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HAL Id: jpa-00210778
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Submitted on 1 Jan 1988

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Random matrix theory and universal statistics for disordered quantum conductors with spin-dependent hopping

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(Reçu le 17 décembre 1987, accepté le 3 février 1988)

Résumé. — Nous étudions les spectres des matrices de transfert de diffuseurs élastiques avec couplages spin-orbite aléatoires, dans la limite d’un grand nombre de canaux de diffusion. La conductance g de ces diffuseurs se déduit simplement de ces spectres. A l’aide d’une première méthode, inspirée de la théorie d’Imry des fluctuations universelles de conductance, nous décrivons ces spectres par une distribution d’entropie d’information maximum. Puis, utilisant une autre méthode, nous déduisons les matrices de transfert d’hamiltoniens microscopiques dont nous donnons la statistique. Nous montrons que la distribution des écarts de niveaux et la rigidité de ces spectres, déduites par ces deux méthodes, sont identiques et correspondent à la statistique de l’ensemble symplectique pour des tailles de diffuseur comprises entre le libre parcours moyen élastique et la longueur de localisation. La théorie d’échelle de la localisation est discutée à l’aide de cette approche nouvelle par la théorie des matrices aléatoires. Des conséquences importantes du couplage spin-orbite, relatives aux fluctuations universelles de conductance et au bruit en $1/f$, sont soulignées en conclusion.

Abstract. — Transfer matrix spectra, which are directly related to the conductance $g$ of a many channel elastic scatterer, are studied in presence of random spin-orbit interactions. A first method, inspired by Imry’s theory of universal conductance fluctuations, leads us to describe these spectra by a distribution of maximum information entropy. Using a second method, we write the transfer matrices in terms of a microscopic Hamiltonian, for which a statistical law is assumed. One shows that the level spacing distribution and the spectral rigidity, yielded by these two different procedures, are identical and agree with the statistics of the symplectic ensemble for sample size smaller than the localization length and larger than the elastic mean free path. Scaling theory of localization is discussed in the light of this new random matrix theory approach. Important consequences of spin-orbit coupling concerning universal conductance fluctuations and $1/f$ noise are emphasized.

A disordered system is by definition so complicated that it is a hopeless task to try to consider a given specimen : only a statistical ensemble, which is more or less arbitrarily defined, can be studied. This contrasts with experiments, which are in general performed on a single sample, but one could hope that, as the system size $L$ becomes large, a law of large numbers would apply, and that the conductance $g(L)$ of different members of the same ensemble would converge towards a unique ensemble-averaged conductance $\langle g(L) \rangle$. A scaling theory [1] of localization has been formulated, the parameter of which being precisely $\langle g(L) \rangle$. However, it is now well understood that $g(L)$ does not obey such a simple law of large numbers in absence of phase breaking process, but remains a random variable, whose value fluctuates significantly from one member to another of the same ensemble, even in the large $L$-limit. This fact has been appreciated a long time ago in topologically one-dimensional systems: Lyapunov exponents of random matrix products, instead of their averaged spectra, are considered in the transfer matrix approach of Anderson localization [2]. Similarly, the typical scaling behaviour of $g$ and the ensemble-averaged conductance have been distinguished by Anderson et al. [3] in one dimension. More recently, even in the metallic phase of two or three-dimensional systems, important fluctuation phenomena have been observed experimentally [4] on the dependence of $g$ as a function of the Fermi energy or the applied magnetic field. Perturbation analysis in the weak disorder limit [5, 6] has shown that the statistical variance of the conductance, averaged over an ensemble of
metal samples with the same macroscopic characteristics satisfies:

\[ \langle g^2(L) \rangle - \langle g(L) \rangle^2 \approx 1 \]  (1)

independent of its size (at \( T = 0 \)), or its average conductance. Therefore, these conductance fluctuations, which have been observed for a wide range of materials where inelastic effects can be neglected (or trivially taken into account), turn out to be both larger than expected by naive classical reasoning [7] and universal. Let us emphasize that relation (1) is valid when transport is diffusive (\( \ell \approx L < \xi \), where \( \ell \) is the elastic mean free path and \( \xi \) is the localization length). Conductance fluctuations are expected to be larger when localization occurs.

Since these fluctuation phenomena have remarkable universal features, it is very natural to ask what are the universality classes, and very plausible to believe that Dyson’s classification [8], which is known to be relevant [9] for our understanding of localization effects, is the answer. Let us recall that, following Dyson, any physical system is represented by a Hamiltonian which must belong to one of three possible ensembles, based mathematically on the orthogonal, unitary or symplectic groups. The orthogonal ensemble is appropriate to potential spin-independent scattering without magnetic field, and must be replaced by the unitary ensemble when time-reversal symmetry is destroyed by an applied magnetic field. The symplectic ensemble applies to odd-spin system without rotational symmetry, such as an electron gas with random spin-orbit scattering.

The fact that the statistic of \( g \), and consequently relation (1), must be changed by the presence of spin-orbit scattering becomes more transparent if one considers the recent work of Muttalib, Pichard and Stone [10] where the joint probability distribution of the eigenvalues of the transfer matrix has been obtained for the orthogonal and unitary ensembles. Let us recall the main ideas of this work before giving the details for disordered systems with spin-orbit coupling changes the symmetries of \( X \), \( A (dX) \) is modified. This yields different statistics for the \( \{X_i\} \) and a different distribution for \( g \). The next step consists to define what is the probability \( P (dX) \) that a matrix \( X \), characterizing a given system, belongs to the volume element \( dX \):

\[ P (dX) = \rho(X) \mu (dX) \]  (3)

and therefore to define a probability density \( \rho(X) \) on the matrix space. Here, we introduce a maximum entropy hypothesis, by taking for \( \rho(X) \) the most uniform probability density, given some constraints which contain all the known specifics of our model [15]. The choice of the constraints is somewhat arbitrary and turns out to be an ansatz. The better
our understanding of the physics of interest, the more appropriate is the choice of our constraints, and the better is our model defined by \( \rho(X) \). Our choice is the same as in reference \([10]\) : we impose a given spectral density \( \sigma(x) \) and assume that all the measurable macroscopic parameters are contained in this function. Let us mention that different constraints have been considered by Mello, Pereyra and Kumar \([16]\) for the same problem without spin-dependent hopping. Then integration of \( P(dX) \) over the variables related to the eigenvectors gives the joint probability distribution \( p(X_1, \ldots, X_N) \) for the eigenvalues.

In part 1, we first define the transfer matrix \( T \) through a disordered system invariant under time-reversal symmetry with spin-dependent hopping. The symmetries of \( T \) are given, implied by current conservation and time reversal symmetry. A matrix \( X \) is defined:

\[
X = U^+ X_c U
\]

\[
X_c = \frac{1}{4} [(T^+ T + (T^+ T)^{-1} - 2 I)]
\]

where \( U \) is a unitary transformation.

The conductance \( g \), using Landauer formula, is shown to be a linear statistic of \( X \)

\[
g = 2 \sum_{i=1}^{n} \left( \frac{1}{1+X_i} \right).
\]

In part 2, the invariant measure \( \mu(dX) \) is derived and expressed in terms of the eigenvalues and eigenvectors of \( X \). The symmetry analysis gives

\[
\mu(dX) \propto \prod_{i} dX_i \prod_{i<j} |X_i - X_j|^{\beta}
\]

with \( \beta = 4 \).

