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Transition to turbulence of convective flows in a cylindrical container

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Résumé. — Le seuil de transition à la turbulence des écoulements convectifs de bas nombre de Prandtl se situe bien plus près du seuil de convection dans une boîte cylindrique étendue, que ne le laisserait penser l'analyse de stabilité de rouleaux droits. La convection met alors en jeu des effets non locaux et une fermeture des échelles spatiales. Nous résolvons ces problèmes en construisant une solution analytique explicite des champs de phase et de leurs écoulements moyens, qui est valable aux ordres dominants de distorsion des structures. Nous aboutissons ainsi à une explication de la faible valeur du seuil de turbulence à bas nombre de Prandtl en géométrie cylindrique ainsi que des mécanismes qui y conduisent.

Abstract. — The threshold of the transition to turbulence of low Prandtl number convective flows occurs much closer to the convective threshold in an extended cylindrical cell, than one could infer from a straight roll stability analysis. Convection then involves non local effects together with a closure of the spatial scales. We solve these problems by constructing an explicit analytical solution of the phase fields and of their mean flow fields, which is valid at the dominant orders of pattern distortions. We hence provide an understanding of the low value of the threshold of turbulence at low Prandtl numbers in a cylindrical cell and of the mechanisms that lead to it.

Introduction.

Among the challenging problems still left in classical physics, one of the oldest but also one of the most puzzling concerns the nature of turbulence. Apart from the studies of fully developed turbulence, a novel way of approaching this difficult question has been traced by investigating the onset of weakly turbulent states, that is the transition to turbulence. For this purpose, continuous media displaying pattern forming systems have been widely considered and especially the hydrodynamic ones. Among these, a great amount of effort has been devoted to the simple and well controlled thermoconvective Rayleigh-Bénard system.

Two widely different situations have been encountered depending on the box aspect ratio \( \Gamma \) (\( \Gamma = L/d \) where \( L \) is a typical horizontal box length and \( d \) the box depth). In low aspect ratio boxes (\( \Gamma < 3 \)), the mean spatial structure is frozen and turbulence involves mainly dynamical modes [1]. The various routes to chaos encountered experimentally have been satisfactorily recovered by the scenarios developed in the framework of low dimensional dynamical dissipative systems [2]. In the present paper, we will focus on the opposite situation of large aspect ratio boxes. Roll patterns then distort themselves so that the spatial modes are inherent to the dynamics. Turbulence thus relates to a wandering of the pattern cells and is called phase turbulence [3].

Experimental studies have shown that in large aspect ratio containers, at least two different behaviours can be encountered on the route to turbulence, depending on the fluid Prandtl number \( Pr \) [3-10]. At high Prandtl numbers, roll patterns reach stationary states regardless of their spatial features, of the box shape and of the box aspect ratio [3, 5]. Turbulence comes in far from the convective threshold, when, due to roll instabilities, the pattern cell no longer consists of a roll but instead displays three-dimensional flows produced by superimposed rolls [8]. On the contrary, at low Prandtl numbers (\( Pr \approx 1 \)), turbulence first arises on roll patterns, at Rayleigh numbers much closer to the value \( Ra_c \) at the threshold of convection for disordered or bent patterns [6, 7] than for straight ones [9]. The
threshold of turbulence therefore depends on the pattern geometry and thus on the box shape but, as shown by experiments in cylindrical containers [10], not much on the box aspect ratio $\Gamma$ above $F = 6$.

At low Prandtl numbers, extremely destabilizing phenomena are then operating, provided that the pattern is distorted. In this paper, we show how they work in a cylindrical container near the threshold of convection, by constructing an analytic solution of the pattern states.

This solution enables us to recover the experimental observations at $Pr = 0.7$ in argon gas [7]. Especially, it leads to an understanding of the small value of the Rayleigh number at onset of time dependence at $Pr = 0.7$ in a cylindrical container and to an identification of the mechanisms involved in this route to turbulence. Since it provides an example of pattern self destabilization near the convective threshold, this result recovers the dynamical behaviour detected in liquid helium [6, 10]. The analytic solution, when it gets unstable, provides an example of a dynamical frustration between roll flows and mean flows, whose concept has been proposed by Manneville for an understanding of weak turbulence [15]. Finally, by investigating the dependence of the stability of our solution on the Prandtl number, we succeed in explaining the strong sensitivity of the route to turbulence on the Prandtl number in the range $1 < Pr < 5$. This result justifies the current definition of low and moderate Prandtl numbers.

A number of studies have linked the destabilizing phenomena occurring at low Prandtl numbers to parallel flows spontaneously generated by the roll flows [11-17]. Each of these flows has its own spatial scale, the box size for parallel flows and the roll size for roll flows. For this reason, convective patterns look like hydrodynamic systems displaying only two spatial scales. Such condition is necessary to produce scale interactions, and it proves here to be sufficient to display their major features: non local effects owing to the long range of parallel flows, a closing problem for the determination of the parallel flow produced by the phase pattern and finally a strong sensitivity to the boundary conditions. Our analytic result may therefore be interesting beyond the area of convection studies, since it provides an example of an exact solution of major hydrodynamic difficulties commonly encountered.

The paper is divided in three parts. In the first part, we briefly describe the main features of the route to turbulence at various Prandtl numbers, and we survey the essential features of parallel flows in order to emphasize their non-local character. Motivated by the observation of patterns at $Pr = 0.7$ in a cylindrical container [7], we construct, in a second part, an analytic stationary solution of the coupled convective flows-parallel flows equations in a cylindrical container, valid at the dominant orders of pattern distortions. We investigate its stability in the third part.

1. Low Prandtl number convection and non-local effects.

1.1 Route to turbulence of distorted pattern in a cylindrical box.

1.1.1 High Prandtl numbers. — In high Prandtl number fluids, the route to turbulence of distorted roll patterns involves stationary asymptotic states [3-5] up to Rayleigh numbers at which straight roll patterns are unstable [8]. Small scale instabilities then develop leading to an elementary convective cell of a different kind from a roll, and consisting of an additional roll-like motion at right angles to the basic roll [8]. The pattern cell therefore consists of a three-dimensional flow. Further increases in the Rayleigh number or the occurrence of pattern defects induce a turbulent behaviour [3, 4], in a way reminiscent of a structure fusion.

Roll patterns cannot therefore be turbulent whatever their spatial features are. The turbulence can only occur so far from the convective threshold that roll themselves are unstable. Those statements relate to any box shape and any aspect ratio.

1.1.2 Low Prandtl numbers : $Pr < 1$. — Surprisingly, an opposite behaviour occurs at low Prandtl numbers. This has first been demonstrated by experiments performed in liquid helium in a range of Prandtl numbers spreading from 0.7 to 3 [6] and extended later to various aspect ratio $\Gamma$ for 0.54 < $Pr < 0.69$ [10]. These experiments showed that turbulence occurs in a cylindrical container at reduced Rayleigh numbers $\varepsilon = (Ra - Ra_c)/Ra_c$ of order 0.1, while the stability limit of straight roll patterns should be of order $\varepsilon = 2.5$ [8], i.e. one order of magnitude greater. This disagreement has been proved by experiments performed in rectangular boxes, where straight rolls are expected: time dependence only comes in at $\varepsilon = 1.5$ and turbulence at $\varepsilon = 2.5$ [9].

The threshold of turbulence therefore depends on the spatial features of the pattern. This strongly indicates that new phenomena responsible for these unexpected dynamical events are brought about by the spatial chaos. However, since no visualization could be achieved in liquid helium, their main features were not identified. Nevertheless, Nusselt number measurements only allowed the mean spatial structure to be determined: in cylindrical cells and for $\Gamma > 6$, nearly parallel rolls are likely to be displayed [18]. Fortunately, as reported below,
further experimental as well as numerical studies have succeeded in visualizing the structures. They led to point out the most important spatial features of low Prandtl number convection.

Owing to the facilities offered by an increase of pressure [7], visualization of convection in gases, at a Prandtl number of 0.7 has been performed. This led to a satisfactory understanding of the spatial mechanisms involved in this transition to turbulence [7, 19, 20]. In a cylindrical container with aspect ratio \( \Gamma = 7.66 \), starting from a straight roll pattern at threshold, a distortion appears continuously as the Rayleigh number increases: two centers of curvature (i.e. foci) take place and force a compression of the rolls in the center of the container (Fig. 1a). Contrary to the high Prandtl number regime where the wavenumber is selected, the convective patterns therefore display a band of wavenumbers (Fig. 3b).

At a threshold value \( Ra_i \) of the Rayleigh number \( (Ra_i = 1.14 Ra_c) \), the rolls become too constrained at the center of the pattern, and a pair of dislocations nucleates (Fig. 1b). These defects climb and glide towards the foci where they disappear. Since a wavelength has been lost in this process, the remaining rolls are less constrained. As none of them is unstable at this time, a stationary state might be expected. But the centers of curvature feed the pattern with new rolls, thus leading the compression to increase again until a new defect nucleation occurs. This process results in a sustained dynamical behaviour of the previous events. As confirmed by measurements, the local wavenumber in the center of the structure reaches periodically a value so large that it becomes unstable versus the Busse and Clever instabilities of straight roll patterns [8]. This explains the occurrence of roll pinchings and of dislocation nucleations. Recently, the road to turbulence observed in argon gas in a cylindrical container has been recovered by Greenside et al. by simulation of a model of convection [21]. Finally, experiments similar to that of reference [7], but with bigger aspect ratios have shown that this route to turbulence was not related to a particular value of the aspect ratio [19, 20, 22].

