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Abstract. — The behaviour of a certain class of periodic minimal surfaces is studied. It becomes evident that simple deformations along the crystallographic axes of these surfaces yield two sets of solutions to the boundary problem; one stable and one unstable. It is shown that the surfaces can be extended only within certain limits. The relationship to analytical minimal surfaces is explored.

Introduction.

Examples of infinite periodic minimal surfaces (IPMS) have been known since the beginning of last century [1], but progress has been rather slow, partly due to the seeming lack of physical applications to this branch of mathematics. During the last few years IPMS have been proven useful for a wide range of structural descriptions [2-4]. Nesper and von Schnering have, using extensive Ewald calculations, shown the link between IPMS and equipotential surfaces [5]. One field of particular interest is the connection between continuous deformations of IPMS and simple tetragonal, orthorombic and rhombohedral distortions of crystal structures.

Minimal surfaces are surfaces of zero mean curvature. It has been shown [6] that such surfaces are critical points for the area function of a surface with a given boundary. It should be noted that these critical points need not be minima and hence that minimal surfaces are not always surfaces of minimal area. The reverse is, however, true. A distinction is often made between minimal surfaces which are also local minima to an area function and these which are not. The former are said to be stable while the latter are called unstable. Stable minimal surfaces can be realized with open soap films whereas unstable ones can not since surface tension tends to minimize area.

If the minimal surface can be continued smoothly and periodically beyond the original boundaries, an IPMS will result. If the surface is bounded by straight lines, continuation by 180° rotation around that line is permitted. If the boundary consists of plane curves, smooth continuation is accomplished by a mirror operation in that plane [7].

One of the most important tools of investigation of IPMS is the Weierstrass parametrisation [8] which gives the Cartesian coordinates of an IPMS in terms of elliptic integrals

\[
\begin{align*}
    x &= x_0 + \text{Re} \int_{\omega_0}^{\omega_1} (1 - \omega^2) R(\omega) \, d\omega \\
    y &= y_0 + \text{Re} \int_{\omega_0}^{\omega_1} i (1 + \omega^2) R(\omega) \, d\omega \\
    z &= z_0 + \text{Re} \int_{\omega_0}^{\omega_1} 2 \omega R(\omega) \, d\omega
\end{align*}
\]

It has recently been shown [9] that the Weierstrass function, \( R(\omega) \), can be determined from the flat points of the IPMS. If the Gauss map of the IPMS is mapped onto the complex plane by a standard stereographic projection the function \( R(\omega) \) is of the form

\[
R(\omega) = \kappa \prod_{\alpha=1}^{n} (\omega - \omega_\alpha)^{-1/b_\alpha}
\]

where \( \omega_\alpha \) are the images in the complex plane of the flat points of the IPMS, \( b_\alpha \) is the degree of the Gauss function.
map at the flat point $\alpha$, defined as the ratio between the angle of intersection between any two geodesics through the flat point ($\gamma$) and the Gauss map image of that same angle ($\phi$). $\kappa$ is a complex constant. Bonnet [10] showed that changing the argument of $\kappa$ corresponds to an isometric bending of the surface. In mathematical parlance two surfaces related by such a bending are associate. If the angle of association ($\arg(\kappa_1) - \arg(\kappa_2)$) is $\pi/2$ they are adjoint.

The catenoid, the surface generated by rotating the catenary $y = a \cdot \cosh(z/a)$ around the $z$ axis is the only minimal surface which is also a surface of revolution. In terms of soap films, it is the connected surface spanning two circular rings. There is a second stable minimal surface solution to this boundary problem; two disjoint planar discs. According Hildebrandt and Tromba [11], an unstable minimal surface is always to be found between two stable ones. In this case a second catenoid may be constructed, having a larger area and a narrower «waist». Whereas the disc solution is always possible, regardless of the distance between the generating circles, no connected minimal surface will form if the two circles are too far apart. If the circles both have radius $r$, the maximum distance between them is about $1.32r$ [11]. For larger distances no minimal surface, and hence, no surface of minimal area exists.

