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Generalized Fréedericksz transition in nematics. Geometrical threshold and quantization in cylindrical geometry

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Résumé. — On étudie les déformations périodique (PD) et apériodique (AD) induites par un champ magnétique dans un nématique confiné entre deux cylindres coaxiaux (géométrie de Couette) avec son directeur fortement ancré sur les surfaces. La méthode utilisée est la théorie du continuum pour l'élasticité de courbure des nématiques dans la limite des petites déformations. Lorsque l'orientation initiale du directeur est azimuthale, et le champ radial, la déformation PD est semblable à l'instabilité statique de Taylor. En absence de champ, le seuil de la déformation périodique PD est semblable à celui prédit auparavant pour AD. Lorsque l'orientation initiale du directeur est axiale, un champ radial peut induire une déformation périodique dont le vecteur d'onde \( q_0 \) des domaines azimuthaux ne peut prendre que des valeurs entières. Dans cette géométrie, on trouve que \( q_0 \) varie de manière discontinue dans certaines gammes de paramètres des matériaux et des rayons de l'échantillon ; cette situation rappelle l'effet Barkhausen des matériaux ferromagnétiques. On discute aussi brièvement une troisième configuration dans laquelle la distortion du directeur au-dessus du seuil peut montrer une modulation azimuthale.

Abstract. — Using the continuum theory of curvature elasticity of nematics in the small deformation approximation, the occurrence of magnetic field induced static periodic deformation (PD) relative to that of the aperiodic deformation (AD) is studied with the nematic (in Couette geometry) assumed to be confined between two coaxial cylinders and the director firmly anchored at the sample boundaries. When the initial director orientation is azimuthal and the field radial, PD is similar to a static Taylor instability. In the absence of a field PD is shown to possess a geometrical threshold like that predicted earlier for AD. When the initial director orientation is axial, a radial field may induce PD whose azimuthal domain wave vector \( q_0 \) can take only integral values. In this geometry, \( q_0 \) is found to vary discontinuously over certain ranges of material parameters and sample radii — a situation somewhat reminiscent of the Barkhausen effect in ferromagnetic materials. A third configuration in which director distortion above threshold may suffer azimuthal modulation is briefly discussed.

1. Introduction.

The Oseen-Frank theory of curvature elasticity [1-7] describes the elastic free energy density of a nematic liquid crystal as a quadratic in the spatial gradients of the unit director vector field \( \mathbf{n} \). The three basic director distortions — splay, twist and bend — are described by the curvature elastic constants \( K_1 \), \( K_2 \) and \( K_3 \), respectively. Owing to the diamagnetic susceptibility anisotropy \( \chi_a \) of a nematic, the orientation of \( \mathbf{n} \) can be affected by impressing an external magnetic field \( \mathbf{H} \).

It is possible to prepare perfectly aligned nematic samples (with \( \mathbf{n} = \mathbf{n}_0 = \text{constant} \)) between parallel plates. When \( \mathbf{H} \) is applied to such a sample in a direction normal to \( \mathbf{n}_0 \), a deformation in \( \mathbf{n} \) is found to occur only when \(|\mathbf{H}| \) exceeds a critical value \( H_c \) known as the Fréedericksz threshold. It is also found that in most nematics, the director deformation is uniform in the sample plane when \(|\mathbf{H}| \geq H_c \). In the present work we shall refer to such a distortion as an aperiodic distortion (AD). Detection of the Fréedericksz transition is a convenient method of measuring the curvature elastic constants of nematics.

An exception to the above rule was discovered by Lonberg and Meyer [8] who studied a polymer nematic (PBG) in the splay geometry (\( \mathbf{n}_0 \) parallel to and \( \mathbf{H} \) normal to the plates). They found that for \(|\mathbf{H}| > \) a well defined threshold \( H_p \), the distortion is

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spatially periodic in the sample plane with the direction of periodicity roughly normal to \( \mathbf{n}_0 \). They showed, by employing the Oseen-Frank theory, that such a periodic distortion (PD), which involves both splay and twist close to the transition, should be more favourable than AD in nematics with \( K_1 > 3.3 K_2 \) when the director is rigidly anchored at the boundaries. This is certainly true for PBG [9]. Not only is PD a new field effect, its occurrence places a severe restriction on using the conventional method of determining \( K_1 \) via the splay Fréedericksz transition.

It has been shown that by using different configurations [10-12] such as the oblique field configuration of Deuling et al. [13], it should be possible to suppress PD in materials such as PBG. The continuum theory does not rule out the occurrence of PD in nematics close to a smectic transition [4, 6, 7, 14] where \( K_2 > K_1 \), provided that the field is now applied in the twist geometry (parallel to the plates and normal to \( \mathbf{n}_0 \), \( \mathbf{n}_0 \) being parallel to the sample planes); there appears to be a symmetry transformation connecting this case to the opposite case of \( K_1 > K_2 \). The range of existence (ROE) of PD may depend critically on the magnitude of director anchoring energy at the sample boundaries [11, 12, 15, 16]. When the ground state \( \mathbf{n}_0 \) is uniformly twisted (twisted nematic), PD may occur as one of two orthogonal modes when \( \mathbf{H} \) is applied normal to the plates [17].

All the above studies have been made on flat samples which are assumed to be infinite in their own plane. Most studies on nematic elasticity concentrate on flat samples as the flat geometry is convenient from a practical viewpoint for optical observation as well as for impressing the destabilizing fields. Apart from a number of studies on defects [3, 4, 6, 7] which use curvilinear coordinates, little work has been done on field induced transitions in curvilinear geometry. Despite practical difficulties which may arise in an actual experiment, one is tempted to consider cylindrical geometry because of the following reason: a cylindrical nematic sample can be imagined to be formed out of a finite, flat nematic sample by a process of bending and joining the ends; this would result in the nematic being confined to the space between two coaxial cylinders. While the flat sample is finite in only one direction (normal to the plates), the cylindrical sample is finite along two directions (radial and azimuthal). This fact can be expected to have some effect on field induced deformations in cylindrical geometry.

Indeed, it has been shown by Leslie [18] that the initial director orientation of a nematic confined between coaxial cylinders may become spontaneously unstable and result in AD when the ratio of sample radii exceeds a critical value which depends on elastic constants; this can be regarded as a geometrical threshold (GT). Atkin and Barratt [19] have also considered some additional cases of AD in cylindrical geometry.

In this communication, the relative occurrence of AD and PD is studied in cylindrical geometry for different elastic ratios, sample radii, field configurations and initial director orientations. In section 2 the differential equations governing hydrostatic equilibrium of nematics are summarized. In section 3 results on AD for three geometries are briefly discussed especially because some of these results are required in later sections; it is also helpful for introducing the concept of GT. Results for PD in geometry 2 are stated in section 4. In section 5, PD in geometry 3 is discussed. Section 6 contains a brief qualitative discussion of the instability mechanisms encountered in sections 4 and 5. Section 7 concludes the discussion by summarizing limitations of the mathematical model used in this work.

2. Governing equations, sample geometry and boundary conditions.

The Oseen-Frank bulk elastic free energy density of a nematic is given by [3]

\[
W = [K_1 (\nabla \cdot \mathbf{n})^2 + K_2 (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + K_3 (\mathbf{n} \cdot \nabla n)^2]/2 .
\]

The equilibrium equations under the action of a magnetic field \( \mathbf{H} \) and the gravitational potential \( G \) can be written as [18]

\[
- (n_{k,i} \partial W/\partial n_{k,i})_j + [\chi_\perp \mathbf{H}_k + \chi_\parallel n_k (n_i H_i)] H_{k,i} = (p + G)_j
\]

\[
(\partial W/\partial n_i)_j - (\partial W/\partial n_i)_j + \chi_\parallel (H_k n_k) H_i + \gamma n_i = 0
\]

where \( p \) is the hydrostatic pressure, \( \gamma \) a Lagrangian multiplier, a comma denotes partial differentiation in cartesian coordinates, repeated indices are summed over and \( \chi_\perp \) the diamagnetic susceptibility normal to the director. As shown in [18], (2) is satisfied provided that (3) holds and \( p \) is restrained by the condition

\[
p = [\chi_\parallel (H_k n_k)^2 + \chi_\perp H_k H_k] - W - G + p_0
\]

with \( p_0 \) an arbitrary constant. We shall come back to (4) in section 7. Due to the generally weak diamagnetic susceptibility of nematics, \( \mathbf{H} \) can be assumed to be unperturbed by the medium. It is thus sufficient to solve (3) with suitable boundary conditions to obtain information regarding director configuration, threshold field, etc. by assuming the validity of (4).

