverse spin diffusion.

We follow the usual linearization procedure: a zero-order local-equilibrium distribution function $g_p^0$ causes the collision integral to vanish. The true solution $g_p = g_p^0 + \delta g_p$ is inserted into the left side of equation (2.3). The second and third terms are then approximated by dropping the correction $\delta g_p$. We cannot drop $\delta g_p$ from the spin-rotation term since that provides its leading contribution. The collision integral is linearized in $\delta g_p$. We know that the solution for $\delta g_p$ must strongly overlap the drift terms so we choose a variational expression based on their form.

We can show that the appropriate local equilibrium distribution function is given by
\[
g_p^0 = \frac{1}{2} (f_p^0 \mathbf{\hat{L}} + \sigma_p^0 \mathbf{\hat{e}} \cdot \mathbf{\sigma})
\]
(3.1)

where
\[
f_p^0 = n_{p_+}^0 + n_{p_-}^0
\]
(3.2a)
\[
\sigma_p^0 = n_{p_+}^0 - n_{p_-}^0
\]
(3.2b)

$\mathbf{\hat{e}}$ is the local direction of the magnetism, and
\[
n_{p_\sigma}^0 = \left[ e^{\beta (\mu_\sigma - \mu_p)} - 1 \right]^{-1}
\]
(3.3)

Here $\mu_\sigma$ is a chemical potential for spin species $\sigma$ that may depend on temperature and position. Only the kinetic energy appears in equation (3.3) because of the form of the energy conservation $\delta$-function in equation (2.6), (see section 4). At this point we simplify the mathematics by making the assumption that a contact interaction (which can be made equivalent to an s-wave approximation) is sufficient.
and take $V(|p|) = V(0) = 2 V_0$. This is a reasonable approximation for the dilute-solution problem. Then the zero-order expressions for the energy terms are

$$
\varepsilon^0_p = \varepsilon_p + \frac{V(0)}{2 h^3} \int dp' f_p^0 = \varepsilon_p + V_0 n
$$

(3.4)

with $n(r, t)$ the particle density and

$$
\mathbf{h}_p^0 = - \frac{\hbar \gamma}{2} \mathbf{B} - \frac{V(0)}{2 h^3} \dot{e} \int dp' \sigma^0_p = - \frac{\hbar \gamma}{2} \mathbf{B} - V_0 \mathbf{m}
$$

(3.5)

where the magnetization at position $\mathbf{r}$ and time $t$ is

$$
\mathbf{m}(r, t) = \frac{1}{h^3} \int dp (n^0_p - n^0_{-p}) \dot{e}
$$

(3.6)

With the substitution of the local equilibrium functions, the $r$ and $p$ gradient terms on the left side of equation (2.14) become

$$
\sum_i \dot{v}_p = \sum_i \dot{v}_p + \sum_i \dot{e} \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} \frac{\partial \mu_\sigma}{\partial \varepsilon_p} \dot{e} + \sum_i \frac{1}{h^3} \int dp \left( \sum_\sigma \sigma n_{p\sigma} \frac{\partial \mu_\sigma}{\partial \varepsilon_p} \dot{e} + V_0 \sum_\sigma \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} \frac{\partial m}{\partial \varepsilon_p} \right)
$$

(3.7)

where $v_p = p/m^*$ and we have dropped the superscript 0 from the local equilibrium functions to simplify the notation.

The use of the Gibbs-Duhem relation (which can be derived from the kinetic equation), given by

$$
\sum_\sigma n_{p\sigma} \nabla \mu_\sigma = 0
$$

(3.8)

and the relation

$$
n_\sigma = \frac{1}{2} (\sigma m + n)
$$

(3.9)

allow us to show easily that

$$
\frac{\partial n}{\partial r_i} = - \frac{\partial m}{\partial r_i}
$$

(3.10)

and

$$
\frac{\partial \mu_\sigma}{\partial r_i} = \sigma \frac{\partial \langle G \rangle}{\partial r_i}
$$

(3.11)

where

$$
d = \sum_\sigma \sigma n_{p\sigma} G_{-\sigma} / \langle G \rangle,
$$

(3.12)

$$
G_\sigma = \frac{1}{h^3} \int dp \left( - \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} \right),
$$

(3.13)

and

$$
\langle G \rangle = \sum_\sigma n_{p\sigma} G_{-\sigma}
$$

(3.14)

The quantities $G_\sigma$ can be evaluated for high degeneracy or for a Boltzmann system; however, we keep the formulas valid for an arbitrary temperature.

If we define the correction $\delta g_\sigma$ to the local equilibrium function as

$$
\delta g_\sigma = \frac{1}{2} (\delta f_p \mathbf{L} + \delta \sigma_p \cdot \mathbf{G})
$$

(3.15)

then the left side of equation (2.14) becomes

$$
\frac{D\sigma_p}{Dt} = - \sum_i \dot{e} \left( v_p \frac{\partial m}{\partial r_i} \sum_\sigma \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} t_\sigma \right) + \sum_i \frac{\partial e}{r_i} v_p \left( \sum_\sigma \sigma n_{p\sigma} + V_0 m \sum_\sigma \frac{\partial n_{p\sigma}}{\partial \varepsilon_p} \right) - \frac{2}{h} \delta \sigma_p \times \left[ \frac{\hbar \gamma}{2} \mathbf{B} + V_0 \mathbf{m} \right]
$$

(3.16)

where

$$
t_\sigma = \frac{1}{\langle G \rangle} [n_{-\sigma} - 2 V_0 n_{\sigma} G_{-\sigma}].
$$

(3.17)

The quantities in this drift term that have the factors $\partial n_{p\sigma}/\partial \varepsilon_p$ are nonvanishing only at the Fermi surfaces in a degenerate system. All longitudinal terms (proportional to $\dot{e}$) are of this nature. On the other hand, there is one term, which is transverse because...
fact that spins tipped away from local equilibrium (while maintaining constant \( |m| \)), will constitute a disturbance everywhere in momentum space that there is a net magnetization including between the two Fermi spheres.

