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Spin glass model for a neural network: associative memories stored with unequal weights

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Abstract. — A long range Spin Glass model is considered in order to study a neural network in which a finite number \( p \) of random patterns has been stored or memorised, each with unequal weight. This is done in order to reflect different degrees of training. The stability of these memories is studied and the possible existence of spurious stable states is investigated.

1. Introduction.

The physical models for neural networks (NN) have been available for a long time [1]. However, it was not until recently that many of the characteristics and processes involved in their collective behaviour started to be understood and as a consequence studies about the capabilities of NN to store information are currently being done [2]. These models describe a memory which is non-local and insensitive to defects as their ordered states (to be defined later), are a consequence of the cooperative behaviour of the infinite system.

This progress has been possible by making use of some existing analogies between NN and a class of magnetic systems with competing interactions called Spin Glasses (SG). In this way, both systems are described by using Ising spins which take two possible values \( S_i = \pm 1 \) depending on whether the neurons are active (spins are up) or inactive (down). On the other hand between a given pair of connected neurons (spins) there can be an excitatory (ferromagnetic) or inhibitory (antiferromagnetic) signal. Finally a NN (SG) can be characterised by a level of noise (temperature) \( T \) which can be a function, among other parameters, of the temperature, energy provided by chemical reactions, etc. If the analogy is taken further, it is possible to assign a Hamiltonian to the NN and some concepts such as order parameters, correlation functions, phase transitions, frustration [3], etc. can be introduced as we would expect them to have a counterpart in large NN. This has already been discussed in the literature [4]. With this in mind, we will use SG and NN terminologies interchangeably in this paper.

Due to the mixture of excitatory and inhibitory interactions, the NN Hamiltonian has local minima in the space of states which act as attractors with a
basin of attraction. In other words, given an initial condition the system relaxes to a local minimum, that is, to a minimum of the free energy which has a large overlap with the initial state [5]. As a consequence the minima of the Hamiltonian can be conceived as memorised states. There can be two kind of memories or ordered states: pure and mixed memories. Mathematically, pure (mixed) memories are associated with one (several) order parameter(s). Therefore a successful model would be one in which the states characterised by having simultaneously more than one order parameter different from zero are not stable as they would correspond to a mixture of memories and their existence would make difficult the process of retrieving information.

In NN it is assumed that the synaptic efficiencies are the result of a learning process. This assumption is supported, for example, by studies in very simple organisms such as aplysia, where direct evidence has been found [6] that the process of learning involves a modulation in the strength of previously existing synaptic connections, the only difference between short and long term memory being the length of time during which the modifications of the synapses persist.

Amit et al. [7] have studied the case in which the synaptic strength $J_{ij}$ between neurons $i$ and $j$ is given by the sum of a finite number $p$ of memorised random patterns (Hebb rule). They found that, for $0 < T \leq 1$ the energy ground states were all related to pure memories, these being the only minima for $0.461 \leq T \leq 1$. They also found other solutions which they classified as symmetrical and asymmetrical states characterised by having an equal or unequal contribution from several memories. The symmetrical solutions appeared for any $T \leq 1$. For $T \leq 0.461$ some of them became stable, and as $T \rightarrow 0$ all symmetric solutions involving an odd number of memories became local minima. The asymmetric solutions appeared only below $T \sim 0.57$ and none of them was found to be stable, at least for $0.461 \leq T \leq 0.57$.

Now, we intend to extend their results by considering non uniform synaptic strengths in order to include adaptation effects. This will be done by considering the synaptic strength $J_{ij}$ as given by the sum of a finite number $p$ of memorised random patterns each stored with unequal weight $J_{ij}$. For this system the partition function is given by

\[ Z = \text{tr} \exp \left\{ -\beta H \right\} \]

with $H$ given by equation (2.1), and $\beta = 1/T$ being related to the noise level $T$ of the neural network, $T$ small corresponding to a low level of noise. The sites can be decoupled by reorganizing the terms. In this way we obtain

\[ Z = \text{tr} \exp \left\{ \frac{\beta}{2N} \sum_{\mu} J_{ij} \left( \sum_{i} \xi_{i}^{\mu} S_{i} \right)^{2} \right\} \]

where a multiplicative constant has been neglected. If we now use the Hubbard-Stratonovich transformation [13], we obtain

\[ Z = \int \prod_{\mu} dq_{\mu} \exp \left[ -N f \{ q_{\mu} \} \right] \]

with

\[ f \{ q_{\mu} \} = \left( \frac{\beta}{2N} \right) \sum_{\mu} J_{ij} q_{\mu}^{2} - \frac{N^{-1}}{2} \sum_{i} \ln \text{tr} \exp \left\{ \beta \sum_{\mu} \xi_{i}^{\mu} S_{i} q_{\mu} \right\} \]

where the $\{ q_{\mu} \}$ are $p$ auxiliary fields introduced in order to linearize the term $\left( \sum_{i} \xi_{i}^{\mu} S_{i} \right)^{2}$.