The interaction term between the eigenvalues is stronger than for systems with spin-independent hopping, without time-reversal symmetry (\( \beta = 2 \) characteristic of the unitary ensemble) or with time-reversal symmetry (\( \beta = 1 \) characteristic of the orthogonal ensemble) \([10]\). Therefore the spectrum of \( X \) is more rigid and the fluctuations of \( g \) are smaller.

In part 3, the amount of information \( I(\rho) \) contained in the probability density \( \rho(X) \) is defined, and among the various densities consistent with an \( a \) priori given spectral density \( \sigma(x) \) for \( X \), we choose for \( \rho(X) \) the one which carries the least amount of information \( I(\rho) \).

Integration over the eigenvectors gives:

\[
P(X_1, \ldots, X_n) = \sum_{i<j} |X_i - X_j|^4 \exp \sum_{k} \lambda_k \]

\[
\lambda(x) = \lambda_0 - \int dy \sigma(y) \ln |y-x|^4
\]

\[
\sigma(x) = \int \prod_{i=1}^{n} dX_i \, P(X_1, \ldots, X_n) \sum_{i=1}^{n} \delta(x - X_i)
\]

(6.3)

Part 4 is based on an entirely different approach. An Anderson model is defined by a microscopic Hamiltonian, for which a statistical law is assumed. \( X \) is deduced from the Hamiltonian and its eigenvalues are studied numerically, for metallic samples \((L \ll L \ll \xi)\), where \( \xi \) may be infinite.

The global variation of the spectrum, characterized by the spectral density \( \sigma(x) \) is studied in part 4.1. Since this function is taken as an \( a \) priori constraint, we sketch how the known behaviour for a disordered metal (Ohm's law, weak localisation correction) and the hypothesized scaling behaviour could be contained in \( \sigma(X) \). A more detailed study \([17]\) in the orthogonal case will be published elsewhere.

The local properties of the rescaled spectrum after discarding the effect of the fast variation of \( \sigma(x) \), are studied in part 4.2. The spacing distribution between consecutive eigenvalues and the spectral rigidity (statistic \( \Delta_3 \)), which are usually considered for characterizing the local fluctuations of the spectrum, are shown to agree accurately with the predictions of part 3, providing an independent test of the validity of the maximum entropy principle.

In part 5, important experimental consequences of the presence of spin-orbit scatterers in a metal, concerning both the universal conductance fluctuations and their possible implications for \( 1/f \) noise are emphasized, giving clear experimental tests for checking the relevance of quantum interference effects in these phenomena.

1. Transfer matrices, symmetries and conductance with spin-orbit coupling.

Let us consider a finite \( d \)-dimensional electronic system where disordered spin-orbit couplings are important in addition to the spin-independent random scattering. Invariance under time-reversal symmetry holds at very low-temperature in the absence of phase breaking processes inside the sample and at zero applied magnetic field. An electric current is supplied from both sides of the sample by two perfect leads with non-random spin-independent interactions, and where electron fluxes are quantized in \( 4n \) conduction channels. The flux amplitude of the \( 2n \)-incoming channels from the left (right) are given by a \( 2n \)-component vector \( \mathbf{a}_l \) :
where $a_t(q_i, \sigma)$ is an incoming flux amplitude from the left characterized by a transverse momentum $q_i$ and a spin $\sigma$ in the leads. Likewise, the amplitude of the outgoing channels are given by $b_t(a_r)$, as sketched in figure 1.

Fig. 1. — A schematic representation of a multichannel scatterer.

In absence of inelastic process in the sample, conduction electrons keep a well-defined Fermi energy $E_F$, and the fluxes standing at the right side are related to the left fluxes by a $4 \times 4$ multiplicative transfer matrix $T$:

$$
T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

(1.1)

$T$, which can be visualized as representing for a given energy $E_F$ a very complicated mesoscopic many-channel disordered elastic scatterer with spin-orbit interactions, cannot be an arbitrary complex matrix, since charge current is conserved ($\Im a_t = \Re b_t = |a_t|^2 - |b_t|^2$), which yields

$$
T^+ I_c T = I_c \\
I_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(1.2)

$\mathbb{I}$ denotes the identity matrix and $T^+$ is the hermitic conjugate matrix of $T$. In addition $T$ is time-reversal invariant, and commutes with $Z_c$. $Z_c$ represents the operation of time-reversal in the channel representation.

$$
[Z_c, T] = 0
$$

(1.3a)

$$
Z_c \begin{pmatrix} a \\ b \end{pmatrix} = K \begin{pmatrix} a^* \\ b^* \end{pmatrix}
$$

(1.3b)

$$
K = \begin{pmatrix} 0 & K_c \\ K_c^* & 0 \end{pmatrix}
$$

(1.3c)

$\sigma$ is the Pauli matrix. Relations (1.3) are equivalent to:

$$
K T^* K^T = T
$$

(1.5)

where $K^T$ is the transposed matrix of $K$, which implies that $T$ is only determined by two $2 \times 2$ matrices $A$ and $B$ when $T$ is time-reversal invariant.

$$
T = \begin{pmatrix} A & B \\ K_c B^* K_c^T & K_c A^* K_c^T \end{pmatrix}
$$

(1.6)

Using (1.2) for having $T^{-1}$, one can show that:

$$
T^+ T + (T^+ T)^{-1} + 2 \mathbb{I} = 4 \begin{pmatrix} A^* A & 0 \\ 0 & K_c A^T A^* K_c^T \end{pmatrix}
$$

(1.7)

Defining the unitary scattering matrix $S$ and the transmission matrix $t$ as usual:

$$
\begin{pmatrix} b_t \\ a_t \end{pmatrix} = S \begin{pmatrix} a_t \\ b_t \end{pmatrix}
$$

$$
S = \begin{pmatrix} t' & t \\ t & t' \end{pmatrix}
$$

(1.8)

one gets, after using the symmetries of $T$ and $S$:

$$
\begin{pmatrix} A^* A & 0 \\ 0 & K_c A^T A^* K_c^T \end{pmatrix}^{-1} = \begin{pmatrix} t^+ t & 0 \\ 0 & t't^+ \end{pmatrix}
$$

(1.9)

which establishes that tr $tt^+$ is a linear statistic of $T^+ T$.

$$
\text{tr } tt^+ = \frac{1}{2} \text{tr } \frac{1}{1 + X_c}
$$

(1.10)

Let us emphasize that the statistics of tr $tt^+$ only are studied in this work, which turns out to be also the statistics of the conductance $g$ (measured in units of $e^2/h$), following a many channel Landaue formula of restricted validity. $g = \text{tr } tt^+$ is a valid relation for a two-probe measurement and breaks down when $L < \ell$ [18] and in absence of phase breaking process outside the studied sample (non local regime of measurement [19, 20]). Therefore, our results cannot be applied directly to the conductance fluctuations, if $g$ is not given by tr $tt^+$.

As noted by Imry [14], two important consequences can be deduced from formula (1.10):

(i) $\langle g \rangle$ indeed measures the number of eigenvalues which are small. More precisely, $X_c$ has only real positive eigenvalues $X_i$ which typically increase as $\frac{1}{2} \cosh 2 \alpha_i L - 1$ with the length $L$ of the sample, for a fixed section $M^2 - 1$. $\alpha_i$ is the inverse localisation length of the $i$th eigenchannel of the disordered sample. Then, $\langle g \rangle$ is roughly equal to the effective number $n_{\text{eff}}$ of active transmission channels for which
\( \alpha, L < 1 \). This number could be large for a good short metallic sample and decreases in an insulator when \( L \) (or the disorder) increases. \( n_{\text{eff}} \sim 1 \) for \( L \approx \xi \).