In the present work the nematic sample is assumed to be confined between two coaxial cylinders of radii \( R_1 \) and \( R_2 (R_2 > R_1) \) with their common axis along \( z \).
In cylindrical polar coordinates \((r, \psi, z)\), solutions are sought for \(n\) in the form

\[
\boldsymbol{n} = [n_r(r, \psi, z), n_\psi(r, \psi, z), n_z(r, \psi, z)]
\]

\[
|\boldsymbol{n}|^2 = 1
\]  

(5)

where \(n_r, n_\psi, n_z\) are physical components. \(\mathbf{H}\) is also defined in terms of its physical components for which one possible choice, satisfying Maxwell’s equations \((\text{div} \mathbf{H} = 0; \text{curl} \mathbf{H} = \mathbf{0})\) is

\[
\mathbf{H} = (H_r, H_\psi, H_z); \quad H_r = A/r; \quad H_\psi = B/r; \quad A, B, H_z \text{ constants}.
\]  

(6)

Here, \(A\) and \(B\) have dimensions of magnetic potential. There exist three simple director alignments (see Fig. 1) which identically satisfy (3). These are:

- **Geometry 1**: \(n_r = 1; \quad n_\psi = 0; \quad n_z = 0\)
- **Geometry 2**: \(n_r = 0; \quad n_\psi = 1; \quad n_z = 0\)
- **Geometry 3**: \(n_r = 0; \quad n_\psi = 0; \quad n_z = 1\)

Starting with a \(\mathbf{n}_0\) from (7), perturbations are imposed on the director field. Governing equations are set up with (3) and solved with suitable boundary conditions.

Though the finiteness of anchoring energy must be taken into account in any realistic calculation \([11, 12, 15, 16]\), the rigid anchoring hypothesis is adopted in this work for the sake of simplicity. Then, the boundary conditions to be satisfied at the two surfaces can be written down for the different cases as follows:

- **Geometry 1**: \(n_r = 1\);
- **Geometry 2**: \(n_r = 0; \quad n_\psi = 1; \quad n_z = 0\) at \(r = R_1, R_2\)
- **Geometry 3**: \(n_r = 0 = n_\psi; \quad n_z = 1\) at \(r = R_1, R_2\)

(7)

All physical quantities are measured in cgs units. The nematic is assumed to have \(\chi_a > 0\). All numerical results have been obtained using the series solution method (see [10] Appendix). For the sake of convenience, \(K_2\) has been fixed at unity and other parameters given suitable values.

3. Aperiodic deformation (AD).

3.1 **Geometry 1.** — Following [18] one puts \(\mathbf{n} = (C_\theta, S_\theta, 0)\) with \(\theta = \theta (r)\), \(S_\theta = \sin \theta\) and \(C_\theta = \cos \theta\). With an azimuthal field \(\mathbf{H} = (0, B/r, 0)\), (1), (3) and (8) yield

\[
W = [f(\theta) \theta^2 - (df/dr)/r + \{K_1 + K_3 - f(\theta)\}/r^2]/2 + \{f(\theta)\theta_r + (df/dr)\theta_r/r + f(\theta)\theta_{rr}/r + [\chi_a B^2 S_\theta C_\theta + (df/d\theta)/2]/r^2 = 0; \quad \theta(R_1) = 0 = \theta(R_2); \quad f(\theta) = K_3 S_\theta^2 + K_5 C_\theta^2; \quad \theta_r = d\theta/dr; \quad \theta_{rr} = d^2\theta/dr^2.
\]  

(9)

The threshold \(B_T\) is given by

\[
\chi_a B_T^2 = [K_3 \pi^2/(\log R_2)^2] + K_3 - K_1; \quad R_2 = R_2/R_1.
\]  

(10)

Equating \(B_T\) to zero one obtains the GT for this case

\[
(R_{21})_{cl} = \exp [\pi \{K_3/(K_1 - K_3)\}^{1/2}] \quad K_1 > K_3.
\]  

(11)

The following observations may be made:

1) When \(\theta = 0, \theta_r = 0\) and \(\theta_{rr} = 0\) in the absence of a field so that the equilibrium equation (9) is identically satisfied. Thus, the director field with \(\theta = 0\) is locally undeformed.
ii) Still, the nematic possesses a (global) free energy density \( W_G = K_1/2 r^2 \) owing to the curvature of the sample. In a flat sample, \( r \to \infty \), \( W_G \to 0 \) as expected. As seen in figure 1a, a nematic radially aligned between coaxial cylinders does present an overall splay distortion; the sample can be regarded as being globally deformed. Integrating \( W_G \) over the sample volume, the global free energy per unit length along \( z \) is found to be \( F_G = K_1 \pi \log R_21 \). The logarithmic increase of \( F_G \) with \( R_21 \) seems to cause GT (11). When \( K_1 > K_3 \), the total free energy can be diminished if the director undergoes a distortion involving the smaller elastic constant \( K_3 \).

iii) It is also clear from (11) that \( K_1 \) and \( K_3 \) have opposing action. This is clear especially in the small deformations limit with \( \theta \) very small. While the difference \( K_1 - K_3 \) is responsible for triggering AD via the torque \( df/d\theta \), \( K_3 \) tends to oppose the formation of AD through the stabilizing elastic torque \( K_3 \theta_{r z} \).

iv) If \( B = 0 \) in (9), there exist only two distinct locally undeformed solutions: \( n_0 = (1, 0, 0) \) and \( n_0 = (0, 1, 0) \), corresponding to geometries 1 and 2, respectively. It is not possible to have a configuration \( n_0 = (\cos \theta_1, \sin \theta_1, 0) \) with \( \theta_1 \) constant, though this is possible in a flat sample.

v) For a given \( R_21 \), (10) cannot exist provided that \( K_1 / K_3 \not\leq 1 + \pi^2 / (\log R_21)^2 \). The condition for the existence of GT (11) also demands that \( K_1 > K_3 \). Such a situation may be possible in discotic nematics [20-22].

vi) With an axial field \( H = (0, 0, H_z) \), it is natural to consider AD with \( n = (C_b \phi, 0, S_b \phi) \); \( \phi = \phi (r) \). The non-linear differential equation (3) for \( H \neq 0 \) is difficult to integrate analytically. Linearizing wrt \( \phi \), (3) and (8) reduce to

\[
r^2 \phi_{,rr} + r \phi_{,r} + \phi [(K_1/K_3) + r^2 (\chi_b H_z^2 H_2/K_3)] = 0 ;
\phi (R_2) = 0 = \phi (R_1) .
\]

This can be solved numerically by effecting the coordinate transformation

\[
r = j + h \xi ; \quad j = (R_1 + R_2)/2 ; \quad h = (R_2 - R_1)/2
\]

and by employing the series solution method. When \( H_z = 0 \), (12) admits the GT

\[
(R_21)_{r z} = \exp \left[ \pi (K_3/K_1)^{1/2} \right]
\]

which exists, in principle, for any \( K_1, K_3 \). When \( H \neq 0 \), the threshold value of \( M_z = \chi_a H_z^2 h^2/K_3 \) can be calculated as a function of \( R_21 \) for different values of \( K_3/K_1 \). For a given material (Fig. 2a), \( M_z \) depends only on the ratio \( R_21 \) and not on the radii \( R_1 \) and \( R_2 \) taken individually. For a given material, \( M_z \to 0 \), as \( R_21 \to (R_21)^{(1)} \).

