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Critical transport and failure in continuum crack percolation

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Résumé. — On étudie les propriétés de transport et de rupture d'un nouveau modèle de percolation continue (fromage bleu) dans lequel le milieu qui supporte le transport est l'espace situé entre des fissures placées aléatoirement. Le comportement critique de la conductivité électrique, de la perméabilité, des constantes élastiques et des seuils de rupture sont, pour la plupart d'entre eux, différents de ceux obtenus en percolation sur réseau discret et avec le modèle de percolation continue du type gruyère. De plus, on suggère que l'assymétrie de la réponse élastique d'une fissure sous contrainte de compression ou d'extension conduit, pour les propriétés élastiques macroscopiques, à un nouveau seuil de percolation pour une concentration de fissures plus grande que celle donnant la percolation géométrique usuelle. Cela est dû à la présence de configurations en crochets ou déferlements dans le squelette de percolation qui transforme localement la contrainte d'extension macroscopique en une contrainte de compression. En dimension deux, on propose une analogie avec la percolation dirigée. Quand les fissures ont une largeur finie, on retrouve le seuil de percolation géométrique habituel et l'analyse des propriétés de transport est analogue à celle développée par Halperin et al. pour le modèle de gruyère. Pour une plaque mince élastique non contrainte dans un plan, les deux lèvres d'une fissure peuvent localement se recouvrir : cela conduit à l'existence de deux modes de déformation supplémentaires (ouverture et flambage) qui donnent des comportements critiques distincts. On analyse le scénario de rupture fragile de type Griffith dans lequel les fissures s'agrandissent sous l'action de contraintes d'extension. Un nouveau comportement critique est annoncé.

Abstract. — Critical transport and failure properties of a new class of continuum percolation systems (blue cheese model), where the transport medium is the space between randomly placed clefts, are discussed. The critical behaviour of electrical conductivity, fluid permeability, elastic constants and failure thresholds are, for most of them, distinct from their counterparts in both the discrete-lattice and Swiss-cheese continuum percolation models. Furthermore, it is argued that the asymmetric elastic response of a crack submitted to compression or extension leads, for the macroscopic mechanical properties, to a new percolation threshold at a crack concentration higher than that of usual geometric percolation. This is due to the presence of hooks or overhang crack configurations which locally transform the macroscopic applied extensional stress into a compressional stress. In two dimensions, an analogy with directed percolation is suggested. When the cracks have a non-vanishing width, one recovers the usual geometric percolation threshold and the analysis of the transport properties is similar to that of the Swiss cheese model developed by Halperin et al. For elastic sheets which are not constrained to lie in a flat plane, the two edges of a crack may overlap : this leads to the existence of two additional elastic deformation modes (crack opening and buckling) which exhibit distinct critical behaviours. The brittle Griffith failure scenario which corresponds to the growth of a crack is analysed and introduces new critical failure exponents.

1. Introduction.

Recently, it was realized that the critical behaviour of electrical conductivity, fluid permeability, elasticity modulus [1] and failure threshold [2] was different in discrete-lattice percolation and in a class of continuous percolation models (Swiss-cheese), pointing out the existence of at least two universality classes of transport phenomena in percolation. This is due to the existence of fluctuations [3] in the micro-bond strengths in the continuous percolation system.

The purpose of this paper is to introduce a third class, the continuous blue-cheese model where the transport medium is the space between randomly placed clefts with small or even vanishing thickness (and not spherical holes as in the Swiss-cheese model). The critical behaviour of the transport quantities and failure thresholds are found, for most of them, to be distinct from their counterparts in...
both the discrete-lattice and Swiss-cheese continuum percolation models (see Tab. I which summarizes the results). This stems both from the existence of micro-bond fluctuations (as occurs in the Swiss-cheese model) and from the peculiar topology of the singly-connected bonds of the blue cheese model (see below).

The relevance of the blue-cheese model is suggested from the crack structure of many mechanical [4] (damaged crystals, solids, ceramics, rocks...) and natural systems [5] (arrays of faults in geology in relation to oil recovery, Geothermics and earthquakes [6]...) which often consists in random arrays of micro-cracks of vanishing thicknesses. Percolation has qualified as a powerful model of disordered media for deriving simple and powerful concepts: in a first step, it is natural to simplify the difficult problem of the analysis of the transport properties of a medium deteriorated by cracks in the large disorder limit by studying its corresponding percolation version. This is my justification for introducing the blue cheese model. As in other percolation models, its simplicity permits to explore geometrical, transport and rupture properties which appear either universal or dependent upon specific details of the systems (see below).

The blue-cheese model is defined as follows. Empty rectangle holes of length \( l_e \) in two dimensions or disk-like holes of diameter \( l_e \) in three dimensions with thickness \( a \) are randomly distributed in a uniform electric or elastic medium. As the number \( N \) of cracks per unit surface (in 2d) or per unit volume (in 3d) increases, a percolation threshold \( N_c \) is reached corresponding to the geometrical deconnexion of the sample in multiple fragments. In three dimensions, the cracks form a percolating structure at a first threshold \( N_1 \) strictly smaller than \( N_c \), before deconnecting the sample in pieces. We are only interested in the regime \( N > N_1 \).

Percolation involving anisotropic objects has been studied [7-10] numerically and theoretically and I will make use of the known geometrical facts for the analysis of transport and failure properties. Note however that, in the blue cheese model, the transport medium is the space left between the randomly placed empty clefts whereas for previous studies of transport properties in overlapping figure stick (or rectangle) percolation models [7], transport is carried on the percolating structure of sticks or disks themselves. This is the main difference between this model and previous studies. It has important consequences for the critical electrical and mechanical properties which are completely different.

One important geometrical result of previous studies is that \( N_c \) is known to decrease as the crack length \( l_c \) increases according to \( N_c \propto l_c^{d} \) [7, 10], resulting in the existence of a quasi-invariant \( N_c l_c^d \approx 5.7 [7-10] \) independent of \( l_c \). The intuitive idea is that each crack defines a sort of excluded volume of the order of \( l_c^d \) around it: then the critical percolation threshold \( N_c \) corresponds to the presence of a constant average number of cracks in each

Table I. — Comparison between Swiss-cheese and blue-cheese continuum transport and failure percolation exponents referred to the corresponding exponents on the discrete-lattice, \( a \) is the width of the cracks in the blue cheese model. \( \neq 0 \) \( \Rightarrow \) case of cracks with a finite width, \( a = 0 \) \( \Rightarrow \) case of cracks with vanishing thickness, \( a^* = 0 \) \( \Rightarrow \) case of cracks with vanishing thickness but whose leaves can overlap in two dimensions, \( \approx \) \( \Rightarrow \) buckling deformation mode, \( G \) \( \Rightarrow \) Griffith mechanism of rupture. The exponents \( \ast \) and \( \ast \) refer respectively to the energy failure criterion (first criterion of [2b]) and to the voltage or strain failure criterion (second criterion in [2b]).