(ii) since \( \langle g^2 \rangle - \langle g \rangle^2 \sim 1 \), independent of \( n_{\text{eff}} \), formula (1.10) shows us that \( X_c \) must possess a rigid correlated spectrum, at least in the part corresponding to the active transmission channels.

It is convenient (appendix 1) to define a new matrix \( P \) by performing a unitary transformation

\[
U = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) \]

on the transfer matrix \( T \):

\[
P = U^\dagger T U \quad (1.12)
\]

\( P \) and \( T \) have the same eigenvalues, but possess different symmetries. Current conservation gives:

\[
P^+ I P = I \quad I = i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \quad (1.13)
\]

and time reversal symmetry yields:

\[
Z P^* Z = P \quad Z = i \left( \begin{array}{cc} K_c & 0 \\ 0 & K_c \end{array} \right) \quad (1.14)
\]

The transfer matrices in the real-space that we shall study in part 4 have the same symmetries than \( P \), while the symmetries of \( T \) correspond to the transfer matrices in the channel representation of the leads (pure system).

\( Q = P^+ P \) is a hermitic positive matrix with two additional symmetries:

\[
Q I Q = I \quad \text{implies that the eigenvalues of } Q \text{ are going by pairs whose the elements are inverse of each other (symplecticity yielded by current conservation)}.
\]

\[
Z Q^* Z = Q \quad \text{implies that the eigenvalues of } Q \text{ are doubly degenerate (Kramers degeneracy yielded by time reversal symmetry)}.
\]

Therefore, any matrix \( Q \) can be diagonalized by a unitary matrix \( A \)

\[
Q = A^+ D A \quad (1.15)
\]

where \( D \) is a diagonal matrix determined by \( n \) positive numbers \( \nu_i \)

\[
D = \left( \begin{array}{ccc} D_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_n \end{array} \right) \quad D_i = \left( \begin{array}{cc} e^{\nu_i} & 0 \\ 0 & e^{-\nu_i} \end{array} \right) \quad (1.16)
\]

and where we have chosen \( \nu_1 < \nu_2 < \cdots < \nu_n \).

The unitary matrix \( A \) satisfies in addition:

\[
A^+ I A = I \quad (1.17a)
\]
\[
Z A^+ Z = A \right. \quad (1.17b)
\]

By our symmetry analysis, we can conclude that the spectrum of the \( 4 n \times 4 n \) matrix \( X_c \) or

\[
X = \frac{1}{4} [Q + Q^{-1} - 2 I] \]

is determined by \( n \) eigenvalues \( X_i \), each of them having a degeneracy equal to 4.

\[
X_i = \frac{1}{2} (\cosh \nu_i - 1) \quad (1.18)
\]

and formula (1.10) gives for \( \text{tr} \, II^+ \).

\[
\text{tr} \, II^+ = 2 \left( \sum_{i=1}^n \frac{1}{1 + X_i} \right) \quad (1.19)
\]

2. Eigenvalue repulsion of the matrix \( X \) yielded by the measure \( \mu (dX) \) of infinitesimal volume element.

In this section, we prove that there is a factor

\[
\prod_{i<j} |X_i - X_j|^4 \quad \text{in the measure } \mu (dX).
\]

This is the same factor which is responsible for the level repulsion found in the Hamiltonians of the usual Gaussian Sympletic Ensemble (G.S.E. [8]). The demonstration is similar to that given by Muttalib, Pichard and Stone for the orthogonal and unitary cases [10], and has been inspired by Dyson's derivation of the distribution of the eigenvalues of the S-matrix. We shall deduce \( \mu (dX) \) from the calculation of \( \mu (dQ) \).

Let us recall the invariance properties of \( Q \):

\[
Q I Q = I \quad (2.1a)
\]
\[
Z Q^* Z = Q \quad (2.1b)
\]
\[
Q^+ = Q \quad (2.1c)
\]

which yield for any infinitesimal matrix \( dQ \)

\[
Q I \, dQ + dQ \, I Q = 0 \quad (2.2a)
\]
\[
Z dQ^* Z = dQ \quad (2.2b)
\]
\[
dQ = dQ^* \quad (2.2c)
\]

Since the \( Q \) do not form a group, the definition of a measure \( \mu (dQ) \) is not entirely trivial. Let us define from \( Q = P^+ P \) an infinitesimal matrix \( dM \)

\[
Q + dQ = P^+ (I + dM) P \quad (2.3)
\]

which obeys (relations (2.2))

\[
dM I + I \, dM = 0 \quad (2.4a)
\]
\[
Z \, dM^* Z = dM \quad (2.4b)
\]
\[
dM = dM^+ \quad (2.4c)
\]

From (2.4a), one sees that \( dM \) is given by two \( 2 n \times 2 n \) matrices \( dM_p \) (\( p = 1, 2 \))

\[
dM = \begin{pmatrix} dM_1 & dM_2 \\ dM_2^* & dM_1^* \end{pmatrix} \quad (2.5)
\]

while (2.4c) yields that:

\[
dM_p = dM_p^+ \quad (2.6)
\]
Using (2.4b), one gets that the matrices $dM_p$ are themselves composed of $n^2 (2 \times 2)$-matrices $(dmp)_{ij}$ which have the form:

$$
(dmp)_{ij} = \begin{pmatrix}
\mu_{ij}^p & \lambda_{ij}^p \\
-(\lambda_{ij}^p)^* & (\mu_{ij}^p)^*
\end{pmatrix}
$$

Therefore, $dM$ is entirely determined by $2n$ real elements $\mu_{ij}^p$ ($i = 1, \ldots, n; p = 1, 2$) and $2(n^2 - n)$ real elements $(\lambda_{ij}^p)_R, (\lambda_{ij}^p)_I$ and $((\mu_{ij}^p))_R$ ($i < j; j = 1, \ldots, n; p = 1, 2$) which vary independently through some small intervals of lengths. (The indices $R$ and $I$ stand for the real and imaginary parts).

Let us define the measure $\mu (dQ)$ of an infinitesimal neighbourhood $dQ$ of $Q$ by the product of the variations of those independent elements of $dM$.

$$
\mu (dQ) = \prod_{p=1}^{n^2} \left[ \left( \prod_{i=1}^{n} d\mu_{ii}^p \right) \times \left( \prod_{i < j} \left( d\mu_{ij}^p \right)_R \left( d\mu_{ij}^p \right)_I \left( d\lambda_{ij}^p \right)_R \left( d\lambda_{ij}^p \right)_I \right) \right].
$$

To make this definition unique, it remains to be proved that $\mu (dQ)$ is independent of the particular $P$ which has been chosen. Let

$$
Q = P^+ P = P'^+ P'
$$

where both $P$ and $P'$ satisfy equations (1.13) and (1.14). The matrix $B = P \cdot (P')^{-1}$ satisfies:

$$
B^+ B = I \quad (2.10a)
$$

$$
B^+ IB = I \quad (2.10b)
$$

$$
ZB^* Z = B \quad (2.10c)
$$

Defining an infinitesimal matrix $dM'$ with relation (2.3), after substituting $P'$ for $P$, one gets:

$$
dM' = B^+ dM B \quad (2.11)
$$

To prove that the measure $\mu (dQ)$ derived from $dM$ is identical to the measure $\mu' (dQ)$ that one would obtain from $dM'$, we need to show that the Jacobian:

$$
J_s = \det \left| \frac{\partial dM'_{ij}}{\partial dM_{kl}} \right|
$$

has absolute value unity. A proof of this is given in appendix II. Thus the volume $\mu (dQ)$ is unique.