It is worth noting that, in a rectangular cell, Manneville has found a similar mechanism for turbulence with a numerical simulation of a model of convection [16]: rolls are constrained until a defect nucleation occurs and the repetition of such events sustains a time dependence.

1.1.3 Moderate Prandtl numbers: \( Pr = 2.5 \) and \( Pr = 5.7 \). — Two experiments performed in water at \( Pr = 2.5 \) and \( Pr = 5.7 \) in a cylindrical cell [23, 24] have displayed a route to turbulence different from that at \( Pr = 0.7 \), near the threshold of convection. Especially it cannot be treated by the analysis of part 2. In the experiment of Heutmaker and Gollub [23], a time dependent behaviour close to onset was sustained by grain boundary motions but was followed by stationary states in the range \( 0.2 \leq \varepsilon \leq 3.5 \). At \( \varepsilon = 3.5 \), a second transition brought into the turbulent domain through the repetitive nucleation of dislocation pairs. This latter transition is

Fig. 1. — Transition to turbulence in a cylindrical container in argon gas \( (Pr = 0.7 \) and \( \Gamma = 7.66) \) [7]: (a) stationary distorted pattern at \( \varepsilon = 0.13 \). Notice the roll compression along the line joining the centers of curvature; (b) above \( Ra = 1.13 Ra_c \), the compression becomes so big that a pair of dislocation nucleates at the center of the pattern. A dynamical behaviour is sustained.
similar to that occurring at $Pr = 0.7$ but arises at higher Rayleigh numbers. Nevertheless, an analysis similar to that performed in [7] in terms of the incompatibility between the roll stability and the roll wavenumber band has been successfully applied. The high threshold of this transition will be explained in part 3. Concerning time dependence close to onset, following the authors of this experiment, we cannot rule out the possibility that it might be due to the lack of uniformity of the layer depth, especially since the experiment of Rehberg et al. [25]. We must also notice that the waiting time may well have been too short in this experiment (40 horizontal times $\tau_H$) to discriminate between transients and turbulent states. Indeed, in a square container, the same authors have observed that a time dependent state at $50 \tau_H$ became stationary at $100 \tau_H$. Similar long transients are currently observed in argon gas [19-21, 26].

In the experiment of Ahlers et al. [24] at $Pr = 5.7$, a similar time dependence near the threshold of convection was noticed, but a clockwise rotation was present for any pattern, even those called stationary. We may then wonder whether such forcing might not be at the origin of time dependence close to onset, where the convective structures are weakest. As a support, in a roll chain, the smallest mean flow amplitude leading to time dependence scales like $\epsilon^{1/2}$ [17].

1.1.4 Conclusion. — The major spatial features specific to the route to turbulence at low Prandtl numbers in a cylindrical container therefore consist in the growth of a wavenumber distribution which finally results in local unstable wavenumbers. These phenomena seem to be the key to understanding phase turbulence and should be included in theoretical models. However, the amplitude of this distribution, together with the pattern distortion, is plausibly dependent on boundary effects, and thus on the boundary shapes, so that a non-local analysis is needed to describe it properly. Such a study, concerning the patterns encountered on the route to turbulence in a cylindrical container at $Pr = 0.7$, is addressed in this paper. It involves the presence, besides the roll flows, of spontaneously generated small amplitude parallel flows, whose features are surveyed in the following section.

1.2 Main features of mean flows.

1.2.1 Theoretical evidences of mean flows. — An important step towards the understanding of the extra phenomena exhibited by the low Prandtl number convection has been performed by Siggya and Zippelius [11]. They pointed out that, provided that a pattern is distorted from a straight and homogeneous geometry, the advection term of the Navier-Stokes equations produces flows of a different kind than the convective ones. These secondary flows correspond to a vanishing harmonic of the basic mainly periodic convective field and thus vary on a spatial scale large compared to the roll diameters. On the roll scale, they appear as parallel flows or mean flows. Their amplitude, scaled by $1/Pr$, is usually quite small compared to that of the convective flows. For this reason, they may be represented as drift flows which enable the fluid to cross the convective pattern from roll to roll by a mean other than molecular diffusion. They thus are reminiscent of the permeation process in liquid crystals [27]. These drift flows produced by the convective field tend in turn to advect it. Owing to this feed-back interaction, they are likely to modify the roll pattern, either spatially or dynamically.

For rigid horizontal boundary conditions, the mean flows must vanish at the horizontal plates of the Rayleigh-Bénard container. They thus vary over the cell depth, but only their resulting effect on a roll gets a pertinent meaning. For this reason, one only takes into account the bidimensional flow field $F$, which results from an average process of the mean flow field over the cell depth. By extension, we will also call it mean flow field. Using a multiscale perturbative expansion of the Boussinesq equations versus both the distance from the convective threshold and the pattern distortion amplitude, namely the inverse of the box aspect ratio $\Gamma$, Cross and Newell have derived a rotationally invariant expression of the mean flow $F$ produced in a slightly distorted pattern [12]:

$$ F = -\gamma k \nabla (kA^3) + \nabla \Pi + o(1/\Gamma) $$

where $A$ is the roll amplitude, $k$ the phase gradient, and $\gamma$ a coupling constant mainly inversely proportional to $Pr$. The pressure gradient $\nabla \Pi$ accommodates the incompressibility condition. The pattern distortion thus creates a density of flow sources directed along the local wavevector. This peculiar direction of flow production may be considered as a consequence of the local reflection symmetry of the pattern versus the wavevector direction, which is displayed by both the curvature and the compression, i.e. by both the dominant distortions. The field produced by the flow sources before the incompressibility condition works must display this symmetry and is thus locally directed along the wavevector direction. The actual mean flow is next determined as a rotational part of this field.

1.2.2 Experimental evidences of mean flows. — A direct measurement of the velocity of mean flows is tremendously difficult to perform, owing to its small magnitude compared to the convective velocity. This arises because mean flow amplitude is proportional to the pattern deformation and to the inverse of the Prandtl number, and, like any convective harmonics, is dominated near the convective threshold by the fundamental convective mode. To overcome these
difficulties, direct experimental evidences of mean flows involve their transport properties with tracer suspensions [28, 29]. This method presents the advantage of integrating the flow effects over long times, and thus of raising the differences between mean flows and roll flows. Owing to the anisotropic features of convective flows, which produce no stream along their axis, the contribution of mean flows may be separated from that of roll flows: along the roll axis, there can only be mean flows. The first direct observation of mean flows has been reported by Willis and Deardorff in the oscillatory regime of convection [28]. The tracer suspension (oil smoke) was introduced in the roll pattern from outside and observed for 10 min i.e. 30 vertical diffusion times $\tau_v$. However, since the overall pattern distortion was not observed and since the method of visualization was perturbative, the origin of these mean flows cannot be clearly established. Another experiment solved these difficulties by using a non-perturbative color printing and by performing a comparison between the transport properties of two patterns, one distorted and one undistorted, at similar Rayleigh numbers [29]. The tracer was observed over one hour i.e. $3 \times 10^4 \tau_v$. As expected from the Siggia-Zippelius mechanism [11], mean flows have only been detected when the pattern was distorted. Their amplitude was evaluated to a value as small as 5 $\mu$m/s at $\varepsilon = 6$ and $Pr = 7$, while the convective flow amplitude was 400 times greater.

As mean flows tend to advect the pattern phase, they modify the phase diffusion coefficients. The phase motions are therefore likely to indicate whether mean flows are present. This can especially be the case for dislocation motion, since it is governed by the far field and not by the defect core [30]. An experimental study of dislocation climbing at $Pr = 70$ has resulted in a measure of the transverse diffusion coefficient $D_\perp$ [31]. Since it is close to the determination of $D_\perp$ performed by Manneville and Piquemal including the spontaneous production of mean flows [13], it has provided an indirect evidence of their presence. A peculiar behaviour of the climbing velocity, for low velocities, introduced discrepancies with the variational behaviour, and could also be linked to the presence of mean flows [31].

1.2.3 Mean flow effects onto a convective pattern. — At low Prandtl numbers, a phase perturbation of a roll pattern generates a mean flow perturbation. This results in an additional tendency of pattern phase advection. This effect, when included in the analytic studies of straight roll stability [11-15] leads to recover the large scale Busse and Clever low Prandtl number instabilities [8]. This strongly indicates that, despite their quite small amplitude near the convection threshold, mean flows may have a huge effect on the convective behaviour. This is confirmed by the numerical experiment of Manneville [16] which showed that mean flows tend to constrain the rolls so much that defects nucleate, hence sustaining a dynamical behaviour.

In order to demonstrate and to study the mean flow effect on a roll structure experimentally, mean flows have been decoupled from the roll flows which are usually their source [17]. They were generated out of the convective domain as small speed parallel flows and then thrown into it. The Prandtl number of the fluid was chosen to be high enough to prevent the spontaneous generation of other mean flows. Instead of being left as a field governed by pattern distortions, mean flows were controlled by the experimentalist. In order to avoid bidimensional phenomena, the convective pattern was an annular chain of rolls with restricted extension along their axis. Mean flows then crossed the chain perpendicularly to the roll axis. When no mean flow was generated, a single value of the roll diameter was displayed, according to the phase diffusion mechanism [32] and in agreement with observations on high Prandtl number fluids. On the contrary, when a mean flow crossed the pattern, a wavenumber distribution was generated and it could even bring about a sustained dynamical behaviour by defect nucleations. The spatial as well as the dynamical features of the low Prandtl number convection were recovered while the fluid Prandtl number was high! Such a result emphasizes the role of mean flows in phase turbulence and proves that the low Prandtl number behaviour is induced by mean flow effects.