The plots of AD threshold vs. the radial ratio \( R_21 = R_2/R_1 \) when the field is axial; \( H = (0, 0, H_z) \), \( h = (R_2 - R_1)/2 \) is the semi sample thickness. \( n \) is the perturbed director orientation. \( H_2 \) is the threshold field.

(a) Geometry 1; \( n = (1, 0, \theta) \) where \( \theta = \theta (r) \). Plot of \( M_z = \chi_a H_z^2 H_2 / K_3 \) vs. \( R_21 \) for different elastic ratios \( K_1/K_3 = (1) 0.1 (2) 0.25 (3) 0.5 (4) 0.75 (5) 0.9 (6) 5 (7) 20 \). When a given material, \( M_z \) decreases to zero when \( R_21 \to \) the GT(14) (Sect. 3.1). (b) Geometry 2; \( n = (0, 1, \theta) \); \( \theta = \theta (r) \). The field is again axial. Plot of \( M_z' = \chi_a h^2 H_z^2 / K_3 \) vs. \( R_21 \) for different elastic ratios \( K_1/K_3 = (1) 4.5 (2) 2.4 (3) 1.7 (4) 1.4 (5) 1.2 (6) 1.1 \). For a given material, \( M_z' \) decreases to zero when \( R_21 \) approaches the GT(18) (Sect. 3.2). The plots depict thresholds determined from a linear stability analysis. To establish that a given distortion actually has lower free energy than the ground state one has to carry out energetics involving non-linear perturbations.

vii) For geometry 1, two GTs, (11) and (14) have been found. In general, (14) is more favourable (lower) than (11). When a spontaneous AD does occur, it seems easier for it to occur in the \( rz \) plane than to occur in the \( r \psi \) plane. This may be due to the sample being closed or limited along \( \psi \) but not along \( z \).

viii) It is possible, in principle, to consider the oblique field case as done in [13] for a flat sample by assuming a deformation \( n = (C_\phi \phi_0, S_\psi \phi_0, 0) \) produced by a field \( H = (0, B/r, H_z) \); \( \theta, \phi \) are assumed to be functions of \( r \). Linearizing wrt \( \theta, \phi \) and using (3), two differential equations result which are coupled by the cross term \( \chi_b BH_z/r \). A numerical solution shows that the thresholds \( H_2 \) and \( B_F \) complement each other.

3.2 GEOMETRY 2. — With the radial field \( H = (A/r, 0, 0) \) and \( n = (S_\theta \phi_0, C_\theta \phi_0, 0) \), the resulting differential equation and its solution are isomorphic to the expressions in (9)-(11) except for an inter-
change of $K_1$ and $K_3$ \[18\]. The field threshold and GT take the respective forms
\[
\chi_s A_\alpha^2 = \left[ K_1 \pi^2/(\log R_{21})^2 \right] + K_1 - K_3; \quad (15)
\]
\[
(R_{21})_{13} = \exp[\pi \{ K_1/(K_3 - K_2) \}]^{1/2}; \quad K_3 > K_1. \quad (16)
\]
The field threshold cannot exist for a given $R_{21}$ when $K_3/K_1 > 1 + \pi^2/(\log R_{21})^2$. This may be possible close to the nematic-smectic transition \[14\] where $K_3$ may diverge. GT \[16\] will not exist when $K_1 > K_3$ and this may hold for certain discotic nematics \[20-22\].

When $H = (0, 0, H_z)$ is axial and $n = (0, C_\phi, S_\phi)$ with $\phi = \phi(r)$, the linearized equation \(3\) along with boundary conditions \(8\) take the form
\[
r^2 \phi_{,rr} + r \phi_{,r} + \phi \left\{ (2 K_3 - K_2)/K_2 \right\} +
+ r^2 (\chi_s H_z^2/K_2) = 0; \quad \phi(R_1) = 0 = \phi(R_2). \quad (17)
\]
With $H_z = 0$, \(17\) admits the GT
\[
(R_{21})_{3} = \exp[\pi \{ K_2/(2 K_3 - K_2) \}]^{1/2}; \quad 2 K_3 > K_2. \quad (18)
\]
When $H_z \neq 0$, \(17\) can be solved numerically. A plot of the threshold value of $M_z = \pi a H_z^2 K_2$ as a function of $R_{21}$ is shown in figure \(2b\) for different ratios $K_3/K_2$. For a given material, $M_z$ is a function of $R_{21}$ for different radii $R_1$ and $R_2$. For $R_{21} \geq 1$, $M_z = \pi^2/4$, as in a flat sample. $M_z$ decreases to zero as $R_{21} \rightarrow (R_{21})_{c4}$.

Thus, in geometry 2, we find two GTs, \(16\) and \(18\). Assuming that both GTs are real, it appears that $(R_{21})_{13} = (R_{21})_{3}$ only if $K_1 < K_2/2$; such a situation may arise close to a nematic-smectic transition. For most nematics it appears that \(18\) will remain lower than \(16\); when AD occurs spontaneously, it would rather occur in the $\omega z$ plane than in the $r\phi$ plane. This can again be traced to the sample being limited along $r$ and $\phi$ but not along $z$.

Exactly analogously with the previous section one can again consider the oblique field case with $H = (A/r, 0, H_z)$ and $n = (S_\theta, C_\theta, C_\phi, S_\phi)$ with $\theta = \theta(r)$ and $\phi = \phi(r)$. In the limit of small perturbations the two linearized equations for $\theta$ and $\phi$, which are coupled by the cross term $\chi_s A H_\omega/r$, can be solved. It is again found that the thresholds $H_\theta$ and $A_\theta$ \(15\) complement each other. We have
\[
3.3 \text{ GEOMETRY 3.} \quad \text{With a radial field}
\]
\[
\bar{H} = (A/r, 0, 0) \quad \text{and} \quad n = (S_\theta, C_\theta, C_\phi, S_\phi)
\]
\[
\theta = \theta(r) \quad \text{equations} \quad (1), \quad (3) \quad \text{and} \quad (8) \quad \text{yield} \quad [19]
\]
\[
W = g(\theta) \theta_{,rr}^2 + K_1 S_\theta C_\theta \phi_{,rr}/r + K_1 S_\theta^2/r^2;
\]
\[
g(\theta) = \theta_{,rr} + r \phi_{,r}/r + \phi_{,r}/r +
+ (\chi_s A^2 - K_1) S_\theta C_\theta/r^2 = 0;
\]
\[
g(\theta) = K_1 C_\phi^2 + K_3 S_\phi^2; \quad \theta(R_1) = 0 = \theta(R_2). \quad (19)
\]

The field threshold is given by
\[
\chi_s A_\alpha^2 = \left[ K_1 \pi^2/(\log R_{21})^2 \right] + K_1. \quad (20)
\]

When the field is azimuthal with $H = (0, B/r, 0)$ one can consider deformation in the $\omega z$ plane such that $n = (0, S_\phi, C_\phi)$, $\phi = \phi(r)$. Equations \(1\), (3) and \(8\) take the form \[19\]
\[
W = K_2 \phi_{,rr}^2/2 + K_2 S_\phi C_\phi \phi_{,rr}/r +
+ S_\phi^2 (K_2 C_\phi^2 + K_3 S_\phi^2)/2 r^2;
\]
\[
K_2 \phi_{,rr}/r + S_\phi C_\phi \times
\]
\[
\times [(x_a B^2 - K_3) + 2 (K_2 - K_3) S_\phi^2]/r^2 = 0;
\]
\[
\phi(R_1) = 0 = \phi(R_2). \quad (21)
\]