Incidently we have established that for a fixed $Q$, the matrix $P$ is indetermined precisely to the extent of a transformation $P \rightarrow B \cdot P$ where $B$ is an arbitrary matrix which satisfies relations (2.10).

We wish now to express the measure $\mu (dQ)$ in terms of the measures $\mu (dD)$ and $\mu (dA)$ defined on the matrices $D$ and $A$ separately (relation (1.15)).

Small neighbourhoods of $D$ and $A$ are given by:

$$
dD = D d\nu \quad (2.12a)
$$

$$
dA = dA C \quad (2.12b)
$$

Here $d\nu$ means the diagonal matrix with elements $(d\nu_1, \ldots, d\nu_n, -d\nu_1, \ldots, -d\nu_n)$ and we define $\mu (d\nu) = \prod_{i=1}^{n} d\nu_i$. The definition of $\mu (dA)$ is given in appendix III. Relation (1.15) yields for $dQ$:

$$
dQ = A^+ dD A + dA^+ DA + A^+ dA - dC D \cdot A \quad (2.13a)
$$

$$
= A^+ [d\nu D + D dC - dC D] \cdot A \quad (2.13b)
$$

where we have used $dC = -dC^+$. This involves for $dM$:

$$
dM = (P^+)^{-1} A^+ d\nu D + D dC - dC D \cdot A P^{-1} \quad (2.14)
$$

Since $\mu (dQ)$ is unique, we are free to choose any $P$ such that $Q = P^+ P$ for defining $\mu (dQ)$. A convenient choice is $P = D^{-1/2} A$, which yields:

$$
dM = d\nu + D^{1/2} dC D^{-1/2} - D^{-1/2} dC D^{1/2} \quad (2.15)
$$

Using formula (2.9), we obtain the measure $\mu (dQ)$:

$$
\mu (dQ) = \prod_{i=1}^{n} d\nu_i \prod_{i=1}^{n} \left( 2 \sin \frac{\nu_i - \nu_j}{2} \right)^4 \times \left( \prod_{i < j} \frac{2 \sin \frac{\nu_i + \nu_j}{2}}{2} \right)^4 \mu (dA) \quad (2.16)
$$

$$
\prod_{i=1}^{n} d\nu_i \quad \text{and} \quad \prod_{i < j} \frac{2 \sin \frac{\nu_i + \nu_j}{2}}{2} \quad \text{come from}
$$

$$
\prod_{i=1}^{n} (d\mu_{ii}^p)_R (d\mu_{ii}^p)_I (d\lambda_{ii}^p)_R (d\lambda_{ii}^p)_I
$$

for $p = 1$ and 2 respectively. $\mu (dA)$ comes from the independent elements of $dC$, and is directly related to the measure $\mu (dA)$, as explained in appendix III.

Now, since $X$ and $Q$ can be diagonalized by the same matrix $A$, it is easy to use relations (1.18) and (2.16) for getting $\mu (dX)$

$$
\prod_{i=1}^{n} 2 \sin \frac{\nu_i}{2} \prod_{i=1}^{n} d\nu_i = \prod_{i=1}^{n} dX_i
$$

$$
\left( \prod_{i < j} \frac{2 \sin \frac{\nu_i - \nu_j}{2}}{2} \right)^4 \left( \prod_{i < j} \frac{2 \sin \frac{\nu_i + \nu_j}{2}}{2} \right)^4 =
$$

$$
= \prod_{i=1}^{n} |X_i - X_j|^4
$$
which exhibits the characteristic repulsion term between the eigenvalues of \(X\), already obtained for symplectic Hamiltonians by Wigner and Dyson.

3. Distribution of maximum information entropy for \(\rho(X)\).

Up to now, we have only used the fact that we are considering a disordered system with spin-dependent hopping. We have not specified whether we are considering a small or a large system, weakly or strongly disordered, two or three-dimensional. This information must enter in the probability density \(\rho(X)\). Then, we shall integrate over the eigenvectors for having the joint probability distribution of the eigenvalues. One has:

\[
P(X_1, \ldots, X_n) = \prod_{i=1}^{n} \rho(X_i) \mu(dX_i) \quad (3.1a)
\]

\[
P(X_1, \ldots, X_n) = \left| X_i - X_j \right|^4 F(X_1, \ldots, X_n) \quad (3.1b)
\]

\[F(X_1, \ldots, X_n) = \int d\lambda \rho(\lambda) \mu(d\lambda). \quad (3.1c)
\]

Incidently, let us note that \(F(X_1, \ldots, X_n)\) is necessary to prevent the eigenvalues from repelling each other too far and to normalize \(P(X_1, \ldots, X_n)\).

We define the amount of information contained in \(\rho(X)\) by [15]:

\[
I(\rho(X)) = \int \rho(X) \ln \rho(X) \mu(dX) \quad (3.2)
\]

and the spectral density \(\sigma(x)\) by:

\[
\sigma(x) = \int dX P(X_1, \ldots, X_n) \sum_{j=1}^{n} \delta(X_j - x). \quad (3.3)
\]

Minimizing (3.2) with the constraint (3.3) gives:

\[
\rho(X) = \prod_{i=1}^{n} \exp \lambda(X_i) = F(X_1, \ldots, X_n) \quad (3.4)
\]

\[
P(X_1, \ldots, X_n) = \left| X_i - X_j \right|^4 \exp \sum_{i=1}^{n} \lambda(X_i) \quad (3.5)
\]

where the Lagrange multipliers \(\lambda(x)\) can be calculated with approximate methods of statistical mechanics. Following Wigner [21], we assume \(n\) to be large and we average \(\ln P(X_1, \ldots, X_n)\):

\[
\ln P(X_1, \ldots, X_n) = \sum_{i=1}^{n} \lambda(X_i) + \sum_{i<j} \ln \left| X_i - X_j \right| \quad (3.6)
\]

\[
\int \prod_{i=1}^{n} dX_i P(X_1, \ldots, X_n) \ln P(X_1, \ldots, X_n) = \int d\sigma(x) \lambda(x) + \int d\sigma(x, y) \ln \left| x - y \right|^4 \quad (3.6)
\]

Neglecting correlations between the positions of the eigenvalues in the large \(n\)-limit \(\sigma(x, y) \sim \sigma(x) \cdot \sigma(y)\), one gets a functional of \(\sigma\). Postulating that the actual density \(\sigma\) makes this functional a maximum consistent with a normalisation requirement, one gets for the confining potential \(\lambda(x)\) the following integral equation:

\[
\lambda(x) = \lambda_0 - \int d\sigma(x, y) \ln \left| x - y \right|^4 \quad (3.7)
\]

where \(\lambda_0\) is a constant.

Formula (3.5) is the major result of our random matrix approach. The probability law for \(g\) is related to \(P(X_1, \ldots, X_n)\) by:

\[
P(g) = \int \prod_{i=1}^{n} dX_i P(X_1, \ldots, X_n) \times \delta \left( g - 2 \sum_{i=1}^{n} \frac{1}{1 + X_i} \right) \quad (3.8)
\]

and the average conductance is given by:

\[
\langle g \rangle = 2 \int d\frac{\sigma(x)}{1 + x}. \quad (3.9)
\]

Since \(\sigma(x)\) is given a priori, it turns out that our approach contains a priori the correct behaviour for \(\langle g \rangle\) (classical Ohm’s law, weak-localization correction, quantum Ohm’s law…), but the higher moments of \(\rho(g)\) are implied by our maximum entropy hypothesis. Let us just note here that:

(i) the most random probability density \(\rho(X)\) which satisfies only a constraint on the spectrum, supposes that nothing particular is assumed concerning the eigenvectors of \(X\). The repulsive correlations of the distribution (3.5) are associated to an absence of exact knowledge on the eigenvectors of \(X\): an exact knowledge would change our matrix ensemble and its volume element;

(ii) in the metallic regime, we are interested in the fluctuations of the number \(n_{\text{eff}}\) of eigenvalues which are in an interval (e.g. \([0,1])\) which is small compared to the total width of the spectrum \((1 \ll n_{\text{eff}} \ll n)\). Therefore, the statistics are determined by local fluctuations which are dominated by the interaction term yielded by \(\mu(dX)\), and not by the confining potential yielded by the constraint. This makes plausible that the statistics derived by Wigner and Dyson could apply to our problem, even if the confining potential here differs from those considered in standard random matrix ensembles.
4. Spectra yielded by an ensemble of Anderson tight-binding Hamiltonians with spin-dependent hopping.