If we carry out a similar local-equilibrium substitution in equation (2.13) we find

\[
\frac{Df_p}{Dt} = \frac{\delta f_p}{\delta t} - \sum v_{pi} \frac{\delta m}{\delta r_i} \times \sum \left( \sigma_{r_{\sigma}} \frac{\partial n_{p_{\sigma}}}{\partial \bar{e}_p} \right) \tag{3.18}
\]

Particle and spin currents may be defined in terms of \( f_p \) and \( \sigma_p \), respectively, according to

\[
J_j = \frac{1}{h^3} \int dp v_{pj} f_p = \frac{1}{h^3} \int dp v_{pj} \delta f_p, \tag{3.19}
\]

\[
J_j(m) = \frac{1}{h^3} \int dp v_{pj} \sigma_p = \frac{1}{h^3} \int dp v_{pj} \delta \sigma_p. \tag{3.20}
\]

Examination of the drift terms leads us to take the following variational form for \( \delta f_p \):

\[
\delta f_p = q \sum v_{pi} \frac{\delta m}{\delta r_i} \sum \left( \sigma_{r_{\sigma}} \frac{\partial n_{p_{\sigma}}}{n_{\sigma} \bar{e}_p} \right) \tag{3.21}
\]

where \( q \) is a variational constant. The form chosen automatically guarantees a vanishing particle current as required in a spin diffusion experiment. The magnetization increment is taken to be

\[
\delta \sigma_p = \delta \sigma_p^l + \delta \sigma_p^\perp \tag{3.22}
\]

where

\[
\delta \sigma_p^l = q \sum v_{pi} \frac{\delta m}{\delta r_i} \left( \sum \frac{1}{n_{\sigma}} \frac{\partial n_{p_{\sigma}}}{\bar{e}_p} \right) \tag{3.23}
\]

is longitudinal, and

\[
\delta \sigma_p^\perp = A \sum v_{pi} \hat{g}_i \left( \sum \sigma n_{p_{\sigma}} \right) \tag{3.24}
\]

is the transverse part. Here \( A \) is a variational constant. The form for \( \delta \sigma_p^p \) corresponds to that used in our previous work [7] and will lead to the same relaxation time \( \tau_1 \) and the same (7) diffusion constant \( D_1 \). The unit vectors \( \hat{g}_i \) in the transverse part are perpendicular to the magnetization direction \( \hat{e} \). To be explicit we can write

\[
\hat{g}_i = x \frac{\partial \hat{e}}{\partial r_i} + y \hat{e} \times \frac{\partial \hat{e}}{\partial r_i} \tag{3.25}
\]

where \( x \) and \( y \) are constants.

We now multiply equation (3.16) by \( v_{pj} \) and integrate over \( p \) to find an equation for the spin current \( J_j(m) \). We find

\[
\frac{\partial J_j(m)}{\partial t} + \alpha_1 \frac{\delta m}{\delta r_j} \epsilon + \alpha_\perp \int \frac{\partial \hat{e}}{\partial r_j} - \gamma J_j(m) \times B + \frac{2}{\hbar} V_0 m \times J_j(m)
\]

whith

\[
\alpha_1 = \frac{1}{m^*} \sum \sigma n_{\sigma}, \tag{3.27a}
\]

\[
\alpha_\perp = \frac{2}{3 m^* \pi} \sum \sigma k_{\sigma} - V_0 n \tag{3.27b}
\]

and

\[
k_\sigma = \frac{1}{h^3} \int dp \bar{p} n_{p_{\sigma}}. \tag{3.28}
\]

The left side of equation (3.26) has roughly the same general form as that found by Leggett [3] and LL [1]. However, there is the important difference that \( \alpha_\perp \) and \( \alpha_1 \) are not equal. Further the present equation is not limited to low polarization as is Leggett's. When the polarization is small and the system degenerate then we find \( \alpha_\perp = \alpha_1 \) and the result becomes equivalent to that given is MMS [5] (which equals Leggett's equation with a constant interaction). Furthermore, when the system is in the Boltzmann regime of temperature, then we again have \( \alpha_1 = \alpha_\perp \) for arbitrary polarization, and the result of LL [1] is reproduced. By finding differing polarization dependences of \( \alpha_1 \) and \( \alpha_\perp \) and their effect on the longitudinal and transverse diffusion constants we are repeating results noted previously by Meyerovich [9].

We will shortly linearize the right side of equation (3.26). However, we anticipate our answer and write down the form that results from our analysis of the collision term. Examination of \( \left( \frac{\partial \sigma_p}{\partial t} \right)_{coll} \) in equation (2.16) tells us what to expect. Linearization will lead to terms in \( \delta f_p \) and \( \delta \sigma_p \). These will necessarily be longitudinal. However, there are other possible forms having factors like \( \delta \sigma_p f_p \), \( \delta \sigma_p f_p f_p \), \( \sigma p(\sigma p \cdot \sigma p') \) which can produce transverse terms. The longitudinal factors are localized on the Fermi surfaces at low temperatures and so the collision integral ultimately produces factors [2b, 7] like \( n_1 n_2 (1 - n_1) (1 - n_2) \) which, together with energy and momentum conservation, restrict all collisions to the Fermi surfaces and lead to the usual \( \tau_1 \sim T^{-2} \) temperature dependence. On the other hand, the transverse quantity \( \delta \sigma_p^\perp \) is not restricted to the Fermi surface and

\footnote{Equation (3.16) contains a previously neglected mean-field term that will change \( D_1 \) a bit.}
different distribution function factors appear in $\tau_{\perp}$. Hence we anticipate $\tau_{\perp} \neq \tau_1$. However, when the polarization is small or in the non-degenerate limit it will turn out that $\tau_{\perp} = \tau_1$.