<table>
<thead>
<tr>
<th>( d )</th>
<th>Conductivity</th>
<th>Permeability</th>
<th>Electric failure</th>
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<td>3/4, 3/2</td>
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(1) Reference [1].
(2) Reference [2].
(3) This work.
typical volume $l_c^d$. This result allows the definition of *singly-connected* micro-bonds to be justified and the electrical and mechanical properties of each micro-bond to be treated separately (see below).

The geometrical critical exponents such as the correlation length exponent $\nu$ are believed to be the same as for isotropic objects and discrete systems. Previous studies of transport properties have been essentially concerned with percolation systems where the transport medium is the connected set of anisotropic objects. For example, in reference [11] the conductivity and elasticity of percolating carbon fibers are studied. In this case, transport is controlled by the percolation structure of the carbon fiber set. Note that in contrast to these studies, I focus here on the connected set complementary to the crack clusters.

A typical sample structure just below $N_c$ is given in figure 1 for $d = 2$. One observes large undeteriorated regions (of size of the order of $l_c$) separated by narrow necks which have the generic topology shown in figure 2. Using the construction shown in figure 2, one can define a mapping of the continuous crack model onto a discrete percolation model in a way similar to the mapping of the Swiss cheese model to a discrete model [13]. In $d = 2$, with each two-crack configuration, one can associate a straight path as drawn in figure 2. Each such path defines a micro-bond in a corresponding percolation lattice model. These necks may sometimes be separated by plaquettes having a polygonal shape of size $\approx l_c$. A set of such micro-bonds connecting the upper side to the lower side of the page is shown in heavy lines in figure 3a. Each narrow neck i.e. equivalent micro-bond such as that in figure 2 is schematized by the sign « = »: this allows us to draw the equivalent discrete lattice and identify the corresponding connectivity shown in figure 3b.

Fig. 1. — A typical random crack structure before geometric disconnection in two dimensions obtained by a computer with the algorithm described in the text (courtesy of C. Vanneste).

Fig. 2. — Illustration of a generalized Voronoi tesselation for a micro-bond configuration in the blue cheese model: the dashed line schematizes the equivalent discrete-lattice micro-bond.

The generalization of this mapping in three dimensions involves planes and is straightforward. Figure 4 shows the corresponding generic neck which involves a three-crack configuration. As in the Swiss cheese model, a micro-bond is determined from the relative position of $d$ neighbouring objects where $d$ is the space dimension (empty holes in the Swiss cheese and empty cracks in the blue cheese) which almost completely intersect each other. Therefore, two linear cracks in $d = 2$ suffice to define a micro-bond as shown in figure 2 whereas three intersecting cracks must be used to construct a generic micro-bond in $d = 3$. Generically, two cracks $(1)$ and $(2)$ intersect with a finite angle. Let us note $P_{12}$ the line formed by the intersection of $(1)$ and $(2)$. A third crack $(3)$ with a random orientation will intersect both $(1)$ and $(2)$ along lines called $P_{13}$ and $P_{23}$ which

Fig. 3. — A small portion of the system shown in figure 1: the heavy line serves as a guide to the eye and shows the connecting paths. The corresponding schematic representation of the discrete lattice structure equivalent to the continuum crack percolation structure is shown in b: the symbol « = » indicates the presence of weak micro-bonds with a topology represented in figure 2.

Fig. 4. — Typical topology of weak micro-bonds in three dimensions. The width $a$ of the cracks has been taken zero.
form finite non-vanishing angles with respect to the first line \( P_{12} \). This result justifies that the generic configuration is of the form depicted in figure 4.

The continuous to discrete mapping allows us to conclude that the blue cheese model has the typical Nodes-Links-Blobs geometrical structure, well known in discrete percolation models \([12, 13]\): in a system of size \( 3 \gg \xi \), it is a hypercube of size \( 3^d \), with nodes separated typically by the percolation correlation length \( \xi \sim (N_c - N)^{-\nu} \), and the nodes are connected by macro-links (there are of course multi-connected blobs decorating the macro-links).

Each macro-link can be viewed as consisting in several sequences of singly connected micro-bonds of total number \( L_1 \approx (N_c - N)^{-d} \) with \( d = 1 \) \([14]\), in series with blobs corresponding to multiple paths connecting the same nodes as depicted in figure 3. Such a continuous to discrete mapping has been used for discussing the transport and failure properties of the continuous Swiss cheese model \([1, 2]\) and is also important in the blue cheese model in order to use the known critical geometrical properties of discrete-lattice percolation.

The difference between the transport and failure properties of the discrete-lattice and the blue cheese percolation models now relies, as in the Swiss-cheese model, on the form of the continuous distribution \( p(\delta) \) of micro-bond strengths (controlled by the bottleneck width \( \delta \) shown in Figs. 2 and 4). The essential ingredient of the analysis is to recognize that the continuous probability distribution \( p(\delta) \) approaches a non-zero limit \( p(0) \) for \( \delta \to 0^+ \). This result can be intuitively visualized as follows. Consider a plausible algorithm for constructing a blue cheese structure in two dimensions (the three dimensional case is similar): one first chooses at random the positions of the crack centers with a given density \( N \). Then, the orientation of each crack must be chosen randomly in the interval \([0, \pi]\). Since the crack centers and their orientation are distributed uniformly, the value \( \delta = 0 \) is not a special point and \( p(\delta) \) is finite. This implies that, in the random blue-cheese percolation model, very small \( \delta \)s can be encountered. The typical minimum value \( \delta_{\text{min}} \) of \( \delta \) along a string of \( L_1 \) singly connected bonds is found by using a Griffith type of argument \([1]\) valid for intermittency in random fields \([3]\):

\[
\int_{0}^{a} p(\delta) \, d\delta
\]

is the probability that \( \delta < \alpha \) and the condition \( L_1 \int_{0}^{a} p(\delta) \, d\delta \leq 1 \) says that less than one event \( \delta < \alpha \) has occurred over \( L_1 \) trials. This allows us to identify

\[
\delta_{\text{min}} = L_1^{-1}.
\]

Due to the presence of these very narrow bonds, transport and failure properties depart from their discrete-lattice analog as we now discuss.