The field threshold is given by
\[
\chi_s A_\alpha^2 = \left[ K_2 \pi^2/(\log R_{21})^2 \right] + K_2. \quad (22)
\]

whose value is formally the same as \(20\) with $K_2$ replacing $K_1$. It is possible to make the following observations:

i) When $n_0$ is aligned along $z$ (Fig. 1c), $W = 0$. Also, \(3\) is identically satisfied. Thus the nematic is undistorted \emph{locally} as well as \emph{globally}. A direct consequence of this is the non-existence of GT.

ii) With an oblique field $H = (A/r, B/r, 0)$ and a distortion $n = (S_\theta, C_\theta, C_\phi, S_\phi)$ with $\theta = \theta(r)$ and $\phi = \phi(r)$, the threshold condition in the limit of small distortions becomes
\[
(x_a A_\alpha^2/K_4) + (x_a B_\alpha^2/K_2) = 1 + \pi^2/(\log R_{21})^2
\]
which is a combination of \(20\) and \(22\). When $R_{21} \gg 2 h = (R_2 - R_1)$, \(23\) goes over to the expression for the threshold field in the case of a flat sample having boundaries $x = \pm h$ if $A_\phi$ and $B_\phi$ are redefined as $(A_\phi/R_1) \rightarrow H_\phi C_\phi$ and $(B_\phi/R_1) \rightarrow H_\phi S_\phi$, respectively, with $H = (H_\phi C_\phi, H_\phi S_\phi, 0)$ (now in cartesian coordinates) as the field acting in the $xy$ plane making an angle $\psi$ with $x$ axis and normal to $n_0 = (0, 0, 1)$, the original planar director configuration.

4. Periodic distortion in geometry 2.

Solutions are considered with $n = (S_\theta, C_\theta, C_\phi, C_\phi, S_\phi)$; $\theta = \theta(r, \psi, \varphi)$; $\phi = \phi(r, \psi, \varphi)$. With a radial field $H = (A/r, 0, 0)$ non-linear coupled partial differential equations result from \(3\). Linearizing these wrt $\theta$ and $\phi$ and using \(8\) the following torque equations and boundary conditions result:
\[
- \Gamma_\theta = K_1 \phi_{,rr} + K_3 \phi_{,\theta\phi}/r^2 + K_2 \phi_{,\theta\varphi} +
+ (K_1 - K_2) \phi_{,\psi} + (K_2 - K_3) \phi_{,rr}/r + K_1 \phi_{,rr}/r
+ \theta (\chi_s A^2 + K_3 - K_1)/r^2 = 0;
\]
\[ \Gamma_\phi = K_2 \phi_{,rr} + K_3 \phi_{,r}^2 + K_4 \phi_{,zz} + \\
+ (K_2 - K_3) \theta_{,r}^2 + (K_1 + K_3 - 2K_2) \theta_{,z}/r + \\
+ K_2 \phi_{,r}^2/r + \phi (2K_3 - K_2)/r^2 = 0; \\
\theta (R_1, \psi, z) = \theta (R_2, \psi, z) = \phi (R_1, \psi, z) = \\
= \phi (R_2, \psi, z) = 0. \quad (24) \]

If, instead of a radial field an axial field is considered, we put \( A = 0 \) in \((24) 1\) and add a term \( \chi A H_2 \) to \((24) 2\).

A comparison of \((24)\) with the equations of \([8]\) for a flat sample is straight forward. Imagine a nematic aligned along \( y \) between plates \( x = \pm h \). The plates are now bent along \( y \) and joined together such that the nematic is confined between two coaxial cylinders \( R_1 \) and \( R_2 = R_1 + 2h \); the initial ground state, in the absence of perturbations is azimuthal (Fig. 1b). The \( x, z \) dependent elastic coupling between \( \theta \) and \( \phi \) in the flat sample is replaced by the \( r, z \) dependent one in the present case. As is evident, there do not exist \( r, \psi, \psi, z \) type of elastic coupling between \( \theta \) and \( \phi \). A significant difference between the flat and cylindrical cases is that in the latter case there occur curvature related terms depending explicitly on the radial coordinate; this has the effect of destroying modal symmetry; \( \theta \) and \( \phi \) are asymmetric wrt the sample centre. As the initial director orientation presents a pure bend due to the sample being closed along the \( \psi \) direction, the bend elastic constant enters the picture as an additional parameter. In principle, the radii \( R_1 \) and \( R_2 \) are two additional free parameters in the problem. It is found, however, that quantities of interest, such as threshold ratio and dimensionless domain wave vector depend only on \( R_{21} \).

Keeping in mind this analogy, it is natural to neglect \( \psi \) dependence and assume that \( \psi, \phi \) depend on \( r \) and \( z \) (this move will be seen to be justified in a later section). Seeking periodic solutions having the \( z \) dependence \( \exp (i q_z z) \) there result from \((24)\) a pair of ordinary coupled differential equations in \( \theta \) and \( \phi \). A numerical solution results in a threshold condition. For a given set of parameters, the lowest possible field potential \( A_N(q_z) \) is determined at a given \( q_z \). When \( q_z \) is varied and \( A_N \) studied as a function of \( q_z \), the neutral stability curve results. If this curve has a minimum \( A_F(q_{zc}) \) at some \( q_z = q_{zc} \), \( A_F \) is regarded as the PD threshold and \( q_{zc} \) as the domain wave vector at PD threshold. \( A_F \) is now compared with the corresponding AD threshold \( A_F \) (Eq. (15); Sect. 3.2). If \( M_R = A_F/A_F < 1 \), PD is considered to be more favourable than AD. As is evident, above PD threshold, the director field should exhibit modulation along the axial direction. If this PD is at all possible, it should develop some what like a static analogue of the Taylor instability in isotropic liquids \([23]\).

The radial field is everywhere directed normal to the sample walls and to the initial director orientation \( n_0 = (0, 1, 0) \). Hence, it is natural to consider the case \( K_1 > K_2 \). Figure 3 depicts the plots of \( M_R \) and the dimensionless threshold wave vector \( Q_{zc} = q_{zc} h \) as functions of reduced elastic constant \( k_1 = K_1/K_2 \) for different \( k_3 = K_3/K_2 \) and \( R_{21} \). The following conclusions can be drawn:

i) When \( k_3 \) is small, curves for different \( R_{21} \) are practically identical (Fig. 3a, 3b). This is natural as, when \( k_3 \) is small, the situation is close to that of a flat sample as long as \( R_{21} \) is not very large. Variation of \( R_{21} \) within reasonable limits will only correspond to changing the sample thickness \( 2h = R_2 - R_1 \) which gets scaled out.

ii) As \( k_1 \) is decreased from a high value, \( M_R \) increases and \( Q_{zc} \) decreases. When \( k_1 \to k_1 c = 3.3 \), \( M_R \to 1 \) and \( Q_{zc} \to 0 \). Thus, for \( k_1 < k_1 c \), AD is more favourable than PD.

iii) When \( R_{21} \approx 1 \), \( M_R \) and \( Q_{zc} \) behave as in figures 3a, 3b (see curves 1, Figs. 3c-3f) even for large \( k_3 \). It is again clear that when the sample radii are large wrt the sample thickness, the situation is close to that of a flat sample with \( k_3 \) not having much effect on the field ratio or wave vector.

iv) When \( k_3 \) is large and \( R_{21} \) sufficiently higher than unity, the variations of \( M_R \) and \( Q_{zc} \) with \( k_1 \) are markedly affected. For instance, \( k_{1c} \) decreases when \( R_{21} \) is enhanced (curves 2, Figs. 3c, 3d). In other cases (Figs. 3c-3f) the behaviour of \( M_R \) and \( Q_{zc} \) is quite different. When \( k_1 \) is diminished from a high value, \( M_R \) increases and \( Q_{zc} \) diminishes initially. However, when \( k_1 \) attains a sufficiently low value, \( M_R \) starts decreasing with \( k_1 \) instead of increasing further to unity; in the same region, \( Q_{zc} \) starts increasing as \( k_1 \) is diminished. When \( k_1 \to k_1 a = 1 \), \( M_R \to 0 \) and \( Q_{zc} \to 0 \). As \( A_F \) the AD threshold is finite and real at \( k_1 = k_1 a \), one must conclude that \( A_F \to 0 \) as \( k_1 \to k_1 a \) for the given \( k_3 = k_3 j \) and \( R_{21} = R_G \). Alternatively, for a material with \( k_1 = k_1 a \) and \( k_3 = k_3 j, R_{21} = R_G \) can be regarded as GT for PD which occurs with wave vector \( Q_{G0} \). This means that a material with \( k_1 = k_1 a \) and \( k_3 = k_3 j \) may undergo PD spontaneously when it is enclosed between coaxial cylinders whose radial ratio \( R_{21} \approx R_G \). Thus, there do exist values of material parameters for which PD has GT lower than that of AD (18).

v) For fixed \( k_3, k_1 a \) increases with \( R_{21} \) for a given \( R_{21}, k_1 a \) increases with \( k_3 \). This can be appreciated by remembering that \( k_3 \) provides the main destabilizing effect for causing a spontaneous deformation and that \( k_3 \) produces the main stabilizing torque.