At high Prandtl numbers, diffusive thermal phenomena dominate the pattern behaviours. As shown by Pomeau and Manneville [32], the phase dynamics of convective patterns is then grasped by a diffusion equation, whose equilibrium states display a single wavenumber. However, as mean flows tend to advect the pattern phase, they produce roll advection, roll distortion, or both of them. This property has been quantitatively studied in the above experiment and led to quite a good agreement [17]. It is worth noting that although this advection tendency produces a quite small modification between a roll and its neighbour, it results in huge wavenumber differences when integrated over a large number of rolls. This emphasizes the importance of large aspect ratio for the appearance of turbulence when mean flows are present. This also shows that large scale phenomena must be taken into account even if their amplitude is quite small.

1.3 MEAN FLOW EFFECTS AS A RESULT OF GAUGE INVARIANCE. — The above mean flow effects may appear as the result of the advection forcing of the pattern phase by a large scale parallel flow. How-
ever, such point of view may hide a deeper meaning of the mean flow effect. Indeed, since a pattern phase can only be defined by comparison to an underlying phase reference which can only bear a relative meaning, the physical description of patterns must be formally invariant in arbitrary phase reference changes [20, 33]. When only roll flows are considered, for instance as in the Pomeau-Manneville phase diffusion equation, the invariance is achieved only for global reference changes. It then expresses global geometrical invariance such as the translational or the rotational one. However it breaks down for local ones. This means that the physical description of the convective pattern, and more generally the physical description of any cellular pattern, is insufficient if it only takes into account the small scale field. Additional fields are needed to achieve the covariance under local reference changes and, for convective patterns, they convey the mean flow effects. Such a result requires the definition of the pattern state not only by its phase field but also by its gauge fields and gives thereby a way to relate different phase fields belonging to the same state [33]. Let us also notice that, even when the gauge invariance does not relate to mean flows, it provides a tractable method for studying defects [34].

This requirement of covariance is similar to an extension of a gauge invariance from a global to a local level. Long-range effects in cellular patterns thereby derive from a local gauge invariance and take the form of gauge fields. From this point of view, mean flows are, for the cellular structures, the analogous of the electromagnetic field for a charged particle. Gauge invariance applied to cellular patterns demonstrates that at least two different spatial scales are needed to grasp their physical properties. It then elucidates the meaning of mean flows.

1.4 Non-local features of mean flows. — The role of mean flows in the pattern behaviour is a non-local problem for several reasons, which are related either to the creation of the mean flows or to their effect on the roll pattern.

Roll distortions behave like local sources of mean flows distributed over the convective domain. However, because of the incompressibility condition, a mean flow produced by a localized source can extend far from it in order to close its streamlines. It may therefore even occur that the mean flows produced in a distorted domain extend into an undistorted one. This phenomenon requires to take into account all the sources of mean flows together, and hence the whole pattern, in order to determine the mean flow field.

This non-local mechanism of mean-flow generation enhances the role of the boundary conditions. When the sidewalls are uncrossable, they act as additional sources which may deeply influence the mean-flow field. For instance, since mean flows must close their streamlines within the sidewall domain, a back flow may thereby be generated. We will see in part 3 that this may be essential. Furthermore, since the boundary conditions govern the pattern phase, even in its bulk, they can also influence, the phase distortions and hence the mean flow field that they produce in this second way.

Once the mean flow is created, its effect on the pattern phase involves the number of rolls on which it acts: the higher the number of rolls in the counter flow direction, the higher the local wavenumber. The whole phase pattern must then be considered to compute the mean flow effect so that a non-local analysis is also required here.

Owing to the non-local features of mean flows, both the whole pattern and its lateral boundary conditions must be taken into account to determine the spatial and dynamical behaviours. Such a difficulty is commonly encountered in hydrodynamics, but it is usually enhanced by the great number of interacting scales. Fortunately, since only two spatial scales are present in convection, we have been able to solve this non-local interaction, at the lowest order in the distortion amplitude, for the patterns encountered on the route to turbulence in a cylindrical container [7]. This solution is presented in the next parts of the paper, but we stress that (except if some general rules can be worked out as guide lines) this kind of analysis may well be strongly dependent on the pattern spatial features and on the box shape.

2. A solution to the non-local convective behaviour. In Rayleigh-Bénard convection at low Prandtl numbers, mean flows are generated by pattern distortions but mean flows in turn can sustain distortions. Such an interaction bears a non-local character. It induces a coupling between the two different spatial scales, i.e. the roll flow and the mean flow, which may result in a sustained dynamical behaviour and thus in phase turbulence. In this part of the paper, we want to treat this coupling and show that it may be exactly soluble at least at the lowest order in the distortion amplitude, within the framework of the phase formalism. Such a solution is performed for the structures encountered at the transition to turbulence in a cylindrical container at $Pr = 0.7$ [35]. It emphasizes the role of the boundary conditions and thus of the boundary shape in the determination of both the mean flow field and the phase field. It finally provides a clear and tractable example of non-local analysis in cellular patterns.

2.1 General framework and procedure.  
2.1.1 General framework.  
(a) The coupled phase-mean flow equations. — The equations governing a phase field $\varphi$ in Rayleigh-
Bénard convection may be written using the Cross-Newell formalism \[14\].

\[
\begin{align*}
\tau \left[ \frac{\partial \phi}{\partial t} + k \cdot F \right] + \nabla (kB) &= 0 + o(1/\Gamma) \quad (1) \\
F &= -\gamma k \nabla (kA^2) + \nabla \Pi + o(1/\Gamma). \quad (2)
\end{align*}
\]

The phase equation (1) involves scalar functions \(\tau(k, Ra, Pr)\) and \(B(k, Ra, Pr)\) which convey the longitudinal and transverse phase diffusion coefficient \(D_1 = -\left(1/\tau\right) \partial (kB)/\partial k\) and \(D_2 = -\left(1/\tau\right) B\). Equation (2) expresses the mean flow \(F\) which is produced, close to the convective threshold, by a phase distortion, within a quasistationary approximation which neglects its dynamics. This approximation is valid until the oscillatory instability regime is approached, at Rayleigh numbers several times greater than the critical one. Equation (2) involves the roll amplitude \(A(k, Ra, Pr)\) and the coupling coefficient \(\gamma\) which is mainly inversely proportional to \(Pr\). The mean flow is determined up to a pressure gradient \(\nabla \Pi\), which enables \(F\) to satisfy both the non-divergence condition and the boundary conditions. Equation (2) is restricted to the vicinity of the onset of convection. Both these equations are valid at order 1/\(\Gamma\), where \(\Gamma\) is the box aspect ratio. They provide a coupled system for the determination of a phase field.

(b) The mean flow boundary conditions. — For rigid lateral boundary conditions, the flow \(F\) must vanish at the sidewalls. However, since it involves a spatial scale in the bulk \((\Gamma d)\) large compared to the cell depth \(d\), it is analogous to a parallel flow in a Hele-Shaw cell. Similar boundary conditions then arise. They come from the fact that, owing to viscous dissipation, the lateral boundary layer scales like the cell height \(d\), i.e. \(O(1/\Gamma) = o(1)\) on the large scale units. Such a layer is therefore too restricted spatially to be handled on the large scale. It thus leads to no boundary conditions on the mean flow \(F\) by its own. Away from this layer, the only condition required is that the mean flow is not divergent, so that its streamlines close on themselves before reaching the sidewall. This requires that the normal component of the mean flow at the sidewall vanishes: \(F \cdot n = 0\), hence giving the boundary condition for the mean flow in a closed box.

2.1.2 The procedure. — The route to chaos in a cylindrical container in argon gas at a Prandtl number of 0.7 has pointed out stationary patterns which display two diametrically opposite foci together with a compression of the rolls in the middle of the pattern \[7, 19, 20\] (Fig. 1). Our aim here is to understand and to describe by the Cross-Newell formalism, the appearance of such distortions and their relation with the Rayleigh number and the Prandtl number. Since this formalism is only valid near the convection threshold, such a study can only be planned because of the proximity of the threshold of turbulence to the onset of convection at \(Pr = 0.7\). We will construct a suitable solution of the Cross-Newell equations in the following way.

As a first step, we model the phase fields for weak distortions in a plausible way, with the use of free parameters to be determined later by the solution of the coupled equations. Then we develop these equations for such phase fields, at the dominant orders in both the distortion amplitude and the aspect ratio, and we look for stationary solutions satisfying the boundary conditions. We find a continuous family of stationary solutions but, introducing a phenomenological treatment of the phase boundary conditions, we succeed in selecting a single solution. In part 3 of the paper, we finally investigate its stability and conclude about the threshold of turbulence.

In the present part of the paper, we determine the stationary states in the following way. We first establish the general features of the perturbative expansion and its consistency. We next determine, from the phase equations, the mean flow \(F\) which is produced by the phase field distortions, with the boundary conditions being taken care of. Then we introduce this flow as an advection term in the phase equation and we look for a criterion of stationarity. The degeneracy of this criterion is next broken by a phenomenological use of the phase boundary condition. This finally allows us to compare our solution to the available experimental results.