2. Electrical transport and failure.

2.1 ELECTRICAL CONDUCTIVITY. — The typical structure of a weak bond in two dimensions is shown in figure 2. We are interested in the conductance of each singly-connected bond which has the shape of a plaquette in 2d or the shape of a polygonal brick in 3d with a typical size \( l_c \). A generic micro-bond configuration is characterized by a finite angle between two neighbouring cracks. Since we are interested in the scaling with \( \delta \), it has the same conductance dependence as that of a configuration of two cracks perpendicular to each other with the same \( \delta \), which itself is equivalent to the problem of an electric current flowing perpendicular to two collinear cracks separated by a distance \( \delta \) in a plate of typical size \( l_c \) (see Fig. 5a). If \( I_0 \) is the total current flowing through the hole, \( j(r) = I_0/\pi r \) is the current density at large distances \( r \gg \delta \) to the hole. From

\[
E = \sigma^{-1} j,
\]

one gets the conductance of the hole

\[
g(\delta) \approx I_0 \left( \int_{\delta}^{l_c} E \, dr \right)^{-1} \approx \sigma / \log \left( l_c / \delta \right) \tag{2}
\]

which vanishes only logarithmically as \( l_c / \delta \to +\infty \), i.e. when the bond thickness \( \delta \) goes to zero. This result has already been reported in \([15]\). To estimate the macroscopic conductivity, we ignore (as in \([1]\)) the resistance of the blobs and approximate the conductance of a string of \( L_1 \) singly connected bonds by

\[
G^{-1} \approx \sum_{i=1}^{L_1} g_i^{-1}.
\]

This yields a lower bound for \( G^{-1} \), therefore an upperbound for \( G \) and a lower bound for the exponent \( t \) (see below). This formula relies on the string-like structure of the \( L_1 \) micro-bonds along a macro-link which allows the rules for conductance series association to be used. Using the fact that the distribution \( p(\delta) \) discussed in paragraph 1 does not vanish as \( \delta \to 0 \), one finds that the sum (3) is not dominated by the weakest bonds since a micro-bond resistance varies less than linearly (in fact only logarithmically) with \( \delta \). Therefore, the macroscopic
conductivity is related to the conductance of a macro-link of length $\xi$ by the classical transport relationship $L = g^2 - d G$. This yields

$$\sum \approx (N_e - N)^t$$ with $t = (d - 2) \nu + 1$

which is the usual lower bound estimate of the conductivity critical exponent whose values is $t \approx 1.3$ [1]. This shows that the blue cheese model is equivalent in 2d to the discrete lattice model or to the Swiss cheese model as far as the conductivity exponent is concerned.

In 3d, the generic structure of a micro-bond is shown in figure 4 (and Fig. 5b for the equivalent hole problem giving the scaling in terms of $\delta$). It is easily seen that the conductance of such a micro-bond is proportional to that of a volume of size $l_1$ almost disconnected in its middle apart from a narrow neck of width $\delta$, since, in order to go from figure 4 to figure 5b, one has only to rotate the micro-bond around the $P_12$ line. The conductance of the micro-bond shown in figure 4 therefore reads:

$$G(\delta) \approx I_0 \left( \int_{\delta_n}^{\delta_m} E \, dr \right)^{-1}.$$ Using $E \approx I_0/\sigma r^2$ from the conservation of the total electric current, one obtains

$$G(\delta) \approx \sigma \delta .$$ (4)

The conductance of a micro-link goes to zero linearly with the neck width $\delta$. The conductance of a macro-link of size $\xi$ is therefore obtained from equation (3) and is written

$$G^{-1} \approx L_1 \sigma^{-1} \int_{\delta_n}^{\delta_m} p(\delta) \delta^{-1} \, d\delta \approx L_1 \sigma^{-1} \log (l_1/\delta_m) .$$

Using $\sum \approx \xi^{2-d} G$, one finds that the macroscopic conductivity is given by

$$\sum \approx (N_e - N)\delta^{-d} \log (N_e - N) .$$ (5)

This is identical to the usual estimation of the discrete case up to a logarithmic correction. In 3d, $t \approx 1.9$ [1]. Contrary to the Swiss cheese model in 3d, the fluctuations of the micro-bond strengths do not lead to a different conductivity critical exponent but only to a logarithmic correction of the scaling. This stems from the different scaling (4) which must be compared to its analog $g(\delta) \approx \delta^{3/2}$ for the Swiss cheese model [1].

2.2 ELECTRICAL FAILURE. — In rupture phenomena, the weakest part of the system fails first and the effect of disorder is markedly more pronounced than in transport phenomena [15-18]. The failure threshold is dominated by extreme fluctuations in contrast to transport and elastic coefficients which are related typically to the second moment of the distributions. This leads to failure exponents which are larger in the Swiss cheese continuum percolation model than in the discrete-lattice model [2]. Now, we briefly discuss this effect on electrical failure in the context of the blue cheese model.

2.2.1 Weakest bond failure threshold. — Consider a system of size $3 \gg \xi$. If a current is injected from one side of the hypercube $3^3$, the number of current carrying paths is $(3/\xi)^d$. However, in each path, there are $(3/\xi)$ macro-links and each macro-link has $L_1$ singly connected micro-bonds. Therefore, the number of current carrying singly-connected micro-bonds is of the order of $R \sim L_1(3/\xi)^d$. The width $\delta_m$ is the minimum of all neck widths out of the total number $R \sim L_1(3/\xi)^d$ in the whole system of size 3 and scales as

$$\delta_m \approx \delta \sim L_1(3/\xi)^d .$$

When the system is submitted to a given current density $j$ (defined for the undeteriorated system i.e. $j = I_0$ is fixed corresponding to a total imposed current $I_0$), the current flowing in each macro-link is of order [15-17]

$$I \approx j \xi^{(d-1)}$$ (6)

and the Ohmic loss in it is written [2b]

$$O \approx g_{\text{min}}^{-1} I^2$$ (7)

where $g_{\text{min}}$ is obtained from $\delta_m^d$ with equation (2) in $d = 2$ and equation (4) in $d = 3$. Assuming that rupture occurs if $O$ becomes larger than a critical value $O_c$ (first criterion of Ref. [2b]) leads to a current density threshold $j_r$:

$$j_r^{(1)} \approx (N_e - N)^{\nu} \left( -d(d-2)/2 + (d-1) \right) + (d-2)/2 \propto \lambda^{-d(d-2)/2} \text{ for criterion i).}$$ (8)