Meyerovich [9] has previously noted what physics results from having differing values of $\tau_1$ and $\tau_{\perp}$. We repeat that argument here in a slightly different form for the sake of completeness.

We will show below that the collision integral gives

$$\frac{1}{h^3} \int dp v_p \left( \frac{\partial \sigma_p}{\partial \tau} \right)_{\text{Coll}} = -\tau_1 \left( J_j(m) \cdot \hat{e} \right) \hat{e} - \frac{1}{\tau_{\perp}} \left( J_j(m) \cdot \hat{g}_j \right) \hat{g}_j.$$

(3.29)

The solution to equation (3.26), with equation (3.29) is found easily to be (Cf. Ref. [1, 3, 9])

$$J_j(m) = -D_1 \frac{\partial m}{\partial r_j} - \frac{D_{\perp}}{\left[ 1 + (\mu m/n)^2 \right]} \times \left[ m \frac{\partial \hat{e}}{\partial r_j} + \frac{\mu m^2}{n} \hat{e} \times \frac{\partial \hat{e}}{\partial r_j} \right].$$

(3.30)

where $\mu$ is the spin-rotation parameter given here by

$$\mu = -\frac{2 nV_0}{h} \tau_{\perp}.$$

(3.31)

$$\left( \frac{\partial \delta q_1}{\partial \tau} \right)_{\text{Coll}} = \left( \frac{2 \pi}{h^2} \right)^2 \int dp_2 dp_1 dp_2 \delta (p_1 + p_2 - p_2^\prime - p_2^\prime) \delta \left( \tilde{e}_1 + \tilde{e}_2 - \tilde{e}_1^\prime - \tilde{e}_2^\prime \right) \times$$

$$\times \left[ \frac{1}{2} \left[ V(|p_1 - p_1^\prime|) \right]^2 \left\{ \text{Tr} (\delta q_2 \delta q_1 + \delta q_2 \delta q_1) \right\} \delta (\tilde{e}_1 + \tilde{e}_2 - \tilde{e}_1^\prime - \tilde{e}_2^\prime) \times$$

$$\times [g_1, g_1], + Tr (\delta q_2 \delta q_2) \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$= \left\{ \text{Tr} (\delta q_2 \delta q_2) \right\} \left\{ \text{Tr} (\delta q_1 \delta q_1 + \delta q_1 \delta q_1) \right\} - \text{Tr} (\delta q_2 \delta q_2 + \delta q_2 \delta q_2)$$

$$(\delta g_p)_{\perp}^1 = \frac{1}{2} \left\{ \delta f_p \right\} + \delta \sigma_p^1 \cdot \mathbf{g}$$

(3.36)

with $\delta f_p$ and $\delta \sigma_p^1$ given in equations (3.21) and (3.23). All matrices are diagonal along the direction of the magnetization and the results are the same as those derived for $\tau_1$ in reference [7]. We examine here only the more interesting transverse case in which

$$\delta g_p^\perp = \frac{1}{2} \delta \sigma_p^\perp \cdot \mathbf{g}$$

(3.37)

and $\delta \sigma_p^\perp$ is given by equations (3.24). We work in the reference frame whose $z$ axis is along $\hat{e}$, the local
magnetization direction. In that frame all the \( n_\sigma \) (local equilibrium distribution functions) are diagonal with matrix elements \( n_{\sigma \sigma} (\sigma = \pm 1) \). The increment \( \delta n_\sigma \), on the other hand, is completely off-diagonal and has the form

\[
\delta n_\sigma = \begin{bmatrix}
0 & s_\sigma^* \sum_\sigma \sigma n_{\sigma \sigma} \\
 s_\sigma \sum_\sigma \sigma n_{\sigma \sigma} & 0
\end{bmatrix}
\]  

(3.38)

We now simply carry out the matrix multiplications indicated in equation (3.35). A typical term is, say,

\[
(\partial S_{\sigma \sigma} / \partial t)_{\text{Coll}} = \sum_\sigma \delta \sigma_{\sigma \sigma} A_{\sigma \sigma} \alpha'.
\]

where

\[
s_\sigma = s_{\sigma x} + i s_{\sigma y} \quad \text{(3.39)}
\]

and

\[
s_\sigma = s_\sigma A_i \quad \text{(3.40)}
\]

We carry out all the matrix multiplications, then multiply the resulting \( (\partial S_\sigma / \partial t)_{\text{Coll}} \) by \( \alpha \) and take the spin trace to give \( (\partial S_{\sigma \sigma} / \partial t)_{\text{Coll}} \). For the sake of simplicity we use the constant potential approximation \( V(p) = V(0) = 2 V_0 \). We find then

\[
J^+ (p) = \frac{1}{\hbar^2} \int dp_{\sigma} \frac{\delta \sigma_{\sigma \sigma}}{\partial t} = A_{\sigma \sigma} \alpha'.
\]

(3.45)

It is easy to test this result in the non-degenerate or low polarization limits. We find in each case that \( \tau_{\perp \perp} \) reduces as expected to \( \tau_{\parallel \parallel} \).