The fact that there is a limit to how far a catenoid can be stretched is of importance to IPMS as well, since these contain units resembling catenoids. IPMS generated by such units are called ring-like by Schoen [12]. The «catenoids» of the ring-like surfaces span, not two circles. But any two closed planar contours. In this paper we will deal with three classical IPMS given by Schwarz [7]. These are $P$, $D$ and $H$. There is a number of ways to generate these surfaces. In figure 1 they are represented by a set of boundaries that emphasize their ring-like properties.

The cubic surfaces $P$ and $D$ have fixed distances between their generating curves while there is no «ideal» $c/a$ ratio for $H$. We will assign the name $P_t$ (tetragonal) to any surface generated by the boundary in figure 1a with a $c/a$ ratio (defined as the ratio between the distance, between the generating polygons and the side of those polygons) differing from the ideal value. The name $D_r$ will be used for the family of surfaces generated by a rhombohedral distortion of $D$.

The surfaces $P$ and $D$ are adjoint. This is also true for their tetragonal, orthorhombic and rhombohedral analogues.

The surfaces $P_t$, $D_r$ and $H$.

The Weierstrass polynomials of $D_r$ and $H$ are elucidated in appendix I. They are:

Fig. 1. — The ring-like minimal surfaces. a) The $P$ surface. b) The $H$ surface. c) The $D$ surface.
For $D_t$ the function $R(\omega)$ was determined recently \cite{13} to be
\[
R(\omega) = \left[ \frac{6 (1 - E^2)^{3/2}}{E(3 + E^2)} \right] \nonumber
\]

\[f(E) = \frac{6 - 14 E^2}{(2 + \sqrt{1 - E^2})} \nonumber\]

\[f_1(E) = \frac{6 - 14 E^2}{(2 + \sqrt{1 - E^2})} \nonumber\]

\[f_2(E) = \frac{E^2}{(1 - \sqrt{1 - E^2})} \nonumber\]

\[f_3(E) = \frac{6 - 14 E^2}{(2 + \sqrt{1 - E^2})} \nonumber\]

For $D_t$, the function $R(\omega)$ was determined recently \cite{13} to be
\[
R(\omega)_{D_t} = [\omega^8 + 6 f(E) \omega^7 - 4 \omega^6 + 14 f(E) \omega^5 + 
+ 6 \omega^4 - 14 f(E) \omega^3 + 4 \omega^2 - 6 f(E) \omega + 1]^{-1/2} \nonumber
\]

\[f(E) = \left( \frac{4 E}{1 - E^2} \right)^2 + 2 \nonumber\]

Since $D_t$ and $P_t$ are adjoint $R(\omega)_{P_t} = R(\omega)_{D_t}$.

The parameter $E$ is equal to the absolute value of the $z$-component of the normalized flat point normal vectors. Hence $0 \leq E \leq 1$. The cubic surfaces $D$ and $P$ are generated by $E = 1/3$ and $E = \sqrt{1/3}$ respectively.

The ring-like surfaces all behave similarly when expanded along the unique axis. They all allow arbitrarily small $c/a$ ratios, but there is a limit to how far they can be stretched. To this critical $c/a$ value corresponds a value of the free parameter $E$ of the $R(\omega)$ function. We will designate this $E_{crit}$. If $E$ is changed beyond $E_{crit}$ the surface contracts along the unique axis, yielding a second minimal surface solution to the boundary problem connected to that particular $c/a$ ratio. This is the unstable minimal surface that always exists between two stable solutions \cite{11}, figure 2. The relationship between $E$ and $c/a$ is shown in figure 3.

**Special features of the $P_t$ surface.**

There is an interesting link between the $P_t$ surface and a well known analytic minimal surface, Scherk’s second surface (Fig. 4).