If rupture occurs following criterion ii) i.e. when the voltage drop over two ends of a micro-bond reaches a critical value $V_c$, one obtains the same type of expression as equation (8):

$$j_r^{(2)} \approx (N_e - N)^{\nu} \left( -d(d-2)/2 + (d-1) \right) + (d-2)/2 \propto \lambda^{-d(d-2)/2} \text{ for criterion ii).}$$

This leads to a scaling form of the first bond rupture threshold : $j_r^{(1)} \approx (N_e - N)^{\nu}$ with

$$E_c = \nu \left( -d(d-2)/2 + (d-1) \right) + (d-2)/2$$

for criterion i. It is important to realize that these scaling laws are valid as long as the system size...
J remains much larger than the percolation length $\xi$.

Will the first micro-bond failure lead to a macroscopic failure? In reference [18], it is argued that the first microscopic failure threshold could in general be distinct from the macroscopic failure threshold for random fuse networks away from the percolation critical point. In the vicinity of $N_c$, the same can hold true depending on the cases as we now discuss.

Using the nodes-links-blobs picture of the percolation structure, one can estimate the failure threshold $j_r^{(2)}$ of the second weakest bond of the total system and compare it with $j_r^{(1)}$. The following argument is developed for the case of a constant applied current. Once the first bond has failed, only $(3/\xi)^d - 1$ macro-links remain and carry the total current flow $j_0^{d-1}$. Therefore, each macro-bond will carry a slightly larger than that before the first rupture by a factor equal to $\left(1 - (\xi/3)^d\right)^{-1}$ (at the basis of this factor is the hypothesis that the current increase due to the breaking of one macro-link is shared homogeneously between all remaining macro-links: this is therefore a kind of mean field argument which neglects possible important local fluctuations). In competition with this increased current per macro-link, is the fact that the second weakest bond is stronger than the weakest one by a factor $\delta_{\text{min}}^{(2)}/\delta_{\text{min}}^{(1)} = \left(1 - (\xi/3)^d\right)^{-1}$. Since the total number of current carrying singly-connected bonds has decreased by the amount $\sim L_1$, now, let us assume that the micro-bond conductance is related to its width $\delta$ by a powerlaw $g(\delta) = \delta^\beta$. Then, one finds that

$$j_r^{(2)} = j_r^{(1)} \left(1 - (\xi/3)^d\right)^{-1}$$

for criterion $i$,

$$j_r^{(2)} = j_r^{(1)} \left(1 - (\xi/3)^d\right)^{-1}$$

for criterion $ii$.

(9)

If $\beta \leq 2$ for criterion $i$ and $\beta \leq 1$ for criterion $ii$, $j_r^{(2)} \ll j_r^{(1)}$ which implies that the second weakest bond will in general fail immediately after the first one. Repeating the argument leads to a macroscopic failure as soon as the weakest bond fails. Equations (2) and (4) show that the blue cheese model belongs to this class. Therefore, macroscopic failure in the blue cheese model is given by equations (8) and (9). In the other cases ($\beta > 2$ for criterion $i$ and $\beta > 1$ for criterion $ii$), $j_r^{(2)} = j_r^{(1)}$ and one observes a succession of increasing failure thresholds ending eventually as a macroscopic failure. Note that for the other boundary condition of a given applied voltage, the results would be drastically changed since a link failure involves the change of the global conductivity and in general reduces the total current. This leads to a distinction between catastrophic failure at imposed current and controlled failure at imposed voltage.

2.2.2 Macroscopic failure threshold. — An upper-bound for the macroscopic failure can be obtained with the current density $j_r$ necessary for all the macro-links to fail. Each macro-link will fail by its weakest singly-connected micro-bond whose width $\delta_{\text{min}}$ scales as $L_1^{-1}$ (see Eq. (1)), which shows that the scaling laws of paragraph 2.2.1 are modified by the absence of the factor $(3/\xi)^d - 1$ in $\delta$. This leads to

$$j_r \approx (N_c - N)(\log |N_c - N|)^{1/2} \quad d = 2$$

$$j_r \approx (N_c - N)^{2} \quad d = 3$$

(10) (11)

with $E_c = 2 \nu + 1/2$ showing a correction equal to $1/2$ to the discrete case $E_c = 2 \nu$ in $d = 3$. If rupture occurs as the voltage drop over the bond becomes larger than a critical value $V_c$ (second criterion of Ref. [2b], one has $j_r \approx (N_c - N)^{\nu}/(\log |N_c - N|)$ in $d = 2$ and $j_r \approx (N_c - N)^{2}\nu$ for $d = 3$ with $E_c = 2 \nu + 1$. These results are summarized in the table I for comparison with those obtained for the discrete-lattice and Swiss cheese models.

2.2.3 Probability distribution of failure thresholds. In addition to leading to a different critical failure exponent (as shown above in paragraphs 2.2.1 and 2.2.2), the existence of bond strength fluctuations also yields a broad distribution of failure thresholds. This effect is discussed in [19] for the Swiss cheese model. A distribution of rupture thresholds stems from the existence of fluctuations 1) of the mesh size $L$ of the macro network around the typical size $\xi$ and 2) of the individual strength $\delta$ of the singly connected micro-bonds along a macro-link, resulting in large current fluctuations. The form of the failure probability distribution for continuum blue cheese percolation model in the critical region can be derived using the method of reference [19] which rests on the knowledge of the probability distributions of the extremes of the fluctuating variables $L$ and $\delta$ and on the expression of the failure current density $j_r$ for a bond of width $\delta$ in a mesh of size $L$ (obtained from Eq. (2) in $d = 2$ and Eq. (4) in $d = 3$).

In two dimensions, $j_r$ has a logarithmic dependence with respect to $\delta$. The corresponding failure distribution is of the form of the exponential of an exponential:

$$P(j) \approx \exp\left(-c_3d\exp(-c'j^{-1}/\xi)\right)$$

(12)

where $c$ and $c'$ are two constants.