The unusual features in \( \tau_{\perp \perp} \) can be seen from the forms appearing in equations (3.43) and (3.44). Energy conservation causes quantities such as \( (++ + +) \) or \( (+- ++) \) to vanish. However, each of the forms written in equation (3.43) has only one different spin component as if spin were not conserved in the collision. Energy conservation does not make the quantity in equation (3.44) vanish. The source of this unusual form is simply that in a transverse process a spin from a neighbouring region enters a given regime, where the local magnetization direction is \( \vec{e} \), with a spin component perpendicular to \( \vec{e} \). It, in a sense, has a foot in each local Fermi sphere. This mixing in \( \delta \sigma_{\sigma \sigma} \) is the source of the mixed spin configurations that occur in \( S_{\sigma \sigma} (p_i) \). Factors as occur in equation (3.44) do not confine the collision to the Fermi surfaces. Suppose for example, that the

where

\[
S_{\sigma \sigma} (p_i) = \frac{2}{\hbar^2} \int dp_{\sigma} dp_{\sigma} dp_{\sigma} \times
\]

\[
\times \delta (p_1 + p_2 - p_1 - p_2) \delta (\vec{e}_1 + \vec{e}_2 - \vec{e}_1 - \vec{e}_2) \times 2 (v_{11} - v_{21}) \left[ (+1 + +) - (+- +) \right]
\]

(3.43)

where the notation is, for example,

\[
(++) = n_{++} - n_{+-}, \quad (++) = n_{--} - n_{-+}.
\]

(3.44)
up-spin Fermi sphere is larger than the down-spin sphere. Then a factor like
\[ n_{1+} n_{2-} \tilde{n}_{1-} \tilde{n}_{2-} = n_{1+} n_{2-} (1 - n_{1-}) (1 - n_{2-}) \]
(3.48)
requires an up spin 1 from anywhere in the up-sphere colliding with a down spin 2 to have both spins end up anywhere outside the smaller down-spin sphere. It is easy to show that energy and momentum conservation put no severe restriction on the collision. Processes far from the Fermi surfaces are easily possible. (Processes in which both final state factors are down-spin, as given in equation (3.48), can be shown easily to dominate all other terms). Because of the large phase space for these processes we expect \( \tau_+ \ll \tau_\|$ as we have mentioned above.

We have not yet made further explicit analytic or numerical reductions of \( \tau_+ \). We plan to treat this matter in a future publication.

We now go back to seek a justification for the kinetic equation used in the preceding analysis.

4. Derivation of the kinetic equation.

Our procedure is a generalization of the Green's function method of Kadanoff and Baym and we need not give all the details here. The main complication is that the Green's functions, which become matrices here, are not diagonal in spin space.

We suppose that our system (either Fermi or Bose) was in thermodynamic equilibrium in the distant past and we have then turned on a perturbation given by
\[
H_{\text{ext}} = \sum_{a\beta} \int dr \, \psi_\alpha^+(r, t) \, U_{a\beta} \, \psi_\beta(r, t)
\] (4.1)
where \( \psi_\alpha^+(r, t) \) creates a particle having spin \( \alpha \) at \( r \) at time \( t \). For spin 1/2 we can take (3)
\[
U_{a\beta} = -\frac{\gamma}{2} \mathbf{B}(r, t) \cdot \sigma_{a\beta}
\]
with \( \mathbf{B} \) a magnetic field and \( \sigma_{a\beta} \) a component of a Pauli matrix. The response of the system to the perturbation \( H_{\text{ext}} \) is given by the non-equilibrium, real-time, Green's functions defined by
\[
G_{a\beta}(1, 1'; \mathcal{U}) = \frac{1}{i} \langle T(\psi_\alpha(1; \mathcal{U}) \psi_\alpha^+(1'; \mathcal{U})) \rangle
\] (4.2a)
\[
G_{a\beta}^*(1, 1'; \mathcal{U}) = \frac{1}{i} \langle \psi_\alpha(1; \mathcal{U}) \psi_\beta^+(1'; \mathcal{U}) \rangle
\] (4.2b)
\[
G_{\alpha\beta}^*(1, 1'; \mathcal{U}) = \frac{\eta}{i} \langle \psi_\beta^+(1'; \mathcal{U}) \psi_\alpha(1; \mathcal{U}) \rangle
\] (4.2c)
in which \( 1 \) stands for \( (r_1, t_1) \), etc, and
\[
\psi_\alpha(1; \mathcal{U}) = v^+(t_1) \, \psi_\alpha(1) \, v(t_1),
\] (4.3)
with
\[
v(t) = T \exp \left\{ -i \int_{-\infty}^{t} d\tilde{r} \, H_{\text{ext}}(\tilde{r}) \right\},
\] (4.4)
and \( T \) is the time-ordering operator. The average \( \langle \ldots \rangle \) in equation (4.2) denotes a grand canonical ensemble average
\[
\langle \mathcal{O}(\mathcal{U}) \rangle = \text{tr} \left\{ e^{-\beta(H - \mu N)} \, \mathcal{O}(\mathcal{U}) \right\} / \text{tr} \left\{ e^{-\beta(H - \mu N)} \right\}.
\] (4.5)

Here \( H \) is the second quantized Hamiltonian of the system, not including \( H_{\text{ext}} \), and \( N \) is the number operator. The trace with a small \( \langle t \rangle \), \( \text{tr} \), is a sum over eigenstates as opposed to the spin trace, \( \text{Tr} \), introduced previously. We take \( H \) to be
\[
H = \sum_{\alpha} \int dr \, \psi_\alpha^+(r) \left( -\frac{V}{2 \, m^*} \right) \psi_\alpha(r) + \frac{1}{2} \sum_{a\beta} \int dr_1 \int dr_2 \, \psi_\alpha^+(r_1) \psi_\beta^+(r_2) \, V(|r_1 - r_2|) \, \psi_\beta(r_2) \psi_\alpha(r_1)
\] (4.6)
with \( V(|r|) \) a spin-independent potential.