When the $c/a$ ratio of $P_t$ tends to zero the surface will approach Scherk’s second surface. This is consistent with the relation found between the adjoint surfaces, $D_t$ and Scherk’s first surface.

**Special features of the $H$ surface.**

In congruence with $P_t$, $H$ is related to a one-periodic minimal surface. Scherk’s second surface can be regarded as two
Fig. 3. — The relationship between elevation parameter, $E$, and $c/a$ ratio for the $D_r$, $H$ and $P_1$ surfaces. The $c/a$ ratio is defined as the ratio between the distance between the generating polygons and the side of those polygons.

planes interpenetrating at right angles. Analogous to this, $H$ approaches a surface consisting of three planes interpenetrating at an angle of $\pi/3$ when the parameter $E$ tends to zero. This surface is one of the saddle towers described by Karcher [14] (Fig. 5).

**Special features of the $D_r$ surface.**

The $D_r$ surface is of particular interest because of its link to the Gyroid ($G$) surface [12], the only known IPMS with neither mirror planes nor two fold axes. The discoverer of this surface, Alan Schoen states that no lower symmetry forms of $G$ exist and proves it for the orthorhombic-tetragonal case. We have been able to show that a hexagonal form of $G$, $G_h$ indeed exists. $G$ is an associate surface to $D$ with an angle of association of $38.015^\circ$. $G_h$ is related to $D_r$ by the same operation. The relation between the $c/a$ ratio of $D_r$ and the elevation parameter $E$ is shown in figure 3.

Fig. 4. — Scherk's second surface.

Fig. 5. — A Karcher saddle tower consisting of three smoothly interpenetrating planes.

$D_r$ is related to the helicoid via a compression along the rhombohedral axis. The helicoid represents the surface $D_r$ with $E$ approaching 0.

**Conclusions.**

The ring-like surfaces exhibit limited extensibility. The surfaces generated by postcritical parameter values will therefore have to be fundamentally different from those generated by precritical values. They are in fact the unstable solutions to the boundary problem. The analysis of the ring-like surfaces yields a collection of new families of IPMS that may prove useful for the description of crystal behaviour.

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**Appendix I. The Weierstrass representation of the surfaces $D_r$ and $H$.**

Inspecting a single fundamental unit of each
surface provides an easy way of determining the position of the flat points as well as their normal vectors. The orthogonality requirements for main curvatures and asymptotes on minimal surfaces yield two simple rules:

- Flatpoints occur when two asymptotes or two planes of maximum curvature intersect at an angle $\neq \pi/2$.
- Flatpoints occur when an asymptote intersects a plane of maximum curvature at right angles.

In both surface elements (Fig. A.I) we find one twelfth of a flatpoint with a fixed, vertical normal vector and one fourth of a flatpoint with variable normal direction. Both surface elements constitute one twenty-fourth of the unit cell of their surfaces and hence each unit cell possesses two flat points with fixed normal vectors and six flat points with variable normal vectors. The Gauss map of the surface elements and all flatpoints in the unit cell are depicted in figure A.II.

Fig. A.II. — The Gauss maps of the $H$ and $D_r$ surfaces. The checked regions are the images of the fundamental surface elements.

The exponent $b_\alpha$ in the Weierstrass function (1) is easily determined from the Gauss map. The angles made by the geodesic boundary lines at the flatpoints (Fig. A.I) are $\pi/6$ ($\gamma_A$) and $\pi/2$ ($\gamma_B$). The corresponding images on the Gauss maps (Fig. A.II) are $\pi/3$ ($\phi_A$) and $\pi$ ($\phi_B$). Thus $b_\alpha = \phi/\gamma = 2$ at all flat points on both surfaces.

The Weierstrass function and the integration domain of each surface element can now be elucidated.