In three dimensions, since the failure threshold scales as a power of $\delta_{\text{min}}$, the failure distribution is controlled by the largest fluctuation field $\delta$ and takes a Weibull form

$$P(j) \approx \exp(-p(0)L_1(\delta/\xi)^{d-1}\xi^{(d-1)/a}j^{1/a})$$

(13)
This is similar to the results already discussed for the Swiss cheese model [19].

3. Critical permeability.

Permeability also exhibits critical behaviour. However, in this paper we study the permeability, not of the crack structures, as would seem natural for example in cracks occurring in geological situations, but of the complementary medium. Therefore one must envision that the cracks are barriers to the flow of a fluid as in a labyrinth structure made of thin long impenetrable walls placed randomly in the transport medium. This system can serve as a model of certain loosely packed rock structures in which impermeable rocks have very anisotropic shapes.

The fluid permeability \( k \) is defined as the flux-rate of fluid flow through the space between the random barriers under a unit macroscopic pressure gradient.

The permeability of a micro-bond of diameter \( \delta \) is proportional to the corresponding conductivity \( g(\delta) \) multiplied by its cross-section \( S = S_\delta - 1 \). Therefore, in 2d, \( k(\delta) = \delta / \log (l_c / \delta) \). The macroscopic permeability is found to vanish as \( N \rightarrow N_c \) according to

\[
k^{-1}_{\text{macro}} \approx (N_c - N)^{-t} \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} p(\delta) \log (l_c / \delta) \delta^{-1} d\delta
\]

leading to

\[
k_{\text{macro}} \approx (N_c - N)^e / \{ \log (N_c - N) \}^2 \quad (14)
\]

where \( e = t = 1.3 \) is equal to the permeability exponent of the discrete-lattice case. In 2d, the blue cheese permeability exhibits only logarithmic corrections with respect to the discrete case.

In 3d, \( k(\delta) = \delta^3 \). This yields \( k^{-1}_{\text{macro}} \approx (N_c - N)^e \) with \( e = e + 2 \) giving a correction to the discrete case equal to 2 (see Tab. I for comparison with the Swiss cheese model).

4. Mechanical properties.

4.1 Elastic properties.

4.1.1 Directed percolation behaviour. — The mechanical properties of system deteriorated by cracks are strikingly different from those of systems weakened by empty holes as in the Swiss cheese model [1].

First, three distinct deformation modes may be defined [27]. Mode I corresponds to an elongation, mode II to a shear and mode III to a flexure and twisting in beams. The analogy is loose since a crack submitted to a parallel shear will exhibit both deformation mode I at its tips and mode II on its sides. In the following, we will only consider deformation mode I and leave the study of the role of modes II and III for future work. These modes are important and should be considered in geological situations since the opening mode I is unlikely to occur on a large scale in the earth because of the large compressional stresses created by gravity.

A simple method of stress analysis in elastic solids with many random cracks has recently been proposed based on a superposition technique and the ideas of self-consistency applied on the average tractions on individual cracks [20]. This method is essentially a mean field theory and gives reasonable results far from the critical percolation point. On the contrary, this paper focuses on the very high disorder limit modeled by the percolation threshold. In this case, another type of analysis is called for and is described below.

An essential feature of crack percolation is the existence of an asymmetry of the response of individual cracks submitted to an externally applied stress. If the stress is tensile, the crack cavity opens up and takes a finite volume leading to a significant softening of the elastic behaviour of the system. If the stress is compressive, the two edges of the cracks are pressed close together and the system responds as in the absence of cracks. This can be summarized by a very asymmetric (stress-strain) characteristic as depicted in figure 6.

![Schematic form of the stress-strain characteristic of a single crack in an homogeneous medium which outlines the large asymmetry of its response with respect to compression or tension.](image)

This feature leads to new behaviours in the macroscopic elasticity of system deteriorated by cracks. Under compression, the system responds as if undeteriorated and no critical scaling behaviour is expected. Under a macroscopic tensile stress, one could think that the elastic constant should vanish at the geometrical percolation threshold \( N_c \). However, this is not true [21], since not all the cracks will be submitted to a local tensile stress but some of them will feel locally a compressional stress. An example of a configuration where this effect occurs is presented in figure 7. This leads to the curious fact that even at the geometrical percolation threshold where a connected path of cracks cuts the system in several pieces, the system still resists against a macroscopic tensile stress. This is due to the existence of hook-like or overhang configurations schematized in fig-
Fig. 7. — Typical hook or overhang configuration in a percolating path. The part of the crack presenting a compressional stress is drawn in heavy line.

"ture 7: the portion of the backbone submitted to a local compression is depicted in heavy line. Therefore, the geometrical percolation threshold \( N_c \) is not critical for the tensile elastic properties of the blue cheese model. However, as the crack density increases above \( N_c \), the typical distance between \textit{hooks} increases until one reaches a new threshold \( N_c^D \) when a set of cracks appears which deconnects the sample into at least two pieces and whose structure does not contain any \textit{hook} or overhang configurations. In the vicinity of this threshold, one expects a critical behaviour for the extensional elastic constant which should vanish as

\[
E_{\text{ext}} \approx (N_c^D - N)^{\nu_1}.
\]

The general description of this new threshold and the corresponding scaling laws is difficult in three dimensions especially since it is related to the physics of oriented random surfaces [22].

In two dimensions however, it is easy to see that \( N_c^D \) corresponds to the \textit{directed percolation} threshold [23]. To prove this assertion, let us define arbitrarily an oriented line \( Ox \) as one of the two directions (left to right or right to left) along a line perpendicular to the macroscopic tensile stress as shown in figure 8. To be concrete, let us assume a vertical tensile stress and choose \( Ox \) from left to right. Then draw on each crack an arrow having a projection on \( Ox \) in the sense of the chosen direction. Now, starting from the left, try to find a path going from left to right without overhangs or hooks. This problem is in close analogy with a continuous version of the directed percolation model. In this model, one starts from ordinary bond percolation on a discrete lattice and introduces a direction to the problem by drawing arrows on the bonds which follow the main direction. The question is then to find a connecting path which follows the arrows [23]. Therefore, at the directed percolation threshold \( N = N_c^{DP} \), there exists a connected set of cracks without hooks or overhangs. Under tension, this set will deconnect the system in two without any elastic resistance. This shows that \( N_c^D \) is equal to the directed percolation threshold \( N_c^{DP} \). For a pure shear, the mechanical threshold should also be given by the directed percolation threshold: one must now choose the \( Ox \) axis making an angle \( \pi/4 \) with respect to the shear. This does not change the percolation threshold due to the isotropy of the random crack structure.