Following reference [13], we get the equations of motion for \( g^\alpha(1, 1'; \mathcal{U}) \) (with \( \mathcal{U} \) dropped from the \( g \) notation)
\[
\left( i \frac{\partial}{\partial t_1} + \frac{V_2}{2 \, m^*} \right) \, g^\alpha(1, 1') - \mathcal{U}(1) \, g^\alpha(1, 1') - \int d\tilde{r} \, \Sigma_{HF}(1, \tilde{r}) \, g^\alpha(\tilde{r}, 1') =
\]
\[
= \int_{-\infty}^{t_1} d\tilde{r} \left[ \Sigma^-(1, \tilde{r}) - \Sigma^+(1, \tilde{r}) \right] \, g^\alpha(\tilde{r}, 1') - \int_{-\infty}^{t_1} d\tilde{r} \, \Sigma^-(1, \tilde{r}) \, g^\alpha(\tilde{r}, 1') - \int_{-\infty}^{t_1} d\tilde{r} \, \Sigma^+(1, \tilde{r}) \, g^\alpha(\tilde{r}, 1')
\] (4.7a)
and
\[
\left( -i \frac{\partial}{\partial t_1} + \frac{V_2}{2 \, m^*} \right) \, g^\alpha(1, 1') - g^\alpha(1, 1') \, \mathcal{U}(1') - \int d\tilde{r} \, \Sigma^+(1, \tilde{r}) \, g^\alpha(\tilde{r}, 1') =
\]
\[
= \int_{-\infty}^{t_1} d\tilde{r} \left[ g^\alpha(1, \tilde{r}) - g^-\alpha(1, \tilde{r}) \right] \Sigma^-(\tilde{r}, 1') - \int_{-\infty}^{t_1} d\tilde{r} \, g^\alpha(1, \tilde{r}) [\Sigma^-(\tilde{r}, 1') - \Sigma^+(\tilde{r}, 1')] \] (4.7b)

\(^{(3)}\) In this section we take \( \hbar = 1 \).
The proper self-energy $\Sigma$ has been split into a Hartree-Fock part and a collisional part according to
\[ \Sigma(1, 1') = \Sigma_{HF}(1, 1') + \Sigma_c(1, 1'). \] (4.8)

$\Sigma_{HF}$ contains a time $\delta$-function, $\delta(t_1 - t_Y)$:
\[ \Sigma_{HF}(1, 1') = \delta(t_1 - t_Y) \Sigma_{HF}(r_1, r_Y, t_1) \] (4.9)
and $\Sigma_c$ is composed of two analytic functions of the time variables in the form
\[ \Sigma_c(1, 1') = \begin{cases} \Sigma^\infty(1, 1') & \text{for } t_1 > t_Y \\ \Sigma^\infty(1, 1') & \text{for } t_1 < t_Y \end{cases}. \] (4.10)

If we subtract (4.7b) from (4.7a) we find
\[
\left( i \frac{\partial}{\partial t_1} + i \frac{\partial}{\partial t_Y} + \frac{v_{11}^2}{2 m^*} - \frac{v_{11}^2}{2 m^*} \right) g^\infty(1, 1') - \left[ U(1) g^\infty(1, 1') - g^\infty(1, 1') U(1') \right] - \\
- \int d\bar{t} \left[ \Sigma_{HF}(1, \bar{t}) g^\infty(\bar{t}, 1') - g^\infty(1, \bar{t}) \Sigma_{HF}(\bar{t}, 1') \right] \\
= \int_{-\infty}^{\infty} d\bar{t} \left[ \left( \Sigma^\infty(1, \bar{t}) - \Sigma^\infty(1, \bar{t}) \right) g^\infty(\bar{t}, 1') - \left[ g^\infty(1, \bar{t}) - g^\infty(1, \bar{t}) \right] \Sigma^\infty(1, 1') \right] \\
- \int_{-\infty}^{t_1} d\bar{t} \left[ \Sigma^\infty(1, \bar{t}) \left( g^\infty(\bar{t}, 1') - g^\infty(\bar{t}, 1') \right) - g^\infty(1, \bar{t}) \left[ \Sigma^\infty(1, \bar{t}) - \Sigma^\infty(1, 1') \right] \right]. \] (4.11)

We change variables to
\[
R = \frac{1}{2} (r_1 + r_Y) \quad ; \quad T = \frac{1}{2} (t_1 + t_Y) \\
r = r_1 - r_Y \quad ; \quad t = t_1 - t_Y \] (4.12)
and write $g^\infty(1, 1')$ as $g^\infty(r, t, R, T)$.