The above results motivate a study of \( M_R \) and \( Q_{zc} \) as functions of \( R_{21} \) for different material parameters \( k_1 \) and \( k_3 \) (Fig. 4). It is seen that as \( R_{21} \) is increased from a value close to 1, \( M_R \) decreases and \( Q_{zc} \) increases. The material parameters are so
Fig. 3. — Geometry 2. Radial field \( \mathbf{H} = (A/r, 0, 0) \). \( M_R = A_P/A_F \) where \( A_P \) is the PD threshold and \( A_F \) is the AD threshold (15). \( Q_{zc} = q_{zc} \) \( h \) is the dimensionless wave vector at PD threshold, \( h \) being the semi gap width. Curves are drawn for radial ratio \( R_{21} = (1) 1.02 \) \( (2) 1.36 \) \( (3) 1.645 \) \( (4) 1.789 \). \( k_1 = K_1/K_2 \); \( k_3 = 1.5 \) in (a, b); \( k_3 = 7 \) in (c, d); \( k_3 = 15 \) in (e, f). \( k_3 = K_3/K_2 \). \( MR \) vs. \( k_1 \) in (a, c, e); \( Q_{zc} \) vs. \( k_1 \) in (b, d, f). For small \( k_3 \), curves for different \( R_{21} \) are close to one another; a small \( k_3 \) implies behaviour approximately like that of a flat sample. Again, for \( R_{21} \approx \) the variation of \( M_R \) or \( Q_{zc} \) is like that in a flat sample. For higher \( R_{21} \), however, the variation of \( M_R \) in the small \( k_1 \) range indicates a possibility of GT which may be lower than GT of AD (Sect. 4).

chosen \( (k_1 > 5) \) that GT for PD > \( (R_{21})_{24} \), the GT for AD. Hence, the results are meaningful only for \( R_{21} < (R_{21})_{24} \) (Eq. (18)). For \( R_{21} \approx (R_{21})_{24} \), AD will occur spontaneously. In this region (which is of academic interest for PD) it is found that \( Q_{zc} \) diminishes as \( R_{21} \) is increased; finally, when \( R_{21} \) attains sufficiently high values, \( Q_{zc} \) decreases rapidly appearing to tend to zero. This is a clear indication that for the material parameters considered in figure 4, PD does not have GT.

From figure 3 it is seen that GT for PD is lower than that for HD when \( k_1 \) is small and \( k_3 \) is high (such systems are known; for instance TMV [24, 25]). To calculate GT for PD, the field is equated to zero. For given material parameters and wave vector \( Q_z \), \( R_{21} \) is varied until the threshold condition is satisfied for \( R_{21} = R_{21}(Q_z) \). When \( Q_z \) is varied the plot of \( R_{21}(Q_z) \) vs. \( Q_z \) yields a neutral stability curve. If this curve has a minimum at \( Q_z = Q_{zc} \), then \( R_G = R_{21}(Q_{zc}) \) is regarded as the GT for PD with \( Q_{zc} \) as the wave vector at PD GT.

In figure 5, \( R_G \) and \( Q_{zc} \) have been studied for different material parameters. As can be seen, for a given \( k_1 \), \( R_G \) increases and \( Q_{zc} \) diminishes when \( k_3 \) is decreased from a high value. When \( k_3 \) -> a lower limit \( k_{3c} \), \( Q_{zc} \) -> 0 and \( R_G \) -> \( R_Gc = \exp[\pi/(2k_{3c} - 1)]^{1/2} \) (see Eq. (18)). Thus, for \( k_3 > k_{3c} \), GT for AD will be more favourable than that for PD. When \( k_1 \) is diminished, \( k_{3c} \) also decreases; this naturally causes augmentation in \( R_Gc \).

At a given \( k_3 \), when \( k_1 \) is increased from a low value, \( R_G \) increases and \( Q_{zc} \) diminishes. When \( k_1 \) -> \( k_{1b} \), \( R_G \) -> \( (R_{21})_{24} \) (see Eq. (18)) and \( Q_{zc} \) -> 0. Thus, for \( k_1 > k_{1b} \), GT for AD will be more favourable than GT for PD; predictably, \( k_{1b} \) is higher for a larger \( k_3 \).

Before going over to the next section it should be remarked that a similar \( r, z \) dependent PD would be possible for nematics with \( K_2 > K_1 \) provided that the field were axial (Fig. 1c). While in the flat sample case there exists a one-one correspondence [10-12] between the two cases \( K_1 > K_2 \) and \( K_2 > K_1 \) for two different field orientations, such a symmetry transformation is non-existent in the present case due to the obviously different natures of the radial and axial
fields. A preliminary calculation shows that with $K_2 > K_1$ and an axial field, the results of figures 3 and 4 can be reproduced qualitatively (the PD threshold $H_{zp}$ is compared with the AD threshold $H_{p}$ of Eq. (17); other details remain the same); quantitatively, however, there remain differences.

5. Periodic domains in geometry 3.

Under the action of a radial field $H = (A/r, 0, 0)$, the distorted director field is assumed to be $n = (S_\theta, C_\theta S_\phi, C_\theta C_\phi); \theta = \theta (r, \psi, z); \phi = \phi (r, \psi, z)$. The non-linear coupled partial differential equations which result from (3) are linearized wrt $\theta$ and $\phi$. These, along with (8) reduce to the following torque equations and boundary conditions:

$$
\Gamma_\theta = K_1 \theta_{,rr} + K_2 \theta_{,\phi\phi}/r^2 + K_3 \theta_{,zr} +
+ (K_1 - K_3) \Phi_{,r}/r + K_1 \theta_{,r} - (K_1 + K_2) \Phi_{,\phi}/r^2 +
+ \theta (\chi_1 A^2 - K_1) /r^2 = 0 ;
$$

$$
- \Gamma_\phi = K_2 \Phi_{,rr} + K_3 \Phi_{,\phi\phi}/r^2 + K_3 \Phi_{,zr} +
+ (K_1 - K_3) \theta_{,\phi}/r + K_2 \Phi_{,r}/r + (K_1 + K_2) \theta_{,\phi}/r^2 -
- K_2 \phi /r^2 = 0 ;
$$

$$
\theta (R_1, \phi, z) = \theta (R_2, \psi, z) = \phi (R_1, \psi, z) =
= \phi (R_2, \psi, z) = 0 . \quad (25)
$$

Fig. 4. — Geometry 2. Radial field $H = (A/r, 0, 0)$; Plots of $M_R$ and $Q_{zc}$ vs. the radial ratio $R_{21} = R_2 / R_1$ for different material parameters. Curves have been drawn for $K_1 / K_2 = k_1 = (1) 15 (2) 10 (3) 5. K_1 / K_2 = k_1 = (a, b) 1.5 (c, d) 7 (e, f) 15. M_R$ vs. $R_{21}$ in (a, c, e); $Q_{zc}$ vs. $R_{21}$ in (b, d, f). The dashed lines correspond to regions $R_{21} > (R_{21})_c$ of (18) where AD is more favourable than PD as AD has a GT. The decrease of $Q_{zc}$ to zero in the high $R_{21}$ range shows that for these material parameters, PD does not have GT (Sect. 4).