2.2 Construction of a solution.

2.2.1 Modeling of the convective structure at the transition to turbulence in a cylindrical container. — To model the phase field \(\varphi(x, y)\) of the patterns observed in a cylindrical container at \(Pr = 0.7\) (Fig. 1a), we take cartesian coordinates with the \(y\) axis along the line joining the foci and we take advantage of symmetry arguments to expand it up to the fourth order in the pattern distortion. The origin of the coordinates takes place at the center of the container. Because of the symmetry of the patterns with respect to the \(y\) axis and the antisymmetry w.r.t. the \(x\) axis, we look at \(\varphi\) fields in the form:

\[
\varphi(x, y) = k_m y [1 + \psi(x, y)]
\]

where \(\psi\) is an even function in both the \(x\) and \(y\) coordinates which vanishes at the origin. The parameter \(k_m\) is therefore the wavenumber at the center of the box. We next write the phase field as:

\[
\varphi(x, y) = k_m y (1 - ax^2/R^2 + by^2/R^2 + cy^4/R^4 + dx^2y^2/R^4)
\]

where \(R\) is the radius of the cylindrical box. The free parameters \(a, b, c, d, k_m\) are the expansion coef-
fects. When applying such a truncation to the phase field \( \varphi \), we assume to capture, as explained below, the dominant modes of distortions. If so, it becomes meaningful to study the phase dynamics in the restricted functional space \( \mathbb{FS} \) spanned by \((y, yx^2, y^3, y^5, x^2 y^3)\).

The first two terms \( k_m y (1 - ax^2/R^2) \) represent indeed a curvature of the phase field. They correspond to an approximation of a hyperbolic phase field in the very vicinity of the medium of its two foci. The parameter \( a \) is related to the ratio between the cell radius \( R \) and the focus distance \( D \) : \( a = (R/D)^2 \). It governs the curvature which increases as the foci get closer to the cell. However, no compression is present in this phase field along the line \( x = 0 \) joining the foci, contrary to experimental observations [7]. The remaining terms in the modelling of \( \varphi \) allow to introduce it in the most general way at the fifth order polynomial development. As the dominant distortion that we want to study is curvature, we will assume that the compression is a next order distortion : the parameters \( b, c, d \) are thus considered to be second order in \( a \). We will show later that this ansatz is consistent with our solutions.

### 2.2.2 Expansion features and consistency of the development.

(a) The expansion features. — The expansions of the geometric fields \( k, \nabla \cdot k, (k \cdot \nabla) k \) in terms of the expansion parameter \( a \) are as follows:

\[
\begin{align*}
  k &= k_m \left[ -2a \frac{xy}{R^2} + 2 \frac{xy^2}{R^4} \right. \\
  k &= k_m \left[ 1 - \frac{x^2}{R^2} + 3b \frac{y^2}{R^2} + 5c \frac{y^4}{R^4} + 3d \frac{x^2 y^2}{R^4} \right] + o(a^2) \\
  \nabla \cdot k &= k_m \left[ -2a y/R^2 + 6b y/R^2 + 20c y^3/R^4 + 2d y^3/R^4 + 6d x^2 y/R^4 \right] + o(a^2/R) \\
  (k \cdot \nabla) k &= k_m \left[ 8a x^2 y/R^4 + 6b y/R^2 + 20c y^3/R^4 + 6d x^2 y/R^4 \right] + o(a^2/R)
\end{align*}
\]

so that

\[
  k = (0, k_m) + O(a), \quad \nabla \cdot k = -2a k_m y/R^2 + o(a/R), \quad (k \cdot \nabla) k = O(a^2/R)
\]

They enable us to determine in the following, both the expansion order of the mean flow \( F \) and of the wavenumber \( k_m \):

The mean flow equation (2) involves the term \( \nabla(kA^2) \), which may be written as:

\[
  \nabla(kA^2) = [A(k_m)^2 + (k - k_m) dA^2/dk_m] x \times \nabla \cdot d + dA^2/dk_m (k \cdot \nabla k) + o(a^2/R).
\]

Its term of order \( a/R \) is thus \( -2a k_m A(k_m)^2 y/R^2 \). It leads to the following contribution of order \( a/R \) to the mean flow \( F \) : \( (0, 2\gamma a k_m A(k_m)^2 y/R^2) \). Since it is a divergent vector field which derives from a potential, it is completely balanced by the pressure gradient. The mean flow \( F \) is therefore at least of order \( a^2/R \).

The phase equation involves the diffusive term \( \nabla(kB) \) is thus proportional to \( B(k_m) \). However, as the mean flow \( F \) is at least of order \( a^2/R \), no stationary solution of the phase equation (1) can be obtained unless \( B(k_m) \) is at least first order in \( a \). On the contrary, the relaxation motion due to curvature cannot be balanced by the mean flow advection. The corresponding pattern evolves until the above condition is satisfied. During this transient phase, the number of rolls present in the cylindrical container varies by continuously either disappearing or arising at both the foci. From now on, we will consider that this transient phase has ended. In the following, the phase fields will therefore display wavenumbers \( k_m \) so close to the wavenumber \( k_0 \) at which \( B \) vanishes, that \( B(k_m) \) is at least first order in \( a : k_m = k_0(1 + \Delta) \) with \( \Delta = o(1) \).

(b) Consistency of the development. — We will develop below both the advection term and the diffusive term to their dominant order, that is \( a^2/R \). However, since the Cross-Newell equations are explicit at order \( 1/R \) only, one has to pay attention that the dropped terms of order at least \( a/R^2 \) may not be comparable to those of order \( a^2/R \) considered here.

For the mean flow \( F \), we note that the terms of order \( a \) of the wavevector \( k \) are second degree polynomial functions. When applying two spatial derivatives, they can only lead at order \( a/R^2 \) to a constant flow, which then conflicts with the pattern symmetry and the boundary conditions. It is there-
fore balanced by the pressure gradient so that the mean flow $F$ is at least $O(a^2)$ at order $1/R^2$.

As shown by Cross and Newell [14], the next higher order term in $1/R$ of the diffusive part of the phase equation is $1/R^3$. It has to be taken into account if $a^2/R = a/R^3$, that is if $a \approx 1/R^2$. The wavenumber $k$ is therefore anywhere quite close to $k_0: k = k_0 + o(1/R)$. However, since the dominant distortion is a parabolic curvature, one can readily show by a higher order analysis similar to that of Cross and Newell, that the dominant order in $1/R^3$ is second order in $a: a^2/R^3$.

Both these remarks show that the expansion in both parameters $a$ and $R$ leads to a consistent result at order $a^2/R$.

2.2.3 Mean flow field. — As the mean flow $F$ is a non-divergent vector field, we take into account this property since the earliest stage, by introducing a stream function $\xi: F = \left(\frac{\partial \xi}{\partial y}, -\frac{\partial \xi}{\partial x}\right)$. We deter-

Fig. 2. — Transition to turbulence in a cylindrical container at $Pr = 0.7$. Experimental observations: (a) straight roll pattern near the convective threshold; (b) stationary distorted pattern at $\epsilon = 0.15$ and $\Gamma = 15$. Solutions of the Cross-Newell equations: (c) mean flow field in a cylindrical box. Notice the back flow along the line joining the centers of curvature; (d) stationary phase pattern determined for $a = 0.6$, $\epsilon = 0.15$, $\Gamma = 15$ and $Pr = 0.7$. Notice the agreement with (b) and especially the roll compression.
mine $\xi$ from the vertical vorticity $\Omega_z$, which may be expressed, on the one hand, in terms of the phase distortion by using the Cross-Newell expression, and on the other hand in terms of $\xi$, by definition of the stream function: $\Omega_z = -\Delta \xi$. We obtain at the lowest order in the phase distortion $a$:

$$\Omega_z = [4 \gamma k_0^2 A^2(k_0)/R^4] \times \left[a^2(1 - 5p) - 3d(1 + p)\right] xy + o(a^2/R^2)$$

and, by integration:

$$\xi = - [(1/3) \gamma k_0^2 A^2(k_0)/R^4] \times \left[\frac{d\log(A^2)}{dk} \right]_{(k_0)} + x y + o(a^2)$$

where $\alpha$, $\beta$, $\gamma$ are integration constants,

$p = k_0^2 \left(\frac{d\log(A^2)}{dk} \right)_{(k_0)}$ and $r^2 = x^2 + y^2$.

We deduce from $\xi$ the expression of the mean flow $F$:

$$F = \gamma \frac{k_0^2 A^2(k_0)}{3 R^4} \left[a^2(1 - 5p) - 3d(1 + p)\right] \times \frac{x}{r^2} + 2xy + o(a^2/R^2).$$

The boundary condition enables us to fix the constants $\alpha$, $\beta$, $\gamma$. It is required that $F \cdot r$ vanishes at the boundary $x^2 + y^2 = R^2$. This is possible only if $\alpha = -R^2$, $\beta = 0$, $\gamma = 0$, so that the mean flow $F$ reads:

$$F = \gamma \frac{k_0^2 A^2(k_0)}{3 R^4} \left[a^2(1 - 5p) - 3d(1 + p)\right] \times \frac{x}{(R^2 - r^2)} + 2xy + o(a^2/R^2) .$$

The mean flow $F$, produced by a distorted phase field $\varphi$ belonging to the functional space FS, and satisfying the lateral boundary conditions is therefore known at order $a^2/R$. This result realizes a closure of the scale interactions, since it enables us to determine what flows on the large scale are produced by the small scale roll flows. This successful analytic determination of $F$ may not be possible for other phase patterns and/or other boundary shapes. Nevertheless, it enables us here to reduce the dynamics of the phase field to a dynamical system for the expansion parameters $a$, $b$, $c$, $d$, $\Delta$.

Whatever the distorted phase field belonging to the functional space FS, the mean flow field that it produces bears the same form. Only its amplitude depends on the parameters $a$ and $d$. This common form is displayed in figure 2c, where the mean flow vector field is plotted at each point of a square lattice.