2. The model.

We will consider the Hamiltonian

\[ H = -\sum_{(i,j)} J_{ij} S_{i} S_{j} \]

where the interaction between the spins (or neurons) $S_{i}, S_{j}$ is a long range interaction given by

\[ J_{ij} = \frac{1}{N} \sum_{\mu} \xi_{\mu}^{i} \xi_{\mu}^{j} \]

with $\mu = 1, \ldots, p$, and $\{ \xi_{\mu}^{i} \}$ are quenched random Ising variables distributed according to

\[ P(\xi_{\mu}^{i}) = \frac{1}{2} \left[ \delta(\xi_{\mu}^{i} - 1) + \delta(\xi_{\mu}^{i} + 1) \right] \]
In the thermodynamic limit \((N \to \infty)\), the integral in equation (2.4a) is dominated by the minima of \(f_\{q_{\mu}\}\) and therefore we can make a Taylor expansion around these points. In this way, the value of the auxiliary fields or order parameters is determined by the conditions \(\delta f / \delta q_{\mu} = 0\), which, after taking the trace imply a set of \(p\) coupled equations given by

\[
q_{\mu} = \left\langle \xi_{\mu}^\dagger \tanh \left( \beta \sum_{\nu} q_{\nu} \xi_{\nu}^\dagger \right) \right\rangle_{\xi}
\]

\(\mu, \nu = 1, 2, \ldots, p\) . \hspace{1cm} (2.5)

with the free energy given by

\[
f_\{q_{\mu}\} = \left( \beta / 2N \right) \sum_{\mu} J_{\mu} q_{\mu}^2 - \left( \ln \left[ 2 \cosh \left( \beta \sum_{\nu} q_{\nu} \xi_{\nu}^\dagger \right) \right] \right)_{\xi} . \hspace{1cm} (2.6)
\]

In these expressions we have used the notation \(\langle \ldots \rangle_{\xi}\) to denote the site average over the random variables \(\{\xi_{\mu}\}\) given by equation (2.3). A second condition must be satisfied in order to have stability, that is, the Hessian of the problem whose elements are given by

\[
H_{\mu\nu} = \delta^2 f / \delta q_{\mu} \delta q_{\nu}
\]

\[
= \left[ \beta J_{\mu} \left( \delta_{\mu\nu} - \beta J_{\nu} \left( \tanh \left( \beta \sum_{\lambda} q_{\lambda} \xi_{\lambda}^\dagger \right) \right) \right) \right]_{\xi} \hspace{1cm} (2.7)
\]

should have positive eigenvalues. Notice that for \(P(\xi_{\mu})\) given by equation (2.3) the problem has the symmetry \(q_{\mu} \leftrightarrow -q_{\mu}\) for all \(\mu\). Therefore we will restrict ourselves to consider the case with all \(q_{\mu} \geq 0\).

There are many solutions to equations (2.5, 2.3), one of them corresponding to all \(\{q_{\mu}\} = 0\). For this case the elements of the Hessian (Eq. (2.7)) are given by

\[
H_{\mu\nu} = \delta_{\mu\nu}(J_{\mu}/T)[1 - (J_{\nu}/T)] ,
\]

therefore, this particular solution is stable only for \(T > J_{1} = 1\), and according to equation (2.6) its energy is equal to zero. On the other hand, from a series expansion of equation (2.6) in powers of \(\langle q_{\mu}\rangle\) we can see that this is the only stable solution in this region, and, therefore, if we had a noise level or temperature \(T > J_{1} = 1\), it would not be possible to retrieve any learned memory. We will call this solution a \(P\)-phase, in analogy to the paramagnetic phase in magnetic materials. In the following sections we will consider the transition from this \(P\)-phase to the others.

3. Pure memories.

There are other solutions to equations (2.5) which are not trivial. Among them, the most important correspond to individual or pure memories. These are given by

\[
q_{\mu} = \tanh \left( \beta J_{\mu} q_{\mu} \right) \quad \mu = 1, \ldots, p
\]

\[
q_{\nu} = 0 \quad \nu \neq \mu \hspace{1cm} (2.8)
\]

for \(\mu\) taking one value among \(1 \leq \mu \leq p\), and \(\nu\) taking all the remaining \(p - 1\) values. For simplicity, from now on, we will mention only the order parameters which are different from zero and all the remaining ones will be assumed to take the value zero.