One can now use the properties which are known about directed percolation for predicting its elastic properties. The directed percolation threshold \( N_c^{DP} \) is known to be larger than the usual percolation threshold: for example in the discrete square bond lattice, the directed percolation threshold fraction is around 0.64 compared to the geometrical threshold occurring at a fraction of occupied bonds equal to 0.5 [21]. Directed percolation is characterized by two correlation lengths \( \xi_1 \) and \( \xi_\perp \), which are the typical distance between overhangs in the direction \( Ox \) and in the direction perpendicular to \( Ox \), respectively. They diverge as \( N \to N_c^{DP}^- \) as

\[
\xi_1 \approx (N_c^{DP} - N)^{\nu_1},
\]

\[
\xi_\perp \approx (N_c^{DP} - N)^{\nu_\perp}.
\]

In \( d = 2 \), \( \nu_1 = 1.7334 \pm 0.0001 \) and \( \nu_\perp = 1.0972 \pm 0.0001 \) [21, 23] which should be compared to \( \nu = 4/3 \). These values which are determined for discrete systems should remain valid in continuous directed percolation due to universality in the geometrical scaling.

Within the directed percolation analogy, one can estimate the critical behaviour of the macroscopic elastic constant with an argument of the type \textit{node-link-blob} adapted to the directed percolation model. Below the directed percolation threshold and from the definition of \( \xi_1 \) and \( \xi_\perp \), there is one \textit{hook} or overhang configuration per area \( \xi_1 \xi_\perp \). Forces are applied on each hook \textit{via} a typical arm of a lever \( \zeta \approx \xi_1 \). The corresponding macroscopic elastic modulus reads

\[
Y_m^{-1} \approx \zeta^2 \xi_1 / \xi_\perp
\]

leading to

\[
Y_m \approx \xi_\perp / \xi_1^3.
\]
Note that due to the existence of two length scales $\xi_I$ and $\xi_\perp$, the usual factor $6^{-(d-2)}$ relating the macro-link elastic constant to the macroscopic effective elastic modulus is changed into $6^{\xi_I/\xi_\perp}$ in two dimensions. One therefore obtains

$$Y_m \approx (N_c^\text{DP} - N)^p$$

with

$$\tau_D = 3 \nu_1 - \nu_\perp \approx 4.1 . \quad (19)$$

Equation (19) gives (as usual in this type of argument) a lower bound for $\tau_D$. It should be compared to the lower bound of $\approx 3.2$ for the elastic modulus exponent in usual vectorial percolation [4] whose values rather turn to be $\tau = 3.96 \pm 0.04$ [4].

Note that this argument can be used to compute a lower bound for the conductivity in a directed electrical problem such as in diode lattices: one obtains the directed electric exponent $t_D = \nu_1 - \nu_\perp$ and therefore $\tau_D = \tau_D + 2 \nu_1$ which is similar to the relation between $\tau$ and $t (\tau = t + 2 \nu)$ in ordinary percolation.

The scaling (19) may be modified outside a narrow critical region if the shear modulus $Y_{\text{shear}}$ is small compared to the compression/extension modulus $Y_{\text{com}}$. In the vicinity of an overhang, a large shear should appear and superimpose over the rotational deformation of the hook. This implies a subdominant scalar correction $\nu_1 - \nu_\perp$ to the scaling which should be important as long as the effect of the lever arm $\xi_\perp^2$ does not win over the ratio $Y_{\text{com}}/Y_{\text{shear}}$.

4.1.2 Usual percolation: case of cracks which have a finite thickness $a (\delta < a)$. — When the cracks have a finite thickness $a$, the preceding anisotropy of the deformation under compressional or extensional stresses disappears at least for sufficiently small deformations. The analysis of the elastic properties then follows closely that developed for the Swiss cheese model. In this case, the elastic percolation threshold is $N_c$ defined in paragraph 1. Under a macroscopic stress, the micro-bonds tend to accommodate the strains in the form of a bending deformation concentrated in the thin portion of width $a$ (see Fig. 9), which shows a magnified view of a neck between two cracks.

![Fig. 9. — Bending mode for cracks having a finite width $a$.](image)

The corresponding bending constant of a micro-bond in two dimensions is that of a plate shown in figure 10 and is written

$$\gamma = Y \varepsilon^3 \delta/a \quad \delta < a . \quad (20)$$

In equation (20) we assume that for $\delta < a$ the bending strain is supported by the weak part of the bond of thickness $\delta$. We can estimate the bending rigidity of the flawed system with the classical formula used in [1]

$$Y_m^{-1} \approx \varepsilon^{d-2} \varepsilon^2 L_1 \int_{\delta_{\text{min}}}^{\delta_{\text{max}}} p(\delta) \gamma(\delta) d\delta . \quad (21)$$

One only gets a logarithmic correction to the discrete result $Y_m \approx (N_c - N)^f$.

In three dimensions, the microscopic bond bending mechanism also controls the macroscopic elasticity when the crack widths $a$ are finite. The bending constant crosses over from $\gamma = Y \delta^4/a$ for $\delta < a$ to $\gamma = Y \delta^2 c_2$ for $\delta > a$. Therefore, the shear stress modulus is $f = f + 3$ with $f = dv + 1$ for $\delta_{\text{min}} < a$, i.e. sufficiently near the percolation threshold. If $\delta_{\text{min}} > a$, $f = f$. As $N$ increases towards $N_c$, one should first observe the discrete-lattice shear modulus exponent and finally cross over to $f = f + 3$ in a narrow critical region $(N_c - N \ll a/l_c)$.

4.1.3 Out of plane two dimensional elasticity ($a \ll \delta$). — In two dimensions, another regime in which the two edges of a crack can go out of the plane can be relevant depending on the experimental conditions. In the elastic plate is kept between two rigid plates, the elastic properties of the system are described in paragraph 4.1.1 or paragraph 4.1.2. However, if the plate can buckle in the third direction, different elastic properties appear. In this regime, two modes of elastic deformation (crack opening and buckling) can be distinguished and lead to distinct critical behaviours. Let us analyse them in turn and discuss their domain of validity.