We note that four vortices are present. This result could have been anticipated by symmetry arguments. Indeed, the vertical vorticity of the mean flow field must be antisymmetric w.r.t. any reflection symmetry of the phase field. Moreover, for the kind of pattern that we consider here, since the phase field is a fifth order polynomial, $\Omega_z$ must be a second order polynomial. Finally, since the phase field is symmetric w.r.t. the $x$ axis and antisymmetric w.r.t. the $y$ axis, $\Omega_z$ must only bear the form $\Omega_z \propto xy$ which leads to four mean flow vortices.

Apart from the value of the vertical vorticity, the actual mean flow field must satisfy the boundary conditions. Because the container is closed, a back flow is produced along the $y$ axis from the walls towards the center of the pattern and is displayed in figure 2c. This effect has important consequences because, according to the study of forced mean flow effects [17], it enhances the roll compression. It is thus likely to drive the pattern towards the phase instability domain.

2.2.4 Criterion of stationarity. — We introduce the above determination of the mean flow field into the phase equation. We next develop it up to second order in $a$ and look for stationary states. A stationary state is reached, provided that

$$\left(\frac{k^2 dB}{\tau dk}\right)_{(k_0)} \left[10 a^2 y^2 + 6b y R^2 + 20 c y^3 R^2 + 6d y^2 R^4 - 2a \Delta \frac{y}{R^3} \right] +$$

$$+ \left[\frac{\gamma k_0^2 A^2(k_0)}{3 R^4} \right] [a^2(1 - 5p) - 3d(1 + p)][3 x^2 y - y^2 y + y^3] = 0 .$$

This condition is fulfilled if the following criterion for the expansion parameters $a$, $b$, $c$, $d$ is achieved:

$$10 a^2 + 6d = \alpha \left[a^2(1 - 5p) - 3d(1 + p)\right]$$

$$6b - 2a \Delta = - (\alpha/3)[a^2(1 - 5p) - 3d(1 + p)]$$

$$20c = (\alpha/3)[a^2(1 - 5p) - 3d(1 + p)]$$

Near the foci, the rolls are parallel to the sidewalls. Since the mean flow is tangent to the sidewall, it is directed in the vicinity of the foci along the roll axis.
No phase advection terms thus occur in these areas. The wavenumber selection mechanism by focus singularities at the wavenumber \( k_0 \) is then likely to operate and to be achieved in the stationary state [36, 14, 13]. We will thus consider that such states display wavenumbers equal to \( k_0 \) at the locations \((0, R)\) and \((0, -R)\). This condition fixes the parameter \( \Delta \) to: \( \Delta = -3b - 5c \). This parameter is therefore second order in \( a \) and may be neglected in the above stationarity criterion.

The solution of this criterion relates \( b, c, d, \Delta \) to \( a \):

\[
\begin{align*}
b &= -a^2[2\alpha/3]/[2 + \alpha (1 + p)] \\
c &= a^2[\alpha/5]/[2 + \alpha (1 + p)] \\
d &= a^2[1/3]\left[\alpha (1 - 5p) - 10\right]/[2 + \alpha (1 + p)] \\
\Delta &= a^2\alpha/[2 + \alpha (1 + p)].
\end{align*}
\]

It shows that \( b, c, d \) are order two in \( a \), in agreement with the previous ansatz.

At this stage, we have obtained, at each Rayleigh number, a continuous family of stationary solutions of the phase field at order \( a^2/R \). However, experimental observations show that, at each Rayleigh number, a single stationary solution is obtained [7].

The breaking of this degeneracy is produced by a phase boundary condition which is present in a closed box at the sidewall, but which has not yet been involved in our analysis.

### 2.2.5 Phase boundary condition.

In argon gas, when the Rayleigh number is increased from its value at onset, one observes that the roll axes tend to become more and more perpendicular to the sidewall [7]. At onset of convection, the rolls are straight so that they end at the sidewalls with various angles (Fig. 2a). Two grain boundaries take place quite close to the sidewalls but, since they disappear as soon as the Rayleigh number is increased, we can disregard them in the following. Nevertheless we must stress that, contrary to experimental observations at moderate Prandtl number [23, 24], such a pattern is stationary. At the threshold of convection, any angle between the rolls and the sidewall is then allowed. On the contrary, above \( Ra = 1.2 Ra_c \), each roll ends perpendicularly to the walls (Fig. 2b). In this experiment, the phase boundary condition is therefore dependent on the Rayleigh number. This effect may be produced by horizontal thermal gradients which are expected from the thermal conductivity difference between the horizontal plates, the fluid and the sidewalls. However, there is another reason specific to this experiment. To avoid an under pressure water circulation system, the under pressure sapphire plate is surrounded by a thermalizing copper plate in closed contact. Because of visualization, a hole has been made in this copper plate in front of the convective medium, hence leading to a non homogeneous thermalization [7].

It would therefore be interesting to determine the phase boundary condition when such thermal effects are likely to be present. However, this task would be difficult to perform while its effect regarding the selection of stationary solutions can be straightforwardly handled in a phenomenological way. For this, we only need to prescribe a phase boundary condition at a given location of the sidewall, for example at \((R/\sqrt{2}, R/\sqrt{2})\). This boundary condition is localized, but sufficient to discriminate among the possible stationary solutions. It then acts in the same way as the complete boundary condition, without being so complex. We therefore hope that it gives a good insight into what an exact treatment should lead to.

The angle \( \theta \) between the modelled phase field wavevector \( k \) and the wall normal \( n \) at \((R/\sqrt{2}, R/\sqrt{2})\) is proportional to \( k \cdot n = k_0 R/\sqrt{2}[1 - 3\alpha/2 + o(a^2)] \). At the convective threshold, since straight rolls are observed, \( a = 0 \). However, experiments show that \( \theta \) decreases to 0 as the Rayleigh number increases, in such a way that no noticeable evolution occurs when the Rayleigh number is greater than 1.2 \( Ra_c \). This behaviour indicates that the curvature parameter \( a \) increases rapidly up to \( O(2/3) \) values as the Rayleigh number grows, but saturates at \( Ra = 1.2 Ra_c \). The detailed knowledge of \( a(\epsilon) \) depends on the actual experimental conditions and should require a specific analysis not performed here. Nevertheless, at each Rayleigh number, the phase boundary condition fixes the value of the curvature and therefore selects among the previous stationary states.

### 2.2.6 Stationary phase fields.

In a closed container, as the Rayleigh number increases, the rolls must approach the walls normally. In order to accommodate this phase boundary condition, the stationary fields must bend. This behaviour does not depend on the Prandtl number [37]. However, the response of the pattern to this geometrical bending does.

When the Prandtl number is infinite (\( \alpha = 0 \)), no mean flow is generated. The equilibrium only involves the phase diffusion process [32]. However, because of the symmetry of the pattern, the curvature produces a slight spread in the wavenumbers. Over the pattern, some wavenumbers have to be different from the equilibrium wavenumber versus bending \( k_0 \). The pattern bending cannot therefore be a stationary distortion and must be balanced by a slight compression. This explains why among the parameters \( b, c, d, \Delta \), one of them does not vanish \( d = -5a^2/3 \). This result shows that, even at infinite Prandtl number, distorted phase fields may be stationary, because of the finite size of the container. This emphasizes the fact that wavenumber selections are only valid in the infinite aspect ratio limit.
When the Prandtl number is finite, mean flows are generated. They lead to a compression of the rolls towards the center of the container. The wavenumber increases along the y axis, from \( k_0 \) at the walls to \( k_{m} > k_0 \) at the center. The equilibrium of the pattern now involves the coupling between the mean flow field and the phase field. A stationary distortion is such that it produces the mean flow field which in turn balances it by a phase advection. Such a stationary distortion involves all the distortion modes \((a, b, c, d, \Delta)\) and is displayed in figure 2d.

In each of the above cases, the distortion is generated at first stage by the phase boundary condition. At the second stage, the bulk of the pattern reacts through various distortion modes depending on the amplitude of the generated mean flows. However, we stress that this pattern behaviour is nothing but a non-local effect whose origin is the phase boundary condition at the walls.

2.2.7 Comparison with experimental results. — In order to compare the stationary phase solutions to the experimental observations, one has to determine the parameters \( \alpha \) and \( \beta \), and thus to infer the expression of the functions \( A, B, \tau \) and \( \gamma \).

When no mean flow is generated, the functions \( A, B, \tau \) are determined at each Prandtl number by a perturbative expansion close to the convective threshold [38]. They are:

\[
A^2 = A_0^2 [\epsilon - \xi_0^2 (k - k_c)^2], \\
B = - \xi_0^2 A^2 (k - k_c)/k_c, \\
\tau = \tau_0 A^2
\]

with

\[
A_0^2 = (0.6995 - 0.0047 Pr^{-1} + 0.0083 Pr^{-2})^{-1}, \\
\xi_0^2 = 0.148,
\]

and

\[
\tau_0 = (0.5117 Pr^{-1} + 1)/19.65
\]

When mean flows are produced by phase distortions, their introduction in the phase equation raises a difficulty. Indeed, this equation is obtained by a solvability condition which involves a projection on the basis straight roll solution. This leads to integrate over the vertical axis \( z \), the product of the vertical dependence of this basis solution with the advection term of the convective variable by the parallel flow \( F(z) \). The result can be expressed in the form \( k \cdot F^* \), where \( F^* \) is related to \( F(z) \) through a complicated \( z \)-integration. The difficulty arises from the fact that the non-divergence condition of the mean flow together with its boundary conditions involves a simple integration over the \( z \) axis, which results in another field \( F \). It would therefore be difficult to express the boundary condition operating on \( F^* \). To overcome this problem, Cross and Newell have shown that the field difference \( F^* - \mu F \), where \( \mu \) is a suitable constant, leads to a phase advection term which can be included in the diffusive part of the phase equation, provided that \( B \) and \( \tau \) are renormalized [14]. We will call \( B^* \) and \( \tau^* \) the corresponding functions. In this way, the phase equation only displays a divergenceless flow field \( \mu F \), which satisfies the actual boundary conditions. By extension, we will also called it \( F \). Such a flow corresponds to the Cross-Newell expression (2) with \( \tau = 0.42 Pr^{-1}(Pr + 0.34)/(Pr + 0.51) \).