The $r, t$ Fourier transforms of these functions are
\[ g^\infty(p, \omega, R, T) = \int dr \int dt e^{-i p \cdot r + i \omega t} \eta i g^\infty(r, t, R, T) \] (4.13a)
and
\[ g^\infty(p, \omega, R, T) = \int dr \int dt e^{-i p \cdot r + i \omega t} i g^\infty(r, t, R, T). \] (4.13b)

For a slowly varying external field, $g^\infty(p, \omega, R, T)$ is a slowly varying function of $R$ and $T$. If we expand equation (4.11) and keep terms up to first order in $\nabla_R$ and $\nabla_T$, we get,
\[
\frac{1}{2} \left\{ \left[ (\omega - E_p) \Delta - U(R, T) - \Sigma_{HF}(p, R, T) - \xi(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\} + \\
+ \left\{ h(p, \omega, R, T), \Sigma^\infty(p, \omega, R, T) \right\} + \\
+ i \left\{ \left[ U(R, T) + \Sigma_{HF}(p, R, T) + \xi(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\} - \left\{ h(p, \omega, R, T), \Sigma^\infty(p, \omega, R, T) \right\} \\
= \frac{1}{2} \left\{ \left[ \Sigma^\infty(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\} + \left\{ \left[ \Sigma^\infty(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\} \\
+ i \left\{ \left[ \Sigma^\infty(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\} \\
- \left\{ \left[ \Sigma^\infty(p, \omega, R, T), g^\infty(p, \omega, R, T) \right] \right\}. \] (4.14a)
and a similar equation for $g^-_r$:

\[ \eta \frac{1}{2} \left\{ \left( (\omega - \bar{\nu}_r) \sum - \sum_{\text{hl}} - \bar{\nu}_r g^+ \frac{\bar{v}}{r} + [b, \sum^+ \frac{\bar{v}}{r}] \right) \right. + \eta i \left\{ \left[ \sum + \sum_{\text{hl}} + \bar{\nu}_r g^- \right] - [b, \sum^- \frac{\bar{v}}{r}] \right\} = \right. \]

\[ = \frac{1}{2} \left\{ [\sum^+, g^- \frac{\bar{v}}{r}], - [\sum^-, g^- \frac{\bar{v}}{r}] \right\} + \frac{i}{4} \left\{ [\sum^+, g^- \frac{\bar{v}}{r}] - [\sum^-, g^- \frac{\bar{v}}{r}] \right\} \]  

(4.14b)

where $[ , ]_c$ represents a commutator (anticommutator) and we define a « Poisson commutator » as

\[ [\mathcal{A}, \mathcal{B}]_P = (\mathcal{A}, \mathcal{B})^P = (\mathcal{B}, \mathcal{A})^P \]

(4.15)

with the generalized Poisson bracket given by

\[ (\mathcal{A}, \mathcal{B})^P = \nabla_r \mathcal{A} \cdot \nabla_p \mathcal{B} - \nabla_r \mathcal{B} \cdot \nabla_p \mathcal{A} + \frac{\partial \mathcal{A}}{\partial \omega} \frac{\partial \mathcal{B}}{\partial T} - \frac{\partial \mathcal{A}}{\partial T} \frac{\partial \mathcal{B}}{\partial \omega}. \]

(4.16)

In equation (4.14) we have introduced the quantities

\[ \sum_{\text{hl}} (p, R, T) = \int dr \, e^{-i p \cdot r} \sum_{\text{hl}} (r, 0, R, T), \]

(4.17)

\[ \xi (p, \omega, R, T) = \int dr \int \frac{d \omega'}{2 \pi} \sum \frac{1}{|t|} \left[ \sum^- (r, t, R, T) - \sum^- (r, t, R, T) \right] = \]

\[ = \int \frac{d \omega'}{2 \pi} \frac{\mathcal{L} (p, \omega', R, T)}{\omega - \omega'}, \]

(4.18)

where

\[ \mathcal{L} (p, \omega, R, T) = \sum^- (p, \omega, R, T) - \eta \sum^+ (p, \omega, R, T), \]

(4.19)

\[ \sum^- (p, \omega, R, T) = \int dr \int \frac{d \omega'}{2 \pi} \sum \frac{1}{|t|} \left[ \sum^- (r, t, R, T) - \sum^- (r, t, R, T) \right] = \]

\[ = \int \frac{d \omega'}{2 \pi} \frac{\mathcal{G} (p, \omega', R, T)}{\omega - \omega'}, \]

(4.20a)

\[ \sum^+ (p, \omega, R, T) = \int dr \int \frac{d \omega'}{2 \pi} \sum \frac{1}{|t|} \left[ \sum^+ (r, t, R, T) - \sum^+ (r, t, R, T) \right] = \]

\[ = \int \frac{d \omega'}{2 \pi} \frac{\mathcal{G} (p, \omega', R, T)}{\omega - \omega'}, \]

(4.20b)

and

\[ \mathcal{B} (p, \omega, R, T) = \int dr \int \frac{d \omega'}{2 \pi} \sum \frac{1}{|t|} \left[ \sum^- (r, t, R, T) - \sum^- (r, t, R, T) \right] = \]

\[ = \int \frac{d \omega'}{2 \pi} \frac{\mathcal{E} (p, \omega', R, T)}{\omega - \omega'}, \]

(4.21)

The spectral density is defined as

\[ \mathcal{G} (p, \omega, R, T) = \sum^+ (p, \omega, R, T) - \eta \sum^- (p, \omega, R, T). \]

(4.22)

By subtracting equation (4.14b) from (4.14a) we get an equation for the spectral density $\mathcal{G} (p, \omega, R, T)$:

\[ \frac{1}{2} \left\{ \left[ (\omega - \bar{\nu}_p) \sum - \sum_{\text{hl}} - \bar{\nu}_p \mathcal{G} \frac{\bar{v}}{r} + i \left[ \sum + \sum_{\text{hl}} + \bar{\nu}_p \mathcal{G} \right] \right] = \right. \]

\[ = \frac{1}{2} \left\{ \left[ \xi^- \mathcal{G} \frac{\bar{v}}{r} \right] - [\mathcal{B}, \xi^- \mathcal{G} \frac{\bar{v}}{r}] \right\} \]

\[ - i \left\{ [\xi^- \mathcal{G}], - [\mathcal{B}, \xi^- \mathcal{G}] \right\}. \]