Fig. 5. — Geometry 2. Field free case. Plot of the geometrical threshold (GT) $R_G$ and threshold wave vector $Q_{zc}$ as functions of $k_1$ and $k_3$, $k_3 = K_1 / K_2, k_3 = K_3 / K_2$.

(a) $R_G$ vs. $k_3$ and (b) $Q_{zc}$ vs. $k_3$ for $k_1 = (1) 1 (2) 2 (3) 3$.

(c) $R_G$ vs. $k_1$ and (d) $Q_{zc}$ vs. $k_1$ for $k_3 = (1) 7.5 (2) 10 (3) 15$. As is clear, GT for PD is more favourable than GT for AD when $k_1$ is small and $k_3$ sufficiently large (Sect. 4).
If, instead of a radial field one considers an azimuthal field, one puts $A = 0$ in (25) and adds a term $X_a B^2 \psi /r^2$ to (25).

Comparing (25) with the equations of [8], one can imagine that the nematic is initially aligned along $z$ between plates $x = \pm h$. The plates are now bent and joined together at their ends in such a way that the nematic is confined between two coaxial cylinders of radii $R_1$ and $R_2$ (where $R_1 + 2 h$); the initial, unperturbed director is axially oriented. The $x$, $y$ dependent elastic coupling between $\theta$ and $\phi$ in the flat sample is replaced by the $r$, $\psi$ dependent one in the present case. There exists no $\psi$, $z$ or $r$, $z$ type of elastic coupling between $\theta$ and $\phi$. Again, as in geometry 2, there occur curvature related terms depending explicitly on the radial coordinate so that no modal symmetry can be expected wrt the sample centre. However, as the initial director is locally as well as globally indistorted, bend enters only via the $z$ dependence of $\theta$ and $\phi$.

Keeping in mind this analogy, $z$ dependence is neglected; $\theta$ and $\phi$ are assumed to be functions of $r$ and $\psi$. As in section 4, seeking $\psi$ dependent periodic solutions of the form $\exp(iq_\psi \psi)$, there result a pair of ordinary differential equations in $\theta$ and $\phi$. A threshold condition results from a numerical solution. The PD threshold $A_P$ is again determined as the minimum of the neutral stability curve at some $q_\psi = q_{\psi c}$. The corresponding AD threshold is found from (20) of section 3.3. The ratio $M_R = A_P/A_F$ helps determine whether PD or AD is more favourable. The following points must be noted:

i) Equations (25) go over identically into the corresponding equations written for an azimuthal field under the symmetry transformation

$$K_1 = K_2; \quad \theta \rightarrow \phi; \quad A \rightarrow B; \quad \psi \rightarrow -\psi.$$  

This is analogous to the symmetry transformation found in a flat sample [10-12].

ii) In all cases involving linear perturbations, the fluctuations are assumed to be of the form of a constant multiplied by a single valued, continuous function of the coordinates. In the case of (25) without $z$ dependence one can write

$$\theta = X_1 f_1(r, \psi); \quad \phi = X_2 f_2(r, \psi)$$

where $f_1$, $f_2$ are analytic and $X_1$, $X_2$ are constants whose relative magnitude alone is known at threshold. This arbitrariness in the individual values of $X_1$, $X_2$ is the price paid for linearization which converts the problem into an eigenvalues problem; the absolute magnitudes of the eigenvectors $\theta$, $\phi$ are not known.

When we make the ansatz

$$\theta = X_1 g_1(r) \exp(iq_\psi \psi); \quad \phi = X_2 g_2(r) \exp(iq_\psi \psi),$$

we have to take into account the fact that the sample is closed along the $\psi$ direction. If we start from some point $(r, \psi, z)$ in the sample and traverse once round in the $r$, $z$ plane we come back to the same point $(r, \psi + 2 \pi, z)$. We naturally require that $\theta$, $\phi$ do not change under this transformation. This simply means that $\theta(\psi) = \theta(\psi + 2 \pi)$; $\phi(\psi) = \phi(\psi + 2 \pi)$. This implies that $q_\psi$ is an integer; loosely, we can say that $q_\psi$ is quantized. This naturally forces us to vary $q_\psi$ in steps of unity even

![Fig. 6. Geometry 3. Radial field $H = (A/r, 0, 0)$. Plots of threshold field ratio $M_R = A_P/A_F$ and $q_{\psi c}$, the dimensionless integral wave vector at PD threshold as functions of $k_1 = K_1/K_2$ and the radial ratio $R_21$. $A_P$ is the PD threshold; $A_F$, the AD threshold, is given by (20). (a) $M_R$ vs. $k_1$ and (b) $q_{\psi c}$ vs. $k_1$ for $R_21 = (1) 1.02$ (2) 1.02/0.9925 (3) 1.02/0.9785 (4) 1.02/0.952. In this $R_21$ range, $M_R$ is practically independent of $R_21$ and depends on $k_1$ alone; hence the single curve in (a). $q_{\psi c}$, however, appears to be a sensitive function of $R_21$ in addition to depending on $k_1$, alone; hence the single curve in (a). $q_{\psi c}$, however, appears to be a sensitive function of $R_21$ in addition to depending on $k_1$. (c) $M_R$ vs. $R_21$ and (d) $q_{\psi c}$ vs. $R_21$ for $k_1 = (1) 15$ (2) 10 (3) 5. While $M_R$ does not vary much in the $R_21$ range chosen, $q_{\psi c}$ is found to change sharply, especially when $R_21 \geq 1$. As curves 2 and 3 lie close to and below curve 1, only the latter has been shown. $q_{\psi c}$ is a measure of the ratio of average circumference to the wavelength of PD at threshold. Due to $q_{\psi c}$ changing in steps of unity, there exist ranges of $k_1$ and $R_21$ over which $q_{\psi c}$ shows discontinuous changes rather reminiscent of the Barkhausen effect in ferromagnetic materials (corresponding to discontinuous changes of magnetization for continuous changes of the field). No attempt has been made to draw a smooth curve even at places where it is possible to do so; this is just to emphasize that $q_{\psi c}$ varies in steps of unity (Sect. 5).
while finding the minimum of the neutral stability curve to locate the PD threshold. If \( q_{\phi c} = 0 \) at PD threshold, this will correspond to 2d extrema of \( \theta \) or \( \phi \) for a single traverse along the \( \psi \) direction round the sample.

Figures 6a, b show variations of \( M_R \) and \( q_{\phi c} \) as functions of \( k_1 \) for different ratios \( R_{21} \) of the cylinder radii. It is seen that \( M_R \) depends only on \( k_1 \) and is independent of \( R_{21} \) for all practical purposes; the different curves for \( M_R \) coincide for the four different \( R_{21} \) chosen in the range \( 1 < R_{21} < 1.05 \). At least in this range of \( R_{21} \), the variation of \( M_R \) with \( k_1 \) is reminiscent of a flat sample or that for geometry 2 in the limit of small \( k_3 \). The dimensionless integral wave vector \( q_{\phi c} \) does show a strong dependence on \( R_{21} \); \( q_{\phi c} \) decreases, at a given \( k_1 \), when \( R_{21} \) is increased; this dependence is pronounced especially in the high \( k_1 \) range. At a fixed \( R_{21} \), \( M_R \) increases and \( q_{\phi c} \) diminishes when \( k_1 \) is decreased from a high value. When \( k_1 \to k_1 d \), \( M_R \to 1 \) and \( q_{\phi c} \to 0 \). It appears that \( k_1 d \approx 3.3 \) is the same for the different \( R_{21} \); it must be remembered that \( k_1 d \) also corresponds to the limit in the case of a flat sample with planar orientation. It is thus clear that for \( k_1 < k_1 d \), AD should be possible.