Let us take a tight-binding Hamiltonian for spin 1/2 electrons with only nearest neighbour spin-dependent interactions. This corresponds to non-interacting electrons in a disordered two-dimensional lattice. Both a random site-potential (independent of spin) and spin-orbit couplings are taken into account in this Hamiltonian. Evangelou and Ziman [22] have shown recently that this model exhibits a true Anderson transition in two dimensions, contrary to the Hamiltonians without spin-orbit coupling. This provides a particular interest to the study of the symplectic ensemble in two dimensions: the metallic regime is not limited by a finite localisation length in the extended phase.

For a strip of width M and length N, the Hamiltonian is written as:

\[ H = \sum_{i,j,\sigma} \left[ W_{ij} |i\sigma\rangle \langle j\sigma| + \sum_{i',j',\sigma'} V_{ij'}^{\sigma\sigma'} |i\sigma\rangle \langle j'\sigma'| \right] \]  \hspace{1cm} (4.1)

for spin components \( \sigma = \pm 1 \). \( W_{ij} \) is the spin-independent value of the potential at the site labelled by \((i,j)\) \((i = 1, \ldots, N; j = 1, \ldots, M)\) and \( V_{ij'}^{\sigma\sigma'} \) is non-zero only between nearest neighbours and can be written in the basis of the identity and Pauli spin matrices

\[ V_{ii'}^{\sigma\sigma'} = \left( 1 + i W_{SO} \mathbf{p}_i^0 \right) \sigma \sigma'. \]  \hspace{1cm} (4.2)

\( W_{SO} \) measures the strength of the random spin-orbit coupling and \( \mathbf{p}_i^0 \) are orbital momenta \((\mathbf{r} \wedge \mathbf{p})\). Let us notice that the motion of the electrons must not be strictly planar, otherwise different spins would not be coupled through \( \mathbf{r} \wedge \mathbf{p} \).

\( H \) is hermitic and invariant under time-reversal symmetry:

\[ H = H^\dagger \]
\[ H = KH^* K^T \]

where \( K \) is still given by (1.3c). This involves for the matrix elements of \( H \):

\[ \langle i\sigma | H | i'\sigma' \rangle = \langle i'\sigma | H | i\sigma \rangle \]  \hspace{1cm} (4.3a)
\[ \langle i\sigma | H | i'\sigma' \rangle = \sigma \sigma' \langle i\sigma | H | i'\sigma' \rangle \]  \hspace{1cm} (4.3b)

The Schrödinger equation:

\[ (E - W_{ij}) \psi_{ij} = \sum_{i',j',\sigma'} V_{ii'}^{\sigma\sigma'} \psi_{i'j'} \]

can be written as:

\[ (E - W_{ij}) U_{ij} = L_{i,i+1} U_{i+1,j} + L_{i-1,j} U_{-1,j} + T_{i,i+1} U_{i+1,j} + T_{i-1,j} U_{i,j-1} \]  \hspace{1cm} (4.4)

where \( U_{ij} \) is the spinor \( \begin{pmatrix} \psi_{ij}^- \\ \psi_{ij}^+ \end{pmatrix} \) and \( W, L \) and \( T \) are 2 \times 2 matrices.

\[ W_{ij} = \begin{pmatrix} w_{ij} & 0 \\ 0 & w_{ij} \end{pmatrix} \]  \hspace{1cm} (site potential)
\[ L_{ij}^{\sigma\sigma'} = V_{ii',j+1}^{\sigma\sigma'} \]  \hspace{1cm} (hopping in the longitudinal direction)
\[ T_{ij}^{\sigma\sigma'} = V_{ii',j+1}^{\sigma\sigma'} \]  \hspace{1cm} (hopping in the transverse direction).

Relation (4.3a) yields

\[ w_{ij} = w_{ij}^* \]
\[ L_{i-1,j}^+ = V_{i,i-1} \]
\[ T_{i,j-1}^+ = V_{ii,j-1} \]

and relation (4.3b) yields that \( L \) and \( T \) have the same symmetry \( \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \).

\( \alpha \) and \( \beta \) can be written in terms of orbital momenta:

\[ \alpha = 1 + i W_{SO} p_i^0 \]
\[ \beta = W_{SO} \mathbf{p}_i^0 + i \mathbf{p}_i^0 \]

The statistical ensemble is characterized by two distributions for the diagonal and off-diagonal elements:

(i) \( w_{ij} \) are taken with a rectangular distribution of width \( W_D \);
(ii) \( p_{ii'}^0, p_{ii'}^1 \) and \( p_{ii'}^2 \) are assumed to be independent random variables uniformly distributed on \([-1/2, 1/2]\).

Equation (4.4) can be written as:

\[ \begin{bmatrix} L_{i,i+1,1} & U_{i+1,1} \\ \vdots & \vdots \\ L_{i,M,1} & U_{i+1,M} \\ U_{i,1} & \vdots \\ \vdots & \vdots \\ L_{i,M} & U_{i+1,M} \end{bmatrix} = M_i \begin{bmatrix} U_{i,1} \\ \vdots \\ U_{i,M} \\ L_{i-1,1} & U_{i-1,1} \\ \vdots & \vdots \\ L_{i-1,M} & U_{i-1,M} \end{bmatrix} \]  \hspace{1cm} (4.5)

where \( M_i \) is a \( 4 \times 4 \) matrix relating the wavefunctions of the \( (i+1) \)th and \( i \)th columns to the wavefunctions of the \( i \)th and \((i-1)\)th columns.

\[ M_i = \begin{pmatrix} \Lambda_i & -I_{2M} \\ I_{2M} \end{pmatrix} \]

\( I_{2M} \) is a \( 2 \times 2 \) identity matrix and

\[ \Lambda_i = \begin{pmatrix} E - W_{i,1} & -T_{i,1} \\ -T_{i,1}^* & E - W_{i,2} \\ \vdots & \vdots \\ -T_{i,M-1} & E - W_{i,M} \\ -T_{i,M-1}^* & E - W_{i,M} \end{pmatrix} \]

for rigid transverse boundary conditions.
In order to get (4.5) in an iterative form, let us define the $4N \times 4N$ matrix $\Gamma_i$

$$
\Gamma_i = \begin{pmatrix}
\frac{L^+_{i,1}}{\det L_{i,1}} & \cdots & \frac{L^+_{i,N}}{\det L_{i,N}} \\
\vdots & \ddots & \vdots \\
\frac{L^+_{i,N}}{\det L_{i,N}} & \cdots & \frac{L^+_{i,1}}{\det L_{i,1}}
\end{pmatrix}.
$$

(4.7)