(4.23)

Were we able to solve this equation, then an equation for the distribution function

\[ \mathcal{G} (p, \omega, R, T) = \int \frac{d \omega'}{2 \pi} \frac{\mathcal{G} (p, \omega, R, T)}{\omega - \omega'}, \]

(4.24)

would follow from equation (4.14a). However, equation (4.23) is too complicated for us to find a general solution easily. As is discussed in reference [13] the determination of a Boltzmann equation involves a further simplification, namely, the neglect of the terms of order $\frac{\partial}{\partial T}$ or $\nabla_r$ that originate from the « collisional » parts of the Green's functions. (In Eq. (4.23), these are on the right-hand side). Then instead of equations (4.14a) and (4.23) we have

\[ \frac{1}{2} \left\{ \left[ (\omega - \bar{\nu}_p) \sum - \sum_{\text{hl}} - \bar{\nu}_p \mathcal{G} \frac{\bar{v}}{r} + i \left[ \sum + \sum_{\text{hl}} + \bar{\nu}_p \mathcal{G} \right] \right] = \right. \]

\[ = \frac{1}{2} \left\{ \left[ \xi^- \mathcal{G} \frac{\bar{v}}{r} \right] - [\mathcal{B}, \xi^- \mathcal{G} \frac{\bar{v}}{r}] \right\} \]

(4.25)
From its definition in equation (4.22) we can show that $a$ satisfies

$$
\frac{1}{2} \left[ (\omega - \bar{\epsilon}_p) \mathcal{L} - \mathcal{U} - \mathcal{H}_0, \mathcal{A} \right] \mathcal{L}^p + \left[ \mathcal{A} - \mathcal{H}_0 + \mathcal{B}, \mathcal{L} \right] \mathcal{L}^p = 0 \quad (4.26)
$$

From its definition in equation (4.22) we can show that $\mathcal{A}$ satisfies

$$
\int \mathcal{A}(p, \omega, R, T) \frac{d\omega}{2\pi} = \mathcal{L} \quad (4.27)
$$

If we assume that the solution to these equations has the form

$$
\mathcal{A}(p, \omega, R, T) = \mathcal{A}(p, \omega, R, T) \mathcal{A}_p(R, T) \quad (4.28)
$$

then we see that a solution of equations (4.25) and (4.26) involves solving the equations simultaneously for $\mathcal{A}$ and $\mathcal{A}_p$. This is still too complicated of a task. If we assume, in equation (4.24), that all quantities are diagonal (i.e. make a local equilibrium assumption with the neglect the small transverse effects) then equation (4.26) can be solved and the form

$$
\mathcal{A}_p = \bar{\epsilon}_p \mathcal{L} + \mathcal{H}_0 \mathcal{A}_p(R, T) + U(R, T) \quad (4.32)
$$

and

$$
\frac{\partial \mathcal{A}_p}{\partial t} \bigg|_{\text{Col}} = \frac{1}{2} \left[ \mathcal{A}_p \mathcal{L}, \mathcal{A}_p(R, T) \right] + \mathcal{U}(R, T) \quad (4.33)
$$

Equation (4.33) has the form of the Landau-Silin equation of Fermi-liquid theory with the addition of the off-energy-shell term $i[\mathcal{L}, \mathcal{U}]$. We discuss this term below.

In Born collision approximation it is easy to write explicit expressions for the quantities in these equations. We find

$$
\mathcal{H}_0(p, R, T) = \int \frac{dp'}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \left\{ V(0) \mathcal{L} \mathcal{U}(p', \omega', R, T) \right\} + \eta V(|p - p'|) \mathcal{A}_p(p, \omega, R, T) \bigg|_{\text{Col}} = \int \frac{dp'}{(2\pi)^3} \left\{ V(0) \mathcal{L} \mathcal{U}(p, R, T) + \eta V(|p - p'|) \mathcal{A}_p(p, R, T) \right\} , \quad (4.34)
$$

One result of the approximation of equation (4.30) is that $\mathcal{A}$ (Eq. (4.21)) is then proportional to $\mathcal{L}$ and so the last term on the left of equation (4.25) drops out. It might be interesting to investigate the physical significance of this term when a less restrictive approximation is made.
and

\[ \Sigma^e(p_1, \omega_1 = \bar{\epsilon}_1, R, T) = \frac{1}{(2\pi)^6} \int dp_x dp_y dp_z \int d\omega_x d\omega_y d\omega_z \delta(p_1 + p_2 - p_y - p_x) \times \]

\[ \times \delta(\omega_1 + \omega_2 - \omega_y - \omega_z) \left\{ [V(|p_1 - p_y|)]^2 \frac{g^e(p_y, \omega_y, R, T)}{ \text{Tr} [g^e(p_2, \omega_2, R, T) g^e(p_x, \omega_x, R, T)]} \right\} \]

\[ + \eta V(|p_1 - p_y|) V(|p_1 - p_x|) \left[ g^e(p_y, \omega_y, R, T) g^e(p_x, \omega_x, R, T) \right] \]

\[ = \frac{1}{(2\pi)^6} \int dp_x dp_y dp_z \delta(p_1 + p_2 - p_y - p_x) \delta(\bar{\epsilon}_1 - \bar{\epsilon}_2) \]

\[ \times [(V(|p_1 - p_y|)]^2 \tilde{g}_1^e \text{Tr} [\tilde{g}_2^e \tilde{g}_2^e] + \eta V(|p_1 - p_y|) V(|p_1 - p_x|) \tilde{g}_1^e \tilde{g}_2^e \tilde{g}_2^e \] \quad (4.35)

where

\[ \tilde{g}_1^e = g_1 \]

\[ \tilde{g}_2^e = \frac{g_2}{L} = \frac{\eta g_2}{L} \] \quad (4.36)