As shown by Manneville and Piquemal [13] and Cross [12], it is worth emphasizing that, even when the mean flow produced by distortion is thoroughly balanced by a pressure gradient, a net effect on the pattern phase remains and is taken into account by the above renormalization of \( B \) and \( \tau \). This occurs especially for a centred axisymmetrical roll pattern in a closed box, where the mean flow \( F \) must vanish, according to symmetry arguments: since the phase pattern is symmetric w.r.t. any line crossing the center, the vertical vorticity must be antisymmetric w.r.t. each of these reflections and therefore vanishes. However, because \( z \)-integrated terms involve the parallel flow \( \tilde{F}(z) \), the diffusion coefficient \( D_\perp = -B/\tau \) is modified despite \( F \) vanishes. It becomes \( D_\perp = -B^*/\tau^* \) and it has been determined by Manneville and Piquemal [13]. Unfortunately, the separate determinations of \( B^* \) and \( \tau^* \) are not available but, for \( Pr = 0.7 \), their ratio \( (-D_\perp) \) is quite close to the value which would be obtained without any mean flow contribution. In order to obtain a determination of the parameters \( \alpha \) and \( \beta \) for \( Pr = 0.7 \), we assume that \( B^* \) and \( \tau^* \) do not differ from \( B \) and \( \tau \). We expect the errors to be of higher order in \( \epsilon \) and \( (k - k_\perp) \). We thus obtain \( k_\perp = k_c \), \( p = 0 \) and \( \alpha = \gamma A_0^2 k_c^2 (\tau_0/\xi_0^2) \epsilon \), i.e. at \( Pr = 0.7 \), \( \alpha = 4.19 \epsilon \).

In order to compare to experimental results, we must consider values of the curvature parameter of order 1. Although this is beyond the validity domain of our development, we assume that the extrapolation of our results gives a correct approximation of what an exact treatment should lead to.

Assuming that \( \epsilon = 0.15 \), the phase boundary condition produces an \( O(2/3) \) value of the curvature parameter \( a \), we determine the corresponding phase field. It is drawn in figure 2d for an aspect ratio \( P = 15 \) and displays a roll compression in the center of the pattern. We notice the agreement with the experimental observation under the same conditions (Fig. 2b) [19, 20].

Another interesting comparison with the experimental observations at \( Pr = 0.7 \) is the growth of the wavenumber band in stationary regimes. Assuming that the curvature parameter grows as \( a = 0.7 (\epsilon/0.2)^{3/2} \) from the convective threshold and saturates above \( \epsilon = 0.2 \), we deduce the growth displayed in figure 3a. As shown in the next section, similar
results should be obtained with other relations between \( a \) and \( \varepsilon \), provided that \( a \) is \( O(1) \) for \( \varepsilon \) around 0.2. The agreement with the experimental results [7] displayed in figure 3b is good, especially concerning the Rayleigh number at which the maximal wavenumber crosses the straight roll stability domain, called the Busse and Clever balloon. However, the minimal wavevector decreases more rapidly than in the experiments, but we must notice that it is localized at the sidewalls, in a region disturbed by the phase boundary conditions. The phase instability mechanisms are therefore likely to be altered so that one cannot decide whether this should produce phase instabilities closer to the convective threshold than for the maximal wavenumbers. However, experimental observations do not reveal instabilities at the sidewalls [7].

The main agreement with the experimental observations is the value of the Rayleigh number for which the maximal wavenumber at the center of the pattern reaches the Busse and Clever balloon. Phase instabilities are then likely to develop. This question is treated in the following part. It leads to emphasize the role of non-local effects in the route to turbulence followed by these convective patterns.

3. Route to turbulence in a cylindrical container.

3.1 DELOCALIZED INSTABILITIES. — The phase patterns observed in experiments have been modeled in a functional space FS which proved to catch the dominant distortion. It has been shown that both the diffusive term and the advection term of the phase equation belong to FS at the dominant orders. This property means that the self dynamical behaviour of a phase pattern belonging to FS is an internal problem of FS. This provides a closure of the expansion and enables us to treat easily the stability of the FS-stationary fields w.r.t. perturbations belonging to FS. However other perturbations external to FS cannot be treated in this way. Some of them give rise to localized instabilities and are studied further.

Linearizing the dynamical equations around a stationary solution leads to the eigenmodes and the eigenvalues of the perturbations. The most dangerous mode is an amplification of the curvature parameter \( a \) and displays the eigenvalue: 
\[
\lambda = 2(a/R)(\xi_0^2/\tau_0)(\alpha (1 - 5\rho) - 10).
\]

The time scale of this instability is the horizontal diffusion time \( \tau_H \propto R^2 \). The instability occurs for \( \alpha > 10 \) when \( \rho = 0 \), i.e. \( \varepsilon > 2.39 \). It arises when the advection term of the phase equation overcomes the stabilizing effect of the diffusive term. It is similar to the skewed-varicose instability of infinite straight roll patterns [12, 8] but it occurs here on distorted rolls and in a closed box.

In the instability domain, the curvature grows together with the other distortion modes. The pattern evolves as a whole, so that the instability is delocalized. However, we want to emphasize below that localized instabilities are likely to develop before the pattern becomes globally unstable. These localized instabilities thus appear to be the first dynamical events encountered on the route to turbulence [7].

3.2 LOCALIZED INSTABILITIES. — For a sufficiently high aspect ratio and away from defects, the distortion of each roll is small. One may therefore consider that at a local level, a convective pattern consists of patches of nearly parallel rolls in each of which the instabilities of a straight roll pattern may occur. As these instabilities do not affect the whole pattern but only a part of it, they may be called localized instabilities. They are linked to the occurrence of local unstable wavenumbers.

Since the roll patches are spatially limited, one could expect the thresholds of the straight roll instabilities to be increased, due to finite size effects. However, considering that the box aspect ratio is sufficiently large to make the threshold shift too
small to be relevant, we will disregard this problem here. As a support, no large shift could be observed when a localized Eckhaus instability was driven by a mean flow in a roll chain with aspect ratio $\Gamma = 61$ [17].

A necessary condition for the pattern stability is the stability of each of its patches: namely a local stability. However, we stress that a global analysis has nevertheless to be performed in order to determine both the global stability and the local values of the phase field and of the mean flow field. Actually, one can never overcome the non-local treatment. As an example, one can only understand the presence of a back flow in the center of the pattern by a non-local analysis including the boundary conditions. However, once the local values of the mean flow field and of the phase field are determined, then the stability of each patch of the pattern may be studied and becomes meaningful.

The band of wavenumber of a stationary pattern grows with the Rayleigh number for two different reasons: the growth of the curvature parameter $a$ and the growth of the roll amplitude $A$. Only the first one is involved in the decrease of the minimal wavenumber $k_\ast = k_0(1-a)$. On the contrary, both of them act to increase the maximal wavenumber: $k_\ast = k_0(1+a^2 \alpha/(\alpha+2))$. The last cause for the wavenumber evolution versus $\varepsilon$ should be the decrease of $k_0$ as the Rayleigh number grows. However, for $Pr = 0.7$, the results of Manneville and Piquemal show that this evolution is rather slow: $k_0(\varepsilon) = k_c(1+0.026 \varepsilon)$ [13]. As it will be dominated by the wavenumber band growth, we will disregard it in the following, taking $k_0$ to $k_c$.

A lot of analytical studies have focused on the straight roll instabilities [11-14]. They led to express the boundaries of the Eckhaus, the zigzag, and the skewed-varicose instabilities in terms of the functions $A, B*, r*, \gamma$. It would therefore be possible to determine analytically the Rayleigh number at which a wavenumber crosses the stability boundaries. However, since this process is driven by the phase boundary condition it depends on the experiments. Moreover, the expressions of the wavenumbers $k_\ast$ and $k_\ast$ have to be extrapolated for values of $a$ of order 1 which are beyond the validity domain of our development. A detailed analytical study would then be irrelevant. Nevertheless, we will emphasize below that the threshold of turbulence is not sensitive to the phenomenological description of the phase boundary condition. Owing to this property, its value will be plausibly related to the various available experimental situations.

For $\varepsilon$ greater than 0.2, the curvature parameter $a$ reaches $O(2/3)$: each roll is nearly normal to the boundaries. Extending the previous results for such values of $a$ as in 2.2.7, we find the following expression of the maximal wavenumber $k_\ast = k_c(1 + a^2 \alpha/(\alpha+2))$, where $a^2 \approx 0.5$, $k_c = 3.117$ and for $Pr = 0.7$, $\alpha = 4.19 \varepsilon$. As can be seen in figure 3b, the phase instabilities for $k > k_c$ and $\varepsilon > 0.2$ occur at $Pr = 0.7$ for $k \approx k_\ast + 0.5$. When $a$ is $O(2/3)$, this is achieved for $a \approx 1$ and so for $\varepsilon_1 \approx = 1/4.19 = 0.2$.