Using that $L^+_i = \frac{L^+_{i,j}}{\det L_{i,j}}$, one gets:

$$
\begin{pmatrix}
U_{i+1,1} & \cdots & U_{i+1,M} \\
U_{i+1,1} & \cdots & U_{i+1,M} \\
L^+_i U_{i,1} & \cdots & L^+_i U_{i,M} \\
L^+_i U_{i,1} & \cdots & L^+_i U_{i,M}
\end{pmatrix} = \Gamma_i M_i 
= \begin{pmatrix}
U_{i,1} & \cdots & U_{i,M} \\
U_{i,1} & \cdots & U_{i,M} \\
L^+_i U_{i,1} & \cdots & L^+_i U_{i,M} \\
L^+_i U_{i,1} & \cdots & L^+_i U_{i,M}
\end{pmatrix}.
$$

Putting on both sides of our disordered sample two non-disordered leads ($W_D = 0$ and $W_{SO} = 0$ for $i \leq 0$ and $i \geq N$), and calling $P_N$ the transfer matrix in the real-space:

$$
P_N = M_N \prod_{p=1}^{N-1} (\Gamma_p M_p)
$$

(4.8)

$$
M_N = \begin{pmatrix}
E - W_{N,1} & -1 & \cdots & -1 \\
-1 & \ddots & \cdots & -1 \\
-1 & \cdots & E - W_{N,M} & 0 \\
+ 1 & 2M & \cdots & 1 & M
\end{pmatrix}.
$$

We can express the wave-functions defined on the columns $N$ and $N + 1$ in terms of the wave-functions defined on the columns $0$ and $1$ by:

$$
\begin{pmatrix}
U_{N+1,1} \\
U_{N+1,M} \\
U_{N,1} \\
U_{N,M}
\end{pmatrix} = P_N 
\begin{pmatrix}
U_{1,1} \\
U_{1,M} \\
U_{0,1} \\
U_{0,M}
\end{pmatrix}
$$

(4.9)

It is easy to verify that $P_N$ has the same symmetries as the matrix $P$ defined in part 1 (relations (1.13) and (1.14)). Then it remains to project the wave-functions defined at the extremities of the disordered sample on the propagation modes of the leads for having the transfer matrix $T$ which appears in formula (1.1).

**4.1 Spectral Density and Scaling.** — Since calculations are more time-consuming in the symplectic case than in the orthogonal case, ($L$ products of $4M \times 4M$ complex matrices instead of $L$ products of $2M \times 2M$ real matrices), we have not yet studied (as carefully as in the orthogonal case) how the $\{X_i\}$ vary with the sample size and shape. Let us however mention in this section what we believe to be correct also in the symplectic case, mainly by extrapolation from our knowledge of the orthogonal case [10], [17]. The reader is free to see the following as a list of well-defined questions, with some preliminary answers. It is more convenient to discuss here the variables $\alpha_i(L)$, related to $X_i$ by:

$$
X_i = \frac{1}{2} \cosh 2L \cdot \alpha_i(L) - 1 \quad i = 1, \ldots, M
$$

since $\alpha_i(L)$ self-averages in the large $L$- limit [23], and becomes the inverse localisation length of the $i^{th}$ channel. For finite values of $L$, $\alpha_i(L)$ fluctuates from one member to another of the same ensemble, and we note $\langle \alpha_i(L) \rangle$ the ensemble average value. As defined previously, $W_{SO}$ measures the fluctuations of the spin-independent site-potentials and $W_{SO}$ characterizes the spin-dependent random hopping. We have studied, for two cases, 250 disordered squares ($L = M = 20$ and $E = 0$).

- **case a:** $W_D = 1$, $W_{SO} = 0$, $\langle g \rangle \sim 10$
- **case b:** $W_D = 1$, $W_{SO} = 1$, $\langle g \rangle \sim 4$

Following Evangelou and Ziman [22], the case b is in an extended phase, since localization occurs at $W_D = 7$ for $W_{SO} = 1$. Case a is in the metallic regime of a localized phase.

In figure 2, one can see that:

- (i) The density of $\{\langle \alpha_i \rangle\}$ is approximately uniform;
- (ii) $\{\langle \alpha_i \rangle\}$ are more equally spaced in average with spin orbit coupling than without.

We have observed in the orthogonal case [17] that larger squares give basically the same results than

![Fig. 2. Ensemble-averaged values $\langle \alpha_i \rangle$ as a function of the relative index $i/M$ for $20 \times 20$ squares ($i = 1, \ldots, M$) without (case a) and with spin-orbit (case b).](image-url)
smaller metallic squares: i.e. \( \langle \alpha_i(L) \rangle \) converges with \( L \to L \)-independent value for a fixed value of the relative index \( I_R = \frac{i}{M} \). This convergence to an asymptotic limit is not a function of \( I_R \) (due to spectral rigidity) and is approximately realized when both \( L \) and \( M \gg \ell \) (metallic regime). A similar effect has been observed also for the inverse localisation lengths \( \alpha_i(\infty) \) when the strip width increases [23]. We assume that a similar convergence to a two-dimensional limit exists also for the symplectic case, as sketched in figure 3.

Fig. 3. — Assumed behaviour of \( \langle \alpha_i(L) \rangle \) as a function of the relative index \( I_R = \frac{i}{M} \) for metallic squares \( (L = M \gg \ell) \) (\( \square \) \( L \), \( \bigcirc \) \( 2L \), \( \longrightarrow \) \( L \to \infty \)).

Then, the knowledge of \( \langle \alpha_i(L) \rangle \), calculated for small metallic samples \( (L \ll \xi) \), provides direct information on the behaviour of very large samples \( (L \gg \xi, \text{ if } \xi \text{ is finite}) \). For instance, our results obtained on very small metallic samples exhibit a difference: extrapolating curve a for \( I_R = 0 \) seems to give a strictly positive value, as sketched in figure 4. This would yield a gap in the spectrum of \( X(x_{\min} > 0) \), in agreement with two-dimensional localisation. On the contrary, curve b does not seem to give such a gap, in agreement with Evangelou and Ziman (absence of two-dimensional localisation for weak disorder, in the symplectic case).

Another important remaining issue is to determine how many parameters are requested to characterize the \( \langle \alpha_i(L) \rangle \), using a finite size scaling analysis of the type developed in reference [23]. If only one parameter is needed, this would imply that our maximum entropy hypothesis yields for large enough systems, presumably large compared to \( \ell \), a one-parameter distribution for \( P(g) \), independent of the microscopic randomness. Therefore \( P(g) \) could be parametrized only by \( \langle g \rangle \) and the renormalization group function \( \beta(\langle g \rangle) \) would have an unambiguous meaning. This important issue has been discussed recently by Shapiro [24] and by Alt'shuler et al. [25].