Crack opening. — In two dimensions, in-plane deformations can only occur via the opening or overlapping of cracks or bending of the micro-bonds. In the case where the width $a$ of the cracks is very small (smaller than the smallest bond width $\delta$), the bending of a micro-bond involves a crack opening or overlapping by a given angle $\theta$ as shown in figure 11. Note that the ability for the crack edges to overlap suppresses the elastic asymmetry of the individual crack shown in figure 6.
Fig. 11. — Crack opening deformation mode in two dimensions when crack edges overlapping is allowed: figure 11a corresponds to the case when the horizontal crack is supposed to remain undeformed whereas figure 11b represents the case when both cracks are deformed.

If in the presence of an external stress, the horizontal crack of the T configuration of a micro-bond (see Fig. 2) did not bend as in figure 11a, the elastic energy $E$ stored in that bond would correspond to that of a disclination of angle aperture (or charge) $s = \theta$. We refer to reference [24] for the definition of the elastic energy and quote the result

$$E \approx Y L^2 \log (L/z) \theta^2$$

for a single disclination in a plate of size $L$ where $Y$ is the Young elastic modulus and $z$ is the solid mesh size. We take $L \approx l_c$ since strains are screened at scales larger than $l_c$ due to the presence of other cracks. Due to the independence of $E$ with respect to $\delta$, no correction to the discrete-lattice percolation case is expected. The reason is that, as $\delta \to 0$, micro-bonds do not weaken in this mode of deformation. However, usually there are no constraints on the deformation of the cracks and in particular, the horizontal crack of the T configuration does bend in general so as to minimize the elastic stress induced by the aperture $\theta$ (as shown in Fig. 11b). This corresponds to the existence of two disclinations of opposite apertures or charges $\pm \theta$ at a distance $\approx \delta$ from one another forming a dislocation of Burgers vector $b = \delta \theta$. In this case, the elastic energy reads [24]

$$E \approx Y \delta^2 \theta^2 \log l_c/z.$$  \hspace{1cm} (23)

The corresponding bending coefficient is $\gamma \approx Y \delta^2 \log l_c/z$. Knowing $\gamma (\delta)$, one can follow reference [1] and compute the macroscopic shear modulus $Y_m \approx \xi^{2-d} K$ with the expression

$$K^{-1} \approx Y^{-1} \xi^2 L_1 \int_{\delta_{\min}}^{\delta_{\max}} p(\delta) \delta^{-2} d\delta.$$  \hspace{1cm} (24)

This gives $Y_m \approx (N_c - N)^f$ with $f = f + 1$ where $f = dv + 1 = 3.7$ is the shear modulus exponent of the discrete case. In this deformation mode, the fluctuations of the micro-bond strengths lead to an important correction for the macroscopic elastic exponent compared to the discrete-lattice case.

Buckling. — If the macroscopic 2d-plate is not rigidly clamped, it can become preferable for the elastic plate not only to have the crack edges to overlap slightly but even to buckle out of the plane in order to relax the elastic stresses as in figure 12. In other words, the major part of the elastic energy is found in the bending and twisting deformation modes. If this is permitted, buckling will always allow the horizontal crack in the T configuration to relax to its optimum shape. One has therefore to analyse only the dislocation case: the in-plane strain will be efficiently screened by the buckling deformation of the plate due to the coupling between in-plane strain and transverse deformation [25]. This screening of the elastic stress by buckling has been estimated in [26]. The diverging expression $Y \delta^2 \log l_c/z$ is replaced by a curvature energy independent of the size $l_c$ of the form

$$E \approx Ye^3 \theta^2$$

leading to a bending coefficient $\gamma \approx Ye^3$ corresponding to a plate of thickness $e$. The independence of $\gamma$ with respect to $\delta$ shows that the buckling relaxes very efficiently the stresses which tend to accumulate in the weak bonds and, as a result, the weak bonds are no longer weak in the asymptotic critical regime $N \to N_c$. The shear modulus and failure exponent are therefore equal to their discrete counterparts.

Fig. 12. — Schematic representation of the buckling of a plate in the presence of a pair of oppositely charged disclinations forming a dislocation. We have used the analogy between figure 2 and figure 5a for replacing the two almost intersecting cracks problem by a two collinear cracks configuration. The buckling is such that the in-plane elastic strain is completely transformed in bending energy.

Note also that, if buckling is allowed, it will always occur as long as $\delta^2 e \log l_c/z \gg e^3$ i.e. for sufficiently thin plates. Otherwise there is a cross-over from a discrete exponent regime for $N_c - N$ not too small to the planar crack opening exponent regime.

4.2 MECHANICAL FAILURE.

4.2.1 Brittle Griffith failure ($d = 2$ and $d = 3$). — In a system containing cracks, brittle rupture following the Griffith mechanism [27] will occur under tensile stress (mode I [27]). This rupture scenario is expected to be prominent for cracks with vanishing thickness. In this failure mechanism, the concen-
tation of the stresses at the apex of a crack tends to open the two edges and lengthen the crack, thus leading to failure. This mechanism must be considered when the tips of the cracks are sharp and no screening due for example to the presence of rounding or of a cavity occurs at the tips of the cracks.

The Griffith criterion states that a crack growth is governed by a balance between the mechanical energy released and the fracture surface energy spent as the crack propagates. More precisely, a crack will grow if the virtual work of the stress field spent as the crack propagates. More precisely, a crack will grow if the virtual work of the stress field

\[ \sigma \approx (p - p_c) \xi^\alpha \]

with \( E_m = 2 \nu + 1/2 \) larger than the discrete case by 1/2.

In 3d, one has to evaluate the concentration of the elastic energy due the presence of the two cracks. Using the analogy between the topology of figures 4 and 5b, it is sufficient to estimate the product \( p \cdot u \) for a disk of diameter \( \delta \). This is not an easy task but one can rely on the heuristic argument (checked with the exact result of the 2d problem) that most of the elastic energy is stored in the vicinity of the disk. Therefore \( p \cdot u \delta^2 \) should be of the order of \( Y^{-1} F^2 \) leading to \( p \cdot u \approx Y^{-1} F^2 \delta^{-2} \). Writing the Griffith criterion \( p \cdot u \approx 2 s \) leads to a stress failure threshold exponent \( E_m = 3 \nu + 1 \). More generally, in arbitrary dimension, one has \( E_m = d \nu + (d - 1)/2 \) with a correction \((d - 1)/2\) to the failure exponent of the discrete-lattice case.