A marked difference in the variations of \( q_{\phi c} \) and \( Q_{zc} \) with \( k_1 \) is seen in the high \( k_1 \) ranges where \( q_{\phi c} \) does not change appreciably with \( k_1 \) (\( Q_{zc} \) is the threshold wave vector for geometry 2). The smallest change that can occur in \( q_{\phi c} \) is unity. In the high \( k_1 \) ranges, this naturally causes \( q_{\phi c} \) to change in discrete jumps rather reminiscent of the Barkhausen effect [26, 27] in ferromagnetic materials corresponding to discontinuous variations in magnetization for a continuous variation of the magnetizing field.

The plots of \( M_R \) and \( q_{\phi c} \) vs. \( R_{21} \) for different materials again bear out the conclusions arrived at in the earlier diagrams. It is seen (Figs. 6c, d) that \( M_R \) hardly changes when \( R_{21} \) is close to unity; \( q_{\phi c} \), however, shows very rapid decrease when \( R_{21} \) is increased from a value close to unity. When \( R_{21} \) attains sufficiently large values, the rate of variation of \( q_{\phi c} \) with \( R_{21} \) again diminishes; owing to change occurring in steps of unity, \( q_{\phi c} \) again shows discontinuous jumps.

6. Discussion of results.

The differential equations for both geometries 2 and 3 do not support pure modes due to the presence of curvature related terms which mix up first and second derivatives. An attempt is made in the following subsections to understand qualitatively (via order of magnitude estimates) the destabilizing mechanism which causes the director to escape into the third dimension.

6.1 GEOMETRY 2. — It seems instructive to write down the elastic free energy density \( W' \) associated with PD for small perturbations \( \theta, \phi \). The total free energy corresponding to \( \theta, \phi \) is got by integrating \( W' \) over the entire volume. Using (1) and ignoring the contribution from the ground state, \( W' \) for geometry 2 can be written as

\[
2 W' = 2 W_A + 2 W_S + 2 W_c;
\]

\[
2 W_A = K_1 \theta \phi + (K_1 - K_3) \phi^2/r^2 + 2 (K_1 - K_3) \theta \phi /r;
\]

\[
2 W_S = K_1 \phi \phi + K_2 \phi \phi + K_2 \phi \phi + (K_2 - 2 K_3) \phi^2/r^2;
\]

\[
W_c = K_1 \phi \phi - K_2 \phi \phi + K_1 \phi \phi /r - K_2 \phi \phi /r + (K_2 - K_3) \phi \phi /r .
\] (26)

If PD has to be more favourable than AD it is clear that when the different components of \( W' \) are integrated, \( W_c \), which consists of cross terms, must contribute a sufficiently large negative value so as to bring the total contribution to a level lower than the contribution from \( W_A \) which corresponds to AD. Essentially, one can say that the excess free energy associated with the splay-bend AD is carried away by the twist with the cross terms serving as conduits. In the above from (26) it is not very obvious as to how this happens.

An alternate way of saying this is by using the related language of torques (for a review on the use of this language to describe hydrodynamic instabilities, see [28]). We can then say that owing to the presence of the cross term \( W_c \), there must exist a destabilizing torque acting out of the \( r, \psi \) plane and that this torque must result from a positive feedback mechanism (PFM). For general, curved samples it is not straightforward to appreciate how PFM works. To see this let us assume that the sample radii are much larger than the sample thickness.

This means that we ignore terms proportional to \( 1/r \) in (24). With the ansatz

\[
\theta = \theta_A (\cos q_z, r) \cos q_z z;
\]

\[
\phi = \phi_A (\sin q_z, r) \sin q_z z;
\]

\[
\Gamma_\theta = \Gamma_A \cos q_z z ; \quad \Gamma_\phi = \Gamma_A \sin q_z z ; \quad \theta_A, q_z, q_z > 0 \] (27)

without loss of generality one finds from (24) that if we start with the \( \theta \) fluctuation, this can lead to a \( \Gamma_\phi \) torque \( \sim (K_1 - K_2) \phi \phi \); this torque causes the formation of a \( \phi \) fluctuation with

\[
\phi_A \sim (K_1 - K_2) \theta_A q_z q_z / (K_1 q_z^2 + K_2 q_z^2) > 0
\]

if \( K_1 > K_2 \). The \( \phi \) fluctuation, in turn, leads to a \( \Gamma_\phi \) torque \( \sim - (K_1 - K_2) \theta_A \phi \) which has the same sign as the destabilizing magnetic torque; thus, the original \( \theta \) perturbation gets augmented and PFM is completed. The ansatz (27), though not very
accurate wrt the r variation of $\theta, \phi$, still gives the right z dependence of the fluctuations; $\theta$ and $\phi$ are exactly out of phase along z.

Two points do become clear. Firstly, if $\theta, \phi$ depend on $\psi$, this may not be energetically favourable. A $\psi$ dependence merely contributes the purely positive contribution $\sim K_3(\theta_0^2 + \phi_0^2)/r^2$ to $W_s$ in (26). As can be seen from (24) the $\psi$ dependence merely results in additional stabilizing torques $\sim -K_3 \theta_0 \phi_0/r^2$ and $K_3 \phi_0 \theta_0/r^2$. A purely formal manifestation of this is a lack of cross terms in (24) involving derivatives wrt r and $\psi$. Secondly, if $K_3 > K_1$, the torque $(K_3 - K_1)/r^2$ will have the same sign as the magnetic torque. When $R_{21}$ is sufficiently large, this elastic destabilizing torque might trigger GT.

6.2 GEOMETRY 3. — In this case AD should occur as a splay distortion $\theta$. The occurrence of PD in preference to HD merely implies that the director escapes out of the r, z plane via a twist. One again expects the free energy to have cross terms; in this case, however, these should involve variations of $\theta$ and $\phi$ with r and $\psi$. Indeed, the different parts of the free energy density $W'$ for PD can be written as

$$2 W_A = K_1(\theta_r + \theta/r^2);$$
$$2 W_S = K_1 \phi^2/r^2 + K_2(\phi^2 + \theta^2 + \phi^2)/r^2;$$
$$W_c = K_1 \phi \phi_0/r - K_2 \phi_0 \theta_0/r + K_1 \theta \phi_0/r^2 - K_2 \theta \phi_0/r^2 + K_2 \phi \phi_0/r.$$

If PD occurs with threshold lower than AD it is implied that on integration, $W_c$ must make a sufficiently large negative contribution to bring down the total free energy to a level less than the total elastic free energy for AD.

In this case it is not possible to ignore curvature related terms fully. To get an idea of PFM in this geometry we use the ansatz

$$\theta = \theta_A \cos(q_\theta \psi) \cos(q_r r);$$
$$\phi = \phi_A \sin(q_\phi \psi) \sin(q_r r);$$
$$\Gamma_\theta = \Gamma_\theta A \cos(q_\theta \psi); \quad \Gamma_\phi = \Gamma_\phi A \sin(q_\phi \psi); \quad \theta_A, q_r, q_\phi > 0 \quad (29)$$

without much loss of generality for a sample with $R_1 > R_2 - R_1$. From (25) it is found that the $\theta$ gives rise to a $\Gamma_\theta$ torque $\sim (K_1 - K_2) \theta_{r^2}/r$ which results in a $\phi$ with $\phi_A \sim (K_1 - K_2) \theta_A q_\theta / K_2 q_\phi^2 > 0$ [we ignore terms $\sim r^{-2}$ in the (25)]. If $K_1 > K_2$, $\phi$ causes a $\Gamma_\phi$ torque $\sim -(K_1 - K_2) \phi_A q_\phi/q_\phi r < 0$ which aids the destabilization by the field. The ansatz (29) again suggests that the perturbations are exactly out of phase wrt $\psi$ variation. To first order at least it is not energetically favourable for $\theta, \phi$ to depend on z as this only brings in the purely positive term $\sim K_3(\phi_0^2 + \theta_0^2)$ into the free energy and contributes purely restoring torques $\sim K_3 \theta_{r^2}$ and $K_3 \phi_{r^2}$.