Assuming that \( \langle \alpha_i \rangle \) varies linearly with \( i \), with a slope \( a \), (Fig. 4), one gets for the spectral density \( \sigma(x) \) in the continuous limit.

\[
\sigma(x) = \begin{cases} 
\frac{M}{2 L \alpha} [x(x + 1)]^{-1/2} & x_{\min} < x < x_{\max} \\
0 & x < x_{\min} \text{ and } x > x_{\max} 
\end{cases}
\]  

(4.1.1)

Let us note that \( \alpha_{\min} \) is by definition the inverse localisation length of the infinite two-dimensional lattice. In addition, let us assume that \( a \) is the inverse elastic mean free path.

\[ \alpha_{\min} = \frac{1}{\xi_{2d}} \quad a = \frac{1}{\ell} \]

One has:

\[ x_{\min} = \frac{1}{2} \left[ \cosh \frac{2 L}{\xi_{2d}} - 1 \right] \]
\[ x_{\max} = \frac{1}{2} \left[ \cosh \frac{2 L}{\xi_{2d}} \left( \frac{1}{\xi_{2d}} + \frac{1}{\ell} \right) - 1 \right] \]  

(4.1.2)

The continuous approximation (4.1.1) yields for the conductance \( g \) (formula (3.9)):

\[ \langle g \rangle = \frac{M \ell}{L} \left[ \left( 1 + \frac{1}{x_{\max}} \right)^{-1/2} - \left( 1 + \frac{1}{x_{\min}} \right)^{-1/2} \right] \]  

(4.1.3)

One obtains the classical Ohm law for \( \ell < L < \xi_{2d} \)

\[ \langle g \rangle \sim \frac{M \ell}{L} \left( \frac{1}{\ell} + \frac{1}{\xi_{2d}} \right) \]  

(4.1.4)

and the quantum Ohm law when \( L \gg \xi_{2d} \)

\[ g(L) \sim \frac{M \ell}{L} \exp \left( \frac{2 L}{\xi_{2d}} \right) \]  

(4.1.5)
if we have two-dimensional localization ($x_{\text{min}} > 0$, orthogonal case).

The question of the weak-localisation corrections is more subtle. It probably requires to use for $\sigma(x)$ a discrete sum instead of a continuous integral and to know how the $\alpha_i$ converge with the sample size towards their limiting values [16]. Let us mention a preliminary result that we have noticed: $\alpha_i$ increases towards its limit in the symplectic case, in contrast with the orthogonal case where $\alpha$ is above its limit for small $L$. This could be consistent with the difference of sign of the 2 d-logarithmic corrections (weak anti-localization and weak-localization).

4.2 LOCAL SPECTRAL FLUCTUATIONS AND MAXIMUM ENTROPY HYPOTHESIS. — We now consider how the different samples of the same ensemble fluctuate around the same global behaviour described in 4.1. For this purpose, we copy the standard spectral analysis developed in nuclear physics [27]. We define a staircase function $N(x)$ for each sample, where we add a step each time we meet an eigenvalue. First, we separate this staircase function into a smooth part $N_{av}(x)$, which is the same for the different samples, and a fluctuating part $N_{fl}(x)$ characteristic of each sample.

One wants to get rid of $N_{av}(x)$ in order to characterize local fluctuation patterns. For this purpose, we unfold the original sequence $\{X_i\}$ to a new rescaled sequence which has on the average a constant mean spacing equal to unity. This method, described in [27], is approximately equivalent here to considering the $\{\alpha_i\}$ instead of the $\{X_i\}$ and discarding the effect of the global variation of the spectral density. Then, one can check directly the effect on the local fluctuations of the interaction term yielded by our symmetry analysis. In addition, since these interaction terms are identical to those obtained by Wigner and Dyson for the statistic of the eigenvalues of the Hamiltonians, one can compare our results for the rescaled spectrum with the known results of the random matrix theory.

Usually, the fluctuation patterns are characterized by:

(i) the spacing distribution between consecutive eigenvalues $p(S)$, which is very well described by the Wigner surmise for the orthogonal ensemble ($\beta = 1$) and for the symplectic ensemble ($\beta = 4$):

$$p(S) = \frac{\pi}{2} S \exp \left( -\frac{\pi}{4} S^2 \right) \quad \beta = 1$$

$$p(S) = \frac{2^{18}}{3^6 \pi^3} S^4 \exp \left( -\frac{64}{9 \pi} S^2 \right) \quad \beta = 4.$$  

(4.2.1)

In these expressions, $S$ is measured in units of constant mean spacing;

(ii) the $\Delta_3$ statistics which measures the spectral rigidity crucial for the universal conductance fluctuations. This statistic has been defined by Dyson and Mehta [28] in order to measure the deviation of a given eigenvalue sequence from a perfect uniform sequence, as a function of the average number of eigenvalues in the sequence:

$$\Delta_3(C) \approx \min_{\lambda, \beta} \left\{ \frac{1}{C} \int_0^C \left( N(x) - Ax - B \right)^3 dx \right\}$$  

(4.2.2)

The approximations known for the orthogonal and symplectic ensembles are [29]:

$$\Delta_3(C) \approx \frac{1}{\pi^2} \ln C - 0.007 \quad \beta = 1$$

$$\Delta_3(C) \approx \frac{1}{4 \pi^2} \left[ \ln \left( \frac{4 \pi C}{\gamma + 1 + \frac{\pi^2}{8} \frac{18}{\beta}} \right) \right] \quad \beta = 4$$  

(4.2.3)

where $\gamma$ is Euler’s constant (0.57721).

Fig. 5. — Level spacing distribution for $X$ yielded by (4.1) compared to the Wigner surmises (formula (4.2.1) continuous curves). The triangles are the numerical results obtained in case a ($\beta = 1$) and the circles correspond to case b ($\beta = 4$).

Fig. 6. — $\Delta_3$ statistics (formula (4.2.2)) as a function of the averaged number $C$ of eigenvalues in the sequence. The continuous curves are the analytical predictions (formula (4.2.3)). The triangles and the circles are the numerical results for cases a and b respectively.
Figures 5 and 6 show $p(S)$ and $\Delta_2$ yielded by the Anderson Hamiltonians in the two cases that we have considered for the rescaled spectra. Let us underline that:

(i) the perfect agreement between our numerical results and the predictions of our symmetry analysis provides an independent test of the validity of the maximum entropy principle on which our analytical derivation is based. Local fluctuations are universal and reflect only the symmetries of $X$;

(ii) case b exhibits in some sense a non-intuitive behavior: the introduction of an additional randomness with the random spin-dependent hopping strongly reduces the spectral fluctuations of $X$.


The presence of spin-orbit coupling in a disordered system yields important measurable effects that we review in conclusion.

If one looks at how Dyson and Mehta [28] obtain the variance of a quantity which is a linear statistic of a spectrum, it is straightforward to see that the power $\beta$ of the interaction term which appears in $p(X_1, \ldots, X_n)$ gives for the variance of $g$:

$$\left(\langle g^2(L) \rangle \right) - \left(\langle g(L) \rangle\right)^2 \sim \frac{1}{\beta}.$$  

The spectral rigidity due to spin-orbit coupling reduces the conductance fluctuations by a factor of 4. This factor, which already appears in the perturbative calculations of Lee, Stone and Fukuyama [7], is characteristic of symplectic statistic. We point out that, to our knowledge, symplectic statistic has been introduced by Dyson in random matrix theory for completeness, but has never been observed until now in a real system. Therefore, quantum transport in solid state physics would provide, for the first time, the opportunity to observe such statistics. The conditions to be satisfied for having orthogonal, unitary or symplectic statistics are the same as those relevant for weak-localisation. A very simple way to produce a transition would consist to take a sample with strong spin-orbit coupling ($\tau_{SO} \gg T$, where $\tau_{SO}$ is the life time due to such scattering and $T$ is the temperature) and to apply a magnetic field. It has been observed that, when the field is strong enough, weak-antilocalisation is destroyed [30], and we predict that this destruction would be accompanied by an increase of the conductance fluctuations ($\beta = 4 \rightarrow \beta = 2$).

Another related issue concerns relaxation processes and noises associated with quantum transport: Feng, Lee and Stone [31] have proposed that the sensitivity of quantum conductance to small configuration changes could explain room-temperature $1/f$ noise in disordered metals. Since we have shown how this sensitivity depends on the universality classes, a very simple way to verify this theory would be to study carefully how $1/f$ noise varies with spin-orbit coupling and magnetic field. This will provide an unambiguous check of the sensitivity of room-temperature noise to quantum interference phenomena.