From the definition of \( \xi \) in equation (4.18) we find, in Born approximation,

\[ \xi(p_1, \omega_1 = \bar{\epsilon}_1, R, T) = \frac{1}{(2\pi)^6} \int dp_x dp_y dp_z \delta(p_1 + p_2 - p_y - p_x) \]

\[ \times \left\{ [V(|p_1 - p_y|)]^2 \frac{1}{\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_y - \bar{\epsilon}_x} \right\} \times \]

\[ \times \left[ V(|p_1 - p_y|) \right] V(|p_1 - p_x|) \left[ \tilde{g}_1^e \tilde{g}_2^e \tilde{g}_2^e \right] \] \quad (4.37)

With these results we arrive at the kinetic equation quoted in equations (2.3) and (2.6) and in agreement with the equation of Silin.

The off-energy-shell term \( i [\xi, g_2] \) has been discussed before [15-17]. The previous discussions were not meant to apply to the degenerate limit. Levy and Ruckenstein [16] claimed that \( \xi \) was cancelled out by a term second order in the \( \epsilon \)-matrix in \( \Sigma_{BH} \). This argument is valid only in the non-degenerate limit. In such a Boltzmann limit, we have

\[ [\xi(1), g_1] = \eta \frac{1}{(2\pi)^6} \int dp_x dp_y dp_z \delta(p_1 + p_2 - p_y - p_x) \times \]

\[ \times \frac{1}{\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_y - \bar{\epsilon}_x} \left[ \text{V}(|p_1 - p_y|) \text{V}(|p_1 - p_x|) \tilde{g}_2^e \tilde{g}_2^e \tilde{g}_2^e \right] \] \quad (4.38)

so that

\[ [\Sigma_{BH}(1) + \xi(1), g_1] = \frac{\eta}{(2\pi)^6} \int dp_x \left[ \text{V}(|p_1 - p_2|) + \right] \]

\[ + \int \frac{dq}{(2\pi)^3} P \left[ \frac{m^*}{(p_1 - p_2)^2 - q^2} \right] \text{V} \left( \left| \frac{p_1 - p_2}{2} - q \right| \right) \text{V} \left( \left| q - \frac{p_1 - p_2}{2} \right| \right) \] \quad (4.39)

The \( T \)-matrix for two particles scattering in vacuum is given by

\[ \langle k_f | t | k_i \rangle = \text{V}(|k_f - k_i|) + m^* \int \frac{dq}{(2\pi)^3} \frac{V(|k_f - q|)}{k_f^2 - q^2 + i\epsilon} \] \quad (4.40)

Laloë has argued [17] that if all higher order terms were considered equation (4.39) would sum simply to

\[ [\Sigma_{BH}(1) + \xi(1), g_1] = \eta \int \frac{dp_x}{(2\pi)^6} \text{Re} \left( \frac{p_1 - p_2}{2} \right) \frac{1}{t} \frac{p_2 - p_1}{2} \right) [g_2, g_1] \] \quad (4.41)
so that $\xi$ (or $I_2$) is gathered harmlessly into the spin-rotation term.

However, the degenerate final state factors in equation (4.37) invalidate such an argument and, moreover, the generalization beyond Born approximation in the degenerate limit would replace $V$'s by many-body $T$ matrices which depend on the non-equilibrium Green's functions and thereby the distribution functions, in contrast to the vacuum $T$-matrix of equation (4.40). We recall that we have neglected first order derivative terms in deriving equations (4.25) and (4.26) and the kinetic equation (4.31). We are unable to say what effect the inclusion of these terms would have on $I_2$. So we feel the question of $I_2$ or $\xi$ is unresolved. We should point out that Miyake [26] has carried out an analysis of a dilute Fermi system, based on the renormalization procedure of Abrikosov and Dzaloshinskii [27], that finds a non-zero off-energy-shell term in spin-rotation.

5. Conclusion.

We have derived a kinetic equation valid in the degenerate regime and equivalent to that of Silin. Our Born approximation result should be valid for dilute solutions of $^3$He in liquid $^4$He. For other systems it is possible to think of $V(p)$ as an effective potential rather than a bare one or as an approximation to replace $V(p)$ by a vacuum $T$-matrix to treat systems interacting \emph{via} a potential having no Fourier transform. We believe the form of our result will not change fundamentally.

The solution of the kinetic equation by a variational method, valid at arbitrary polarization leads to both longitudinal and transverse relaxation times and diffusion constants. In highly polarized and degenerate Fermi systems these quantities will not be the same for the longitudinal and transverse cases. We have shown how the transverse relaxation time can be much smaller than the longitudinal one resulting in a smaller diffusion constant and a smaller spin-rotation parameter. These results may explain previously anomalous spin-echo experiments.

Several items remain for future work. We must reduce our analytic formulas, for $\tau_\perp$, for example, to numerical values to provide comparison with experiment. Several aspects of the derivation of the kinetic equation, including questions of the use of a more accurate spectral density and the nature of $I_2$, remain to be investigated more fully.

Acknowledgments.

We would like to thank Dr. F. Laloë for many useful conversations, and Dr. L. Mayants for translating a portion of Silin's book from the Russian. W. J. Mullin would like to acknowledge the hospitality of Ecole Normale Supérieure where a portion of this work was done.

References


[26] MIYAKE, K., private communication.