Above $\varepsilon = \varepsilon_1$, no defectless patterns such as those belonging to FS can be stationary at $Pr = 0.7$. Moreover these patterns produce propagating singularities i.e. dislocations. Although we cannot rule out that more complicated patterns could be stationary, we note that none of them have been observed close to $\varepsilon_1$ [26]. Our result shows that, owing to mean flow effects, phase turbulence at $Pr = 0.7$ can occur for values of $\varepsilon$ far below the highest allowable value $\varepsilon_B$ of $\varepsilon$ of the Busse and Clever balloon ($\varepsilon_B \approx 2.5$). The validity of this statement does not depend on the precise phase boundary conditions, provided that perpendicular rolls are achieved for $\varepsilon > O(0.1)$, and that the box is closed. Nevertheless it is essential that the box shape be cylindrical. This result finally proves that turbulence at low Prandtl numbers may occur close to the convective threshold in agreement with the experiments of Ahlers and Behringer [5]. In particular, we note that, since Nusselt number measurements are consistent with nearly straight roll patterns near the onset of convection [17], the spatial structures involved in these experiments could be relevant to our study.

This result finally confirms the interpretation of Manneville of the occurrence of weak turbulence in extended geometry in terms of a dynamical frustration [15]. Indeed the strong amplitude ($\gamma$) of the generation of mean flows at low Prandtl number produces a lack of compatibility between the mean flow field and the pattern stability.

### 3.3 Routes to Turbulence at Different Prandtl Numbers.

#### 3.3.1 High Pr.

— At high Prandtl numbers, the mean flow amplitude is too small to induce noticeable modifications to the route to turbulence of infinite Prandtl numbers fluid. The band of wavenumbers is even too small to be measured in experiments [3-5]. Various wavenumber selection mechanisms then dominate pattern evolution [30, 31, 36, 39, 40] and, as expected in a variational situation, they are compatible [30, 40], except may be for the sidewall effects [41]. No permanent evolution can arise because convection obeys a variational principle. Stationary roll patterns are thus obtained after transients, unless the highest allowable value of $\varepsilon$ of the Busse balloon, $\varepsilon_B$, is reached. Pattern cells of another kind than roll then arise and lead, for further increases of the Rayleigh number, to a turbulence reminiscent of a structure fusion.
3.3.2 Low Prandtl numbers. — On the contrary, at low Prandtl numbers, the mean flow amplitude is sufficiently large to strongly modify the route to turbulence. In a cylindrical cell and for patterns displaying two opposite foci, our study shows, in agreement with experiments, that the band of wavenumbers grows so much with the Rayleigh number that it crosses the Busse and Clever balloon by its sides, hence for values of $\varepsilon$ of order 0.1 far below $\varepsilon_B$.

3.3.3 Moderate Prandtl numbers. — At Prandtl numbers slightly higher than 0.7, one may wonder in what extent is the route to turbulence of patterns displaying two opposite foci modified. Indeed, one has to take into account the dependence on the Prandtl number of both the Busse and Clever balloon and the amplitude of mean flow. In the following, we use the extrapolated and interpolated determination of Kolodner et al. at $Pr = 3.5$, which is similar to that obtained by Bolton et al. at $Pr = 2.5$ by direct calculation [42]. Owing to the dependence on the Prandtl number, the parameter $\alpha$ is 0.629 $\varepsilon$. Extrapolating the expression of the maximal wavenumber $k_+$ to high values of $\varepsilon$, we note in figure 4 that this wavenumber crosses the instability boundaries at the decreasing branch in the $Ra, k$ plane of the skewed varicose instability. The amplitude of mean flows is thus too small to enable the pattern to cross the Busse and Clever balloon by its sides. The instability domain can only be reached by the top, as in the high Prandtl number case, and thus far from threshold, for values of $\varepsilon$ of order $\varepsilon_B$.

At high values of $\varepsilon$ ($\varepsilon = O(1)$), experiments in cylindrical containers show that patterns display more complicated spatial features than two opposite umbilics. However, when the transition to turbulence occurs as in reference [23] through roll pinchings induced by umbilics, we may infer that its basic mechanism is related to our study. It therefore seems to us highly plausible that if patterns with two opposite umbilics are stable, then no sustained time dependence can be obtained by a roll pinching mechanism. This gives an explanation for the high value of the threshold of turbulence ($\varepsilon = 3.5$) in the Heutmaker and Gollub experiment [23]. On the other hand, despite the presence of some defects in the convective field, time dependent patterns above $\varepsilon = 3.5$ mainly display in this experiment two opposite umbilics which produce roll pinching in the center of the cell. Owing to this great similarity, the mechanisms of time dependence must be quite analogous to those occurring at the transition to turbulence at $Pr = 0.7$ and are thus involved in the study of part 3.4.

It is worth noticing that in the same fluid and despite the rectangular shape of the container, Gollub et al. have sometimes observed patterns displaying opposite foci [43]. Rolls were here too compressed in the middle of these foci. Roll pinchings were also noticed in this experiment and linked to the skewed-varicose instability. It therefore seems that similar phenomena as those studied in a cylindrical container in the present paper could also arise in containers of other shapes.

3.3.4 Conclusion. — The comparison between the $Pr = 0.7$ and $Pr = 3.5$ regimes indicates a significant evolution of the features of the route to turbulence in this range of Prandtl numbers. Especially the threshold of turbulence arises at $\varepsilon$ of order 0.1 ($\varepsilon \ll \varepsilon_B$) at $Pr = 0.7$ and at $\varepsilon$ of order 1 ($\varepsilon = \varepsilon_B$) at $Pr = 3$. Nevertheless, in both cases and contrary to the high Prandtl number regime, the wavenumber band is noticeable. It leads the pattern wavenumber to cross the balloon by the top for $Pr > 3$ and by the sides for $Pr < 1$, but, in both cases, turbulence is sustained by a roll pinching mechanism. These results provide a comprehensive definition of moderate and low Prandtl numbers.

3.4 Non-local mechanisms on the route to turbulence. — As shown by experiments, the transition to turbulence at $Pr = 0.7$ occurs when the maximum wavenumber at the center of the pattern crosses the straight roll marginal stability. Roll compression is therefore the distortion by which the
instability arises. However, several mechanisms act simultaneously to end this phenomenon. They emphasize non-local features and the role of the phase boundary conditions. We summarize them in the following.

The phenomenon which is at the birth of the route to turbulence is the formation of curvature due to sidewall effects onto the rolls [37]. However, because horizontal thermal gradients are induced by sidewalls, the curvature grows continuously from the convective threshold in experiments [7].

At finite Prandtl number, the curvature generates a mean flow field which participates to the pattern dynamics. However, the actual mean flow field depends on the boundary condition so that in a closed container a back flow crossing the rolls has to be present. This enhances the compression and brings the wavenumber at the center of the cell into the instability domain.

The transition to turbulence at finite Prandtl number therefore appears to be produced by a non-local response to boundary conditions: the phase boundary condition for the generation of curvature, and the mean flow boundary condition for the generation of a back flow. We then conclude that, in cellular structures, the route to turbulence is parametrized by the boundary conditions, either by their shapes or by any other suitable feature. As a support, we may notice that when a straight roll pattern is sustained by a sidewall forcing at the shortest sides of a rectangular container, the transition to turbulence occurs for $\varepsilon \approx 2.5$ [19, 20] in argon and at a similar large value of $\varepsilon$ in liquid helium [9]. This indeed corresponds to the experimental realization of the numerical studies of Busse and Clever [8] and emphasizes the role of the phase boundary condition.

At first sight, since the transition to turbulence occurs from localized instabilities, one could think that it bears weak similarity with the mechanisms of the skewed varicose instability. However, concerning the way the mean flows act to destabilize the pattern it is quite similar. Indeed, the study of the straight roll pattern instability has shown that mean flows produced by curvature act in a stabilizing way [11-13], while the mean flows produced by compression are likely to destabilize the roll pattern [12]. However, these destabilizing flows need a bidimensional distortion to operate; otherwise, they are cancelled by a pressure gradient in order to satisfy the mass conservation. When the skewed-varicose instability arises, the curvature thus enables mean flows to work in compression, and participates in the mechanisms of instability. Although the direct influence of curvature is stabilizing, it is at the origin of the skewed varicose instability by allowing the destabilizing mean flows to operate [12]. A similar mechanism occurs in the present route to turbulence: the curvature produces mean flows which would be stabilizing if a back flow was not produced in turn owing to the boundary conditions.

3.5 DYNAMICAL REGIME. — When the maximal wavenumber $k_*$ becomes unstable, at a value that we denote $k_*(\varepsilon)$, a phase instability develops at the center of the pattern and pinches a roll pair. A dislocation pair then nucleates, so that the phase pattern no longer belongs to the functional space FS. Its evolution cannot be treated by the above derivation but is determined by the experimental observations: both dislocations climb to the sidewall and glide towards the foci where they disappear [7].

At this time, since the phase pattern is defectless, it belongs once again to FS. At order $a^2/R$ of the expansion and thus on a time scale $\tau_1 \propto R^2/a$, its evolution is therefore a relaxation towards one of the stationary phase patterns determined previously. Since a roll pair has been expelled from the pattern, we expect the maximum wavenumber $k_0$ of this stationary pattern to be shifted from $k_1$ by the amount $\mu/R$ with $\mu = O(1)$: $k_0 = k_1 - \mu/R$. In fact, at higher orders of the expansion of the dynamical system, and thus on longer time scales, the pattern still evolves, but we have not handled these higher-order evolutions. Nevertheless, we may expect the quasistationary phase patterns (i.e. the patterns stationary on the time scale $\tau_1 \propto R^2/a$) to evolve on the time scale $\tau_2 \propto R^3/a$ towards a final state whose maximal wavevector is noted $k^*$ ($k^* > k_1$). If the dynamical behaviour were not altered by a localized instability, the maximal wavenumber $k_*$ of the pattern should follow a relaxation evolution towards the value $k^*$: $k_* (t) = [k^* - k_0][1 - \exp (-t/\tau_2)] - k_0$, the origin of time being the previous dislocation elimination. However, a localized instability is triggered when $k_* = k_1$ so that a new defect nucleation occurs at $T \approx -\tau_2 \log \left[ (k^* - k_1)/(k^* - k_0) \right]$.