In summary, we have shown for disordered systems with spin-orbit coupling that statistics in the metallic phase are correctly described by our distribution of maximum information entropy. Statistics in the localized phase ($W_D > 7$) deserve further studies: the $X_i$ increase exponentially with the size and the interaction is certainly weakened by an exponentially large spacing.

Acknowledgments.

Long, extensive and friendly discussions with A. Douglas Stone, K. Muttalib and Y. Imry are gratefully acknowledged. We thank also B. Derrida, O. Bohigas and U. Sivan for helpful remarks and suggestions.

Appendix I.

One could think that we have arbitrarily introduced the transformation $U$ in part 1. Let us show, in this appendix, that such a transformation would appear necessarily further in our derivation, in part 2.

It would be possible, as we do for $Q = P^+ P$ in part 1, to define the ensemble of matrices $Q_c = T^+ T$ and a measure $\mu (dQ_c)$ on this ensemble. Unicity of this measure is also easy to show: following the same method as for $Q$, one considers $V$, which satisfies like $T$ the relations (1.2), (1.3) and $Q_c = V^+ V$. The study of the Jacobian of the transformation shows us that the measure defined from $V$ is the same as $\mu (dQ_c)$ defined from $T$.

Difficulties only show up when one tries to express $\mu (dQ_c)$ as a function of the eigenvalues and eigenvectors of $Q_c$. Writing $Q_c = R^+ D R$, where $D$ is diagonal, one cannot take $\sqrt{D} \cdot R$ as a particular $V$, since $\sqrt{D} \cdot R$ does not satisfy both relations (1.2) and (1.3) — because $D$ does not —. Then, at this point of the derivation, one cannot go on without a transformation. A way to solve this difficulty, consists in considering a new Ensemble of matrices. Noticing that $D$ satisfies

\[
\begin{align*}
D^* I D &= I \\
(2D^* Z) &= D 
\end{align*}
\]

we consider $\sqrt{D} \cdot RU$, where $U$ is the unitary matrix $\frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i 1 \\ -i 1 & 1 \end{array} \right)$. $\sqrt{D} \cdot RU$ satisfies relations (a) too, like $P$ defined by (1.12). Therefore, the matrix $Q = P^+ P$ is diagonalizable by $Q = (RU)^+ D (RU)$ where $\sqrt{D} \cdot (RU)$ and $P$ satisfy...
the same relations. As a consequence, on our new ensemble of matrices $Q$, it is now possible, not only to define a unique measure $\mu (dQ)$ but also to express it as a function of the eigenvalues and eigenvectors of $Q$.

Let us finally note that there would be another way, following the method used by Mello et al. in the orthogonal case. They study directly the ensemble of matrices $T$ (instead of matrices $Q_2$ or $Q$) with Bargmann's decomposition $T = W \wedge Y$ [16]. The measure on their ensemble can be expressed in terms of eigenvalues of $Q$, as soon as one notices that $\Lambda^2$ and $Q$ have the same eigenvalues.

### Appendix II.

Let us show that $\mu (dQ) = \mu' (dQ)$ i.e. that:

$$\prod_{i,j}^{\alpha} dM_{ij} = \prod_{i,j}^{\alpha} dM'_{ij}$$

(b)

where $\alpha = \pm 1$:

$$dM_{ij}^\alpha = \text{Re } dM_{ij} \quad \text{and} \quad dM_{ij}^\alpha = \text{Im } dM_{ij}.$$

In each product of (b), we consider independent terms only. Since $p$ and $p'$ are both sympletic (see 2.10a and 2.10b), $dM$ and $dM'$ have the same symmetries. Therefore both products in (b) contain the same indices and we can forget other indices:

from now on, we assume $\frac{\partial dM_{ij}^\alpha}{\partial dM_{kl}^\beta} = \delta_{ik} \delta_{jl} \delta_{\alpha \beta}$. In order to show that the Jacobian $J_a = \det \left| \frac{\partial dM_{ij}^\alpha}{\partial dM_{kl}^\beta} \right|$ has absolute value unity in general, it is sufficient to consider the case $B = 1 + \Omega$ when $\Omega$ is infinitesimal and satisfies $\Omega = - \Omega^+$, as a consequence of (2.10c). Then (2.11) becomes:

$$dM' = dM (1 + \varepsilon \Lambda)$$

where $\varepsilon \Lambda = \Omega - (dM)^{-1} \Omega dM$ and $\varepsilon \ll 1$. Let us notice that $\text{tr } \varepsilon \Lambda = 0$.

This involves, on the matrices elements:

$$dM_{ij}^\alpha = \sum_k (dM_{ik}^\alpha (\delta_{k,j} + \varepsilon \Lambda_{kj}) - \alpha dM_{ik}^\alpha \varepsilon \Lambda_{kj}^{-1})$$

where $\Lambda_{kj}$ and $\Lambda_{kj}^{-1}$ stand for $\text{Re } \Lambda_{kj}$ and $\text{Im } \Lambda_{kj}$ respectively.

Hence

$$\frac{\partial dM_{ij}^\alpha}{\partial dM_{kl}^\beta} = \delta_{ik} \delta_{\alpha \beta} \delta_{jl} + \varepsilon \delta_{ik} (\Lambda_{lj} - \alpha \delta_{\alpha \beta} \Lambda_{lj}^{-1})$$

and

$$J_a = \det (1 + \varepsilon \phi)$$

with $\text{tr } \varepsilon \phi = 0$.

Therefore

$$J_a = 1 + 0 (\varepsilon^2).$$

### Appendix III.

Matrices $A$, defined by their unitarity and relations (1.17), form a group. Hence $\mu (dA)$ is the invariant measure of this group. The infinitesimal matrix $dC$ satisfies:

$$[dC, I] = 0 \quad \Rightarrow \quad Z dC Z = dC$$

$$dC = - dC^+$$

which involves that $dC$ is given by two $2n \times 2n$ matrices $dC_p$ ($p = 1, 2$)

$$dC = \left( \begin{array}{cc} dC_1 & dC_2 \\ - dC_2 & dC_1 \end{array} \right).$$

The $dC_p$ have the same symmetries as the $dM_p$, excelled that $dC_1 = - dC_1^+$ instead of $dM_1 = dM_1^+$. Writing the $2n^2 (2 \times 2)$ matrices $(dC_p)_{ij}$ under the form (2.7), one gets for $\mu (dA)$ an expression which has exactly the same structure as (2.9) but which is, in addition, multiplied by $\mu (dG)$

$$\mu (dG) = \prod_{i,j}^{\alpha} g_{ij}^\alpha g_{ij}^\beta g_{ij}^3,$$

This additional factor $\mu (dG)$ comes from the $(n (2 \times 2)$ matrices $(dC_1)_{ij}$ whose the imaginary diagonal and complex off diagonal elements vary independently through small intervals of lengths $g_{ij}^1$, $g_{ij}^2$ and $g_{ij}^3$ respectively.

In fact, only $\mu (dA) / \mu (dG)$ is interesting because the decomposition (1.15) is not unique: $Q$ is unchanged by the substitution $A \rightarrow G . A$ where $G$ is any unitary sympletic and diagonal by blocs matrix. Then only equivalence classes $\tilde{A}$ have some importance. This is the reason why $\mu (d\tilde{A}) = \mu (dA) / \mu (dG)$ appears in the formula 2.16.

### References

[26] Imry, Y., private communication.