6.3 CYLINDERS BETWEEN POLES OF MAGNET. — Before concluding the discussion mention must be made of a fourth configuration. Here, a nematic in geometry 3 is subjected to the action of a magnetic field $H = (H_x, 0, 0)$ (cartesian coordinates) applied normal to z along the direction $\psi = 0$. When we transform to cylindrical polars, the field takes the form

$$H = (H_r, H_\phi, 0) \quad \text{with} \quad H_r = H_x S_\psi; \quad H_\phi = -H_x S_\phi.$$

This field is everywhere normal to the unperturbed director orientation and can be expected to couple with fluctuations to produce a destabilizing torque which should produce a deformation above some threshold. If, under perturbations $\theta$ and $\phi$, $n = (\theta, \phi, 1 - \theta^2/2 - \phi^2/2)$, one again obtains (25) except that $A = 0$, a term $x_s S_\psi H_2^2(\theta C_\psi - \phi S_\psi)$ is added to (25)1 and a term $x_s S_\phi H_2^2(\phi S_\psi - \theta C_\psi)$ is added to (25)2. In a region around $\psi = 0$ (corresponding to x axis in cartesians) the sample is in the splay geometry; in the vicinity of $\psi = \pi/2$ (y cartesian axis) the sample is in the twist geometry. As the driving term is a periodic function of $\psi$, one is forced to seek solutions of the form

$$(\theta, \phi) = \sum_{k=1}^{N} f_k^{(\theta, \phi)}(r) \sin k \psi + \sum_{k=0}^{N} g_k^{(\theta, \phi)}(r) \cos k \psi$$

where N is an integer which is large enough to ensure convergence. The magnetic field term produces couplings between the different k and $k \pm 2$ functions and also between their derivatives. This leads to a system of $(2N + 1)$ coupled linear differential equations which are solved with the boundary conditions (for rigid anchoring) that the functions $g$ and $f$ must vanish at $r = R_1$ and $r = R_2$. This should lead to a threshold corresponding to deformations $\theta$ and $\phi$ which are modulated along the azimuth. Apart from such a threshold depending upon both $K_1$ and $K_2$, it is clear that azimuthal modulation may be possible in any nematic regardless of the anisotropy between splay and twist. As the total distortion is a sum of various harmonics it is difficult to guess either threshold or the nature of deformation without an exact calculation. This will be presented in a future communication.

7. Conclusions, limitations of mathematical model used.

It has been shown that when a nematic, confined between two coaxial cylinders of different radii, is subjected to a radial field it may suffer PD of
different kinds depending upon the initial orientation, material constants and geometric parameters. When the initial orientation is azimuthal (Geometry 2) and \( K_2 \gg K_1 \), the resulting PD has periodicity along the common cylindrical axis (\( z \)) and may appear similar in form to a Taylor instability in isotropic liquids. When the radial ratio \( R_{21} \) is sufficiently high, PD may exhibit GT which is lower than GT for AD over certain ranges of material parameters; the existence of GT is due to the global deformation energy associated with the initial orientation in geometry 2. In the linear approximation it seems unlikely that modulation of deformation along \( \psi \) will exist. The familiar language of torques used to understand hydrodynamic instabilities can be used to get some insight into the PFM responsible for the occurrence of PD.

When the initial director orientation is axial (Geometry 3) and \( K_3 \gg K_2 \), PD that results may have periodicity along the azimuthal direction. Due to the sample being closed along \( \psi \), the dimensionless PD wavevector \( q_{\phi c} \) is integral. Because of this reason, \( q_{\phi c} \) shows discontinuous jumps over certain ranges of geometrical and material parameters reminiscent of the Barkhausen effect in ferromagnetic materials. In this case it appears that neither PD nor AD has GT as the initial director orientation is locally as well as globally undeformed. To first order it appears that \( K_3 \) may have no effect on AD and PD thresholds; it also seems unlikely that PD will have \( z \) modulation.

A related geometry is one in which the nematic which has initial axial orientation is subjected to a constant magnetic field directed normal to the common axis of the cylinders. An examination of the governing equation suggests that a \( \theta, \phi \) deformation may occur above a threshold which should be determined by both \( K_1 \) and \( K_2 \); the deformation above threshold should have azimuthal modulation for any nematic though the extent and nature of this modulation may vary with the elastic ratio \( K_1/K_2 \); this is simply because the applied field itself shows azimuthal modulation when transformed to cylindrical polars. A complete discussion of this case will be presented elsewhere.

Geometry 1 has not been treated at all. By following the thought experiment of preparing a cylindrical sample from a flat sample it becomes clear that geometry 1 is the coaxial cylindrical analogue of the homeotropic geometry in a flat sample. It appears [10] that PD may not be possible in a homeotropic flat sample. In the present case again the same conclusion should be possible for \( R_{21} \gg 1 \). Detailed calculations for higher \( R_{31} \) should establish whether or not PD in any form is possible in geometry 1 when the field is applied along \( z \) or \( \psi \).

Results have been obtained using the small perturbation approximation which results in the solution of an eigenvalue problem with the perturbation amplitudes as eigenvectors. This makes it possible to calculate the relative magnitude and not the absolute magnitudes of the perturbations at threshold. It is difficult to predict the nature of PD above threshold or to show that the total free energy of PD is less than that of AD or the ground state; only non-linear perturbation calculations can achieve this. However, for geometry 2 or 3 with a radial field it appears that the existence of AD threshold is sufficient to ensure that AD is less energetic than the ground state. Hence, if PD has lower threshold than AD one can say, with some assurance, that PD is less energetic than the ground state.

Perturbations have been assumed to be rigidly anchored at the boundaries. It is now well known [11, 12, 15] that the magnitude of surface anchoring energy plays an important role in determining the range of existence of PD relative to that of AD. A similar result cannot be ruled out for PD in cylindrical geometry.

That (4), or equivalently (2), has been assumed to be satisfied must not be lost sight of. This means that the elastic stresses which are created due to the distortions in the director field are assumed to be balanced by a hydrostatic pressure \( p \) whose spatial variation is given by (4). The validity of (4) may be unquestionable for thin samples. The doubt still remains whether (4) can be assumed to be valid for thick samples too.

Mention must be made of practical difficulties which may arise in an actual experiment. One will naturally be the difficulty of imparting proper surface treatment to the inner and outer surfaces of the outer and inner cylinders, respectively. The second difficulty is to impress a destabilizing radial field on the sample. The third is of course connected with the ease with which optical observations can be made.

In terms of convenience it appears easier to impress a destabilizing electric field on the sample (assuming, of course, that the sample has positive dielectric susceptibility anisotropy). This should be possible by coating the sample boundaries with a conducting material. In terms of theoretical analysis, however, the electric field case is much more complicated than the magnetic field one. Firstly, it is not possible to think of electric field effects without bringing in flexoelectricity [29, 30]. Secondly, considering that the effect of flexoelectricity is to impart large additional elastic anisotropy to the material in certain geometries [31] it seems all the more important to include this effect while considering electric field induced PD; after all, the relative occurrence of PD and AD does depend critically on the elastic anisotropy. Lastly, the perturbation of the electric field by the perturbations has to be taken into account fully; this is not a very simple task (see, for instance, [5]; also [32] and references there in).
Hence, the possibility of electric field induced PD will be dealt with separately.

Finally, it must be stressed that the PFM dealt with in section 6 merely indicates why the director field has a propensity to move out of the plane determined by \( n_0 \) the ground state and \( \mathbf{H} \) the magnetic field. When the transition does occur, it will involve dynamics. A full discussion of the transition is possible only by including viscous effects.

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