Owing to the dependence of the wavenumbers $k^*$ and $k_1$ upon $\varepsilon$, the basic period $T + \tau_1$ of the roll pinching mechanism is related to $\varepsilon$. At $Pr = 0.7$, we estimate from our stationary solutions and from the Busse and Clever balloon the following evolution versus $\varepsilon$: $k^* = k_1(\varepsilon_1) + \left[ k_1(\varepsilon_1) + 2 \varepsilon / (x + 2) \right] (\varepsilon - \varepsilon_1)$ i.e. $k^* = k_1(\varepsilon_1) + 1.2 (\varepsilon - \varepsilon_1)$ and $k_1(\varepsilon) = k_1(\varepsilon_1) - 0.7 (\varepsilon - \varepsilon_1)$, where $\varepsilon_1$ is the threshold of time dependence. This enables us to deduce the following relation between $T$ and $\varepsilon$: $T(\varepsilon) \approx -\tau_2 \log \left[ 2(\varepsilon - \varepsilon_1) / \left( 2(\varepsilon - \varepsilon_1) + \mu/R \right) \right]$. At the threshold of time dependence $\varepsilon_1$, the basic period is infinite but it decreases rapidly as $\varepsilon$ increases. It becomes similar to $\tau_2$ for $\varepsilon - \varepsilon_1 \approx \mu/(2R(\varepsilon_1 - 1))$ where $x = O(1)$ and $\mu = O(1)$ and so for $\varepsilon - \varepsilon_1 = O(10^{-3})$ for $R = O(10)$. Since at onset of time dependence $a$ has reached $O(23)$
values, we expect $\tau_2$ to be of order $R^3$. This result agrees with the prediction of Cross and Newell [14]; according to these authors, the earliest time to reach complete equilibrium scales like $R^3$.

Moreover, we emphasize that outside this range of $\epsilon$, only short periods ($T < \tau_2$) can be displayed. For sufficiently small $T$, the dislocation nucleations could even occur before the previous dislocations have disappeared. This should result in the generation of many defects in the convective pattern and hence in a chaotic behaviour. Especially, we notice that for $\Gamma = 57$, this range of $\epsilon$ becomes as small as a few $10^{-4}$, so that a chaotic behaviour is likely to appear just above the threshold of time dependence, as observed experimentally [6]. Vanishing of the range of $\epsilon$ of the long period regime in the limit of infinite $R$ emphasizes the dependence of the onset of chaos on the aspect ratio.

The vanishing frequency of this scenario as $\epsilon$ tends to $\epsilon_1$ is in agreement with the type II periodicity reported in liquid helium and in a cylindrical container for $R \geq 6$ [10]. These periodic evolutions also display the features of relaxation oscillations, in agreement with the above determination. A fit of the law $T(\epsilon)$ is uneasy to perform because the data have mainly been obtained at fixed heat current rather than at fixed $\epsilon$. Nevertheless, for $R = 9.002$ and $Pr = 0.54$, the periods of the oscillations close to the threshold of time dependence are consistent with the following values of $\tau_2$ and $\mu$: $\tau_2 = 0.3 R^3 \tau_\nu$ and $\mu = 0.16$.

In argon, the vanishing of the frequency of oscillation at the onset of time dependence has not been observed [7]. Instead, complicated windows involving stationary states, various periodic behaviours or chaos have been noticed and will be reported elsewhere [20, 26]. Quite near $\epsilon_1$ and for $R = 7.66$, the period is constant and leads to the value $T = 20$ min, while $\tau_\nu = 3.3$ s and $\tau_H = 191$ s. A small amplitude noise on $\epsilon$ could explain that no large period have occurred near $\epsilon_1$. Indeed, since $T$ should be similar to $\tau_2$ for $\epsilon - \epsilon_1 \geq O(10^{-3})$, the expected regime $T > \approx \tau_2$ would have been strongly perturbed if the noise amplitude has reached such values. Within this interpretation, the data are consistent for $\mu = 0.16$ with the following value of $\tau_2$: $\tau_2 \approx 0.7 R^3 \tau_\nu$.

Compared to the observations in liquid helium, the time scale $\tau_2$ is greater in argon. This discrepancy could indicate that the spatial patterns in liquid

Fig. 5.—Difference between phase fields just before and just after dislocation nucleations : (a) from stationary solutions of the Cross-Newell equations, one phase pattern displaying one more roll pair than the other ($\epsilon = 0.15$ and $Pr = 0.7$). Dashed and blank regions display opposite senses of roll displacement. Theses senses indicate that rolls travel towards the center of the container; (b) from digitized images of experimental observations at $\epsilon = 0.15$, $\Gamma = 15$ and $Pr = 0.7$. Since the contrast has been enhanced, the sharp phase oppositions which occur along the rolls are related to changes in the roll displacement sense. This enables us to define the same regions as in (a). We notice a good agreement with the picture (a). This visualization of the roll displacement sense indicates that the centers of curvature feed the pattern with new rolls.
helium are different from nearly straight roll structures. For instance these patterns might be composed of axisymmetric rolls as already observed in cylindrical cells [4, 5, 29, 44]. However, we notice that heat flux measurements are rather consistent with a nearly straight roll structure [18].

The dynamical regime provides a way to compare our determination of quasistationary phase patterns to the experimental observations. In order to enhance the phase evolution, it is helpful to subtract the stationary background. We then consider the field \( \Phi(x, y) \) obtained by subtracting the quasistationary phase pattern observed just after the dislocation elimination \( \varphi(x, y, \tau_1) \) from that observed just before dislocation nucleation \( \varphi(x, y, \tau_2) \): \( \Phi(x, y) = \varphi(x, y, \tau_2) - \varphi(x, y, \tau_1) \). As this is a difference between two periodic fields with close spatial frequencies, beatings arise and tell the way the pattern distortion has evolved. In figure 5b, we show the image difference of experimental fields. As the contrast has been enhanced, the beating gets sharp spatial variations. For this reason, we notice well defined domains which are in phase opposition, meaning that the sense of the roll displacements was opposite. The same behaviour is recovered in the functional space FS. Figure 5a shows the domains of equal roll displacement sense, obtained from two stationary phase patterns, one of them having one more roll pair than the other. This agrees with the observations and shows that the foci tend to feed the pattern with rolls. This phenomenon results from the existence of a black flow, i.e. a non-local effect which includes the influences of the boundary conditions. The mechanisms involved are therefore more complex than those described in [45].

4. Summary and conclusion.

Rayleigh-Bénard convection displays two interacting spatial scales: the roll scale and the box scale. It thus provides an elementary model system of scale interactions, the study of which may lead to the knowledge of the inner mechanisms involved in hydrodynamic phenomena and especially in turbulence. In this paper, three hydrodynamic problems still far from being understood and treated have been encountered: non-local effects, closing problem, and the influence of boundary conditions. They have been solved for nearly straight roll patterns occurring in a cylindrical container near the threshold of convection. This has provided an understanding of the transition to turbulence in this situation.

All the main features observed on the route to turbulence at \( Pr = 0.7 \) in a cylindrical container have been recovered with a good agreement: stationary distorted phase fields, localized instabilities at the pattern center as first dynamical events, and the threshold of turbulence at an \( \varepsilon \) of order \( 10^{-1} \), this provides an understanding of the appearance of the low Prandtl number turbulence in argon gas [7]. Since it proves that turbulence can occur spontaneously close to the convective threshold. This result enables us to recover the observations performed in liquid helium [6, 10]. It also gives an insight into the mechanisms at work on the route to turbulence. In particular, we notice their large dependence on the boundary conditions. In a closed container and for solid sidewalls, the boundary conditions produce on the one hand the phase distortions which generate mean flows, and on the other hand a back flow which compresses the rolls. Such a compression finally overestabilizes the pattern and leads to turbulence. When they are extrapolated to high Rayleigh numbers, these mechanisms give a partial understanding of the transition at moderate Prandtl numbers [23]. Finally, the instability of our solution illustrates the concept of dynamical frustration proposed by Manneville for an understanding of weak turbulence [16].

The strong dependence of the destabilizing mechanisms on the boundary conditions raises important questions. To what extent should similar routes occur for other sidewall shapes, other sidewall materials, or other basic roll patterns? Clearly, this calls for other experimental as well as theoretical studies in various situations. Another related problem involves the non-local features of disordered patterns: how should similar mechanisms operate when the basic pattern is divided into several nearly straight roll patches? Moreover, our study shows that, above a Rayleigh number threshold, no topologically simple phase patterns in a cylindrical container can be stationary solutions to the Cross-Newell equations supplemented by suitable boundary conditions. Defects are thus likely to appear, therefore introducing small scale singularities into the convective field. What non-local features do they bring into a disordered convective field? Could their motion be determined from a large scale behaviour or should a small scale be needed too? These few questions represent major problems to be solved in order to understand not only the transition but also the own regime of turbulence.

Our study has proved that non-local effects of spatial scales interactions could be solved at least in simple topological situations. We hope that such solutions may lead to a better understanding of the inner mechanisms of phase turbulence and that they could provide a guide to deal with more complex situations including disordered patterns containing defects, more complex geometries or more than two scale interactions. Steps of understanding in these areas would be especially exciting since they would provide new tools to attack the fascinating problem of turbulence.
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