Note that it is not clear whether this failure threshold for the weakest crack’s local configuration coincides with the macroscopic failure threshold, due to complex screening and enhancement interactions. Further attention should be given to this difficult problem.

4.2.2 Bond flexion failure: case of cracks with finite thickness. — This corresponds to the case treated in paragraph 4.1.2 for the elasticity. The analysis closely follows reference [2b]. Bending failure will occur on the weakest bond and is characterized by a rupture threshold given by

\[ \sigma \approx (N_c - N)^{\xi^\alpha} \]

with \( E_m = 2 \nu + 1/2 \) for the critical elastic energy criterion in 2d and \( E_m = 2 \nu + 1 \) for the critical strain criterion in 2d since the bending constant scales as \( \delta \) (for \( \delta < a \)) as given by equation (12).

In 3d, the failure exponent is \( E_m = 3 \nu + 2 \) for the critical elastic energy criterion and \( E_m = 3 \nu + 4 \) for the critical strain criterion since the bending constant \( \gamma \) scales as \( \delta^4 \) in the critical regions (\( \delta < a \)). Outside this critical region (\( \delta < a \)), one recovers the discrete-lattice exponents.

4.2.3 Out of plane two-dimensional rupture.

Crack opening. — In this case, the failure problem can be analysed with the tools of reference [2]. 2d mechanical failure is dominated by the weakest bond. If a stress \( \sigma \) is applied at the boundary, the force \( F \) supported by a macro-link is \( F = \sigma \xi (d - 1) \) and the torque \( M \) which is transmitted in the macro-link is \( M = F \xi = \sigma \xi^d \).

Assuming that rupture will occur if the bending elastic energy \( E = \gamma^{-1} M^2 = \delta^{-2} \sigma^2 \xi^{2d} \) in that bond becomes larger than a threshold value \( E_c \) (first criterion of [2b]), one obtains the failure threshold

\[ \sigma \approx (p - p_c) \xi^\alpha \]

with \( E_m = 2 \nu + 1/2 \) larger than the discrete result \( E_m = 2 \nu + 1 \) by one.

If rupture occurs when the angle \( \theta \) by which the neck is bent is larger than a threshold value \( \theta_c \) (second criterion of [2b]), one has \( E_m = 2 \nu + 1/2 \). Buckling failure. — The screening of the strain by buckling is so efficient that one recovers the discrete-lattice exponents.
4.2.4 Failure distribution. — Due to fluctuations in the bond strengths, the failure thresholds are distributed according to a probability distribution taking a Weibull form when the expression of the stress failure threshold of a micro-bond of width \( \delta \) is of the form

\[
\sigma_f \approx \delta^{-\alpha}.
\]

(30)

The analysis is similar to that developed for the electrical failure in paragraph 2.2 and is not repeated. Generally, the appearance of a Weibull failure distribution can be tracked back to the existence of a power-law distribution of micro-bond strengths. In the case of the blue cheese percolation model, the powerlaws are created by the flat distribution of bond-widths in the presence of the power-law dependence of the failure stress with respect to the micro-bond strengths.

5. Concluding remarks.

An analysis of the transport and failure properties of a new class of percolation model (the blue print of stick or disk percolation [7]) for systems deteriorated by thin cracks has been presented. Several deformation modes have been discussed. Of particular importance is the asymmetry of the stress-strain characteristic of a single crack. This leads to a new percolation problem which in two dimensions corresponds to a continuous version of the directed percolation model. In three dimensions, one has to analyse the connectivity properties of random disks or plaquettes, a problem related to the statistical properties of random surfaces [22]. This difficult problem is left for future work.

Finally, let us note the following points:

1) the results of this paper and of references [1, 2] can be contrasted with the exponent for geometrical properties such as \( \nu \) which have been confirmed numerically to be identical for both discrete and continuum systems [13]. The difference between continuum and discrete-lattice exponents stems from the fact that transport and failure properties are influenced by the weak bonds of the distribution of bonds strengths. The difference between the Swiss-cheese and blue-cheese continuum percolation models stems from the different topology of the micro-bonds of the percolating structure. The blue-cheese model can be viewed as another way for taking the limit of a discrete-lattice mesh size \( z \) going to zero. In the Swiss-cheese model, \( z/b \to 0 \) (where \( b \) is the spherical hole radius) in contrast to the blue-cheese model where \( z/t_c \to 0 \) and the discrete case where \( b/z \approx l_c/z = 1 \). We can thus trace back the non-universality of the transport and failure critical properties to the different geometrical limits in the different models;

2) table I summarizes the results of [1, 2] and this work. A striking fact is the wealth of exponents in elasticity and mechanical failure corresponding to the different modes of deformation. What will be the mechanical failure exponent in a given experimental set-up? The answer is not simple and depends sensitively on the precise conditions which apply to the system under study. In 2d, if out of plane buckling is forbidden (by clamping the plate between rigid plates) but crack edges overlapping is allowed as in paragraph 4.1.3 and if the widths of the cracks are vanishingly small \( (a < \delta_{min}) \), crack opening rupture will dominate. When buckling is permitted, one recovers the discrete-lattice exponent.

In 3d, Griffith rupture dominates for thin cracks \( (a < \delta_{min}) \) but leaves room for micro-bond bending failure for cracks with a finite thickness.

This study therefore demonstrates that, in systems damaged by cracks, the Griffith failure mechanism will not always occur. This is in contrast with the brittle failure mechanism occurring when the system is damaged by a single crack. In the presence of many cracks, bond bending failure must be considered (this is also suggested by the analysis [30] of the experiments reported in [31]). This problem provides another example of the appearance of a novel qualitative behaviour characteristic of large disorder;

3) the different critical exponents summarized in the table should be observed in a narrow interval \( N \to N_c \) as suggested by recent experimental and numerical works [31, 16]. This implies facing the difficult problem of finite size effects. Note that the cracks must be really distributed at random in order to create the narrow necks which are so crucial for the distinction between continuum and discrete systems. See for example [32] for a careful discussion of the numerous experimental pitfalls that must be avoided the critical transport.

These predictions are currently being checked in our laboratory on model experiments with sheets of different materials which are deteriorated by cracks. This work will be reported subsequently.

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References

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