The $H$-surface's flat point normal vectors are of the form:

$$
\left(\begin{array}{c}
0 \\
\pm \sqrt{1-E^2} \\
-1/2 \sqrt{1-E^2}, \pm \sqrt{3/2 \sqrt{1-E^2}}, \pm E
\end{array}\right)
$$

Since one normal vector $(0, 0, 1)$ will coincide with the North pole in the Gauss map rotation of the whole Gauss map is necessary prior to the stereographic projection. The Gauss map is therefore rotated one quarter turn clockwise around the $y$-axis.

This yields a new set of normal vectors:

$$
\left(\begin{array}{c}
\pm 1 \\
0 \\
0
\end{array}\right), \left(\begin{array}{c}
\pm E \\
0 \\
\sqrt{1-E^2}
\end{array}\right), \left(\begin{array}{c}
\pm \sqrt{3/2 \sqrt{1-E^2}} \\
-1/2 \sqrt{1-E^2}, \pm 1/2 \sqrt{1-E^2}
\end{array}\right)
$$

Standard stereographic projection onto the complex plane results in:
Applying equation (1) yields the Weierstrass polynomial:

\[ R(\omega)_H = \left[ \omega^8 + (f_1(E) - f_2(E) - 1) \omega^6 + (f_3(E) - f_1(E) f_2(E) - 1) \omega^4 - (f_2(E) f_3(E) + 1) \omega^2 + f_2(E) f_3(E) \right]^{-1/2} \]

\[
\begin{align*}
  f_1(E) &= \left( \frac{6 - 14 E^2}{(2 + \sqrt{1 - E^2})^2} \right), \\
  f_2(E) &= \left( \frac{E^2}{(1 - \sqrt{1 - E^2})^2} \right), \\
  f_3(E) &= \left( \frac{(3 + E^2)^2}{(2 + \sqrt{1 - E^2})^2} \right).
\end{align*}
\]

The integration domain is the stereographic projection onto the complex plane of the Gauss map image of the original surface element. This is the domain \( \text{Re} \, k \geq 0, \, \text{Im} \, k \geq 0, \quad (\text{Im} + \sqrt{1/3})^2 + \text{Re}^2 \leq 4/3 \). (Compare Fig. A.III.)

The \( D_t \)-surface flatpoint normal vectors are of the form:

\[
\pm \left( \begin{array}{ccc}
  0 & 0 & 1 \\
  \sqrt{1 - E^2} & 0 & -E \\
  1/2 \sqrt{1 - E^2} & \sqrt{3/2} \sqrt{1 - E^2} & E
\end{array} \right).
\]

The coincidence problem with a flatpoint normal vector image and the Gauss map North pole reoccurs here and it is solved in exactly the same way as in the \( H \)-surface. A one quarter turn rotation around the \( y \)-axis results in a new set of flat point normal vectors:

\[
\pm \left( \begin{array}{ccc}
  1 & 0 & 0 \\
  E & 0 & \sqrt{1 - E^2} \\
  -E & \pm \sqrt{3/2} \sqrt{1 - E^2} & 1/2 \sqrt{1 - E^2}
\end{array} \right).
\]

The result of stereographic projection is:

\[
\pm \frac{1}{1 - \sqrt{1 - E^2}}
\]

\[
(\pm \sqrt{3/2} \sqrt{1 - E^2})/\left(1 + 1/2 \sqrt{1 - E^2}\right)
\]

\[
(1 - 1/2 \sqrt{1 - E^2}).
\]

Applying equation (1) produces the function \( R(\omega) \):

\[
R(\omega)_{D_t} = [\omega^8 + 6 f(E) \omega^7 - 4 \omega^6 + 14 f(E) \omega^5 + 6 \omega^4 - 14 f(E) \omega^3 + 4 \omega^2 - 6 f(E) \omega + 1]^{-1/2}
\]

\[
f(E) = \frac{5(1 - E^2)^{3/2}}{E(3 + E^2)}.
\]

The integration domain is the same as for the \( H \) surface. (Compare Fig. A.IV.)

The Cartesian coordinates of the surfaces are then easily calculated by solving the Weierstrass equations numerically.
References