From equation (2.8) we find the transition temperature from the paramagnetic to the \(1\) \(q\)-phase to be given by \(T_{P\to 1q} = J_{1}\). In this case only the diagonal elements of the Hessian (Eq. (2.7)) are different from zero and the stability conditions can be summarized as (see Appendix)

\[
J_{\mu} \geq T \geq J_{1}(1 - \tanh^2 \left( \beta J_{\mu} q_{\mu} \right)) \hspace{1cm} (2.9)
\]

where the first inequality is needed for the existence and the second for the stability of this phase. Figure 1 shows the value of the order parameter \(q_{\mu}\) for four of those pure solutions corresponding to four different values of \(J_{\mu}\). The solid (dotted) line corresponding to the region of stability (instability) of these solutions. The chain curve is the stability line for the \(1\) \(q\)-phase given by \(T_{S} = 1 - q_{1}^2(T_{S})\). For \(T > T_{S}\) the order parameter \(q_{\mu}\) represents an unstable solution. As can be seen, the memory related to \(J_{1} = 1\) is stable for all \(T < J_{1}\). On the other hand, solutions associated with \(J_{\mu} < J_{1}\) exist only for \(T < J_{\mu}\); however, they need even lower noise levels given by \(T < T_{S}\) in order to become stable, all of them being stable as \(T \to 0\), with \(q_{\mu} \to 1\). In this limit, we also find that, according to equation (2.6),

\[
f_{\mu} = - \frac{1}{2} J_{\mu} .
\]

Therefore

\[
f_{1} < f_{2} < \cdots < f_{p} < 0
\]

where \(f_{\mu}\) is the free energy corresponding to the pure state associated with \(q_{\mu} \neq 0\).
4. Mixed memories.

The remaining $p - 1$ families of solutions to equations (2.5), are related to mixtures of several memories. In order to find these solutions for $n \geq 3$ it was used an iterative improvement method to minimize, until $F = 0$, the function

$$F = \sum_{\mu} \text{ABS} \left( q_{\mu} - f \left( q_{\lambda}, T \right) \right)$$

where the argument of $\text{ABS}$ is given by equations (2.5). This was done, starting from a given set of initial values $\{ q_{\mu} \}$. Due to the nature of this problem, the choice of these initial values is very important, as we have a high probability to end up in the minima with the wider basins of attraction. In order not to miss any solution, we obtained these initial values by using the known results for the phase with the immediate lower value of $q$'s different from zero: we bracketed the $nq$-phase from the high temperature side by calculating the $(n - 1) q \rightarrow nq$ transition temperature, and from the low temperature side by using the analytical results at $T = 0$. We found two different kinds of transitions or branching of the solutions: 1) Smooth transitions $(n - 1) q \rightarrow nq$.

There can be a nesting or branching of the solutions as $T$ decreases (see Fig. 4). 2) Abrupt $P \rightarrow nq$ transitions with high values for the order parameters. These solutions appear at low temperatures for $n \geq 3$ (see Fig. 3).

After finding the solutions we analysed their stability by computing the eigenvalues of the Hessian. We found the following results:

4.1 2-$q$-states. — These are states corresponding to the mixture of $2q's$ different from zero. The order parameters for these solutions are given by

$$q_{\mu} = \frac{1}{2} \left( \tanh \left\{ (q_{\mu} J_{\mu} + q_{\nu} J_{\nu})/T \right\} + \tanh \left\{ (q_{\mu} J_{\mu} - q_{\nu} J_{\nu})/T \right\} \right)$$

$$q_{\nu} = \frac{1}{2} \left( \tanh \left\{ (q_{\mu} J_{\mu} + q_{\nu} J_{\nu})/T \right\} - \tanh \left\{ (q_{\mu} J_{\mu} - q_{\nu} J_{\nu})/T \right\} \right)$$

(2.10)

with $\mu \neq \nu$ and taking one value among $1 \leqslant \mu, \nu \leqslant p$. This solution would be stable provided the following conditions were satisfied (see Appendix):

$$T > J_1 \left( 1 - q_1^2 - q_2^2 \right). \quad (2.11a)$$

$$\left( T - J_\mu \left( 1 - q_\mu^2 - q_\nu^2 \right) \right) \times \left( T - J_\nu \left( 1 - q_\mu^2 - q_\nu^2 \right) \right) > 4 J_\mu J_\nu q_\mu^2 q_\nu^2. \quad (2.11b)$$

We found that, from the high temperature side, there is a $1q \rightarrow 2q$ transition ($q_\mu \rightarrow q_\mu$, $q_\nu$) at a temperature given by

$$T_{\mu \rightarrow \mu \nu} = J_\nu \left[ 1 - \tanh^2 \left\{ q_{\mu} J_{\mu}/T \right\} \right] = J_\nu \left( 1 - q_\mu^2 \right). \quad (2.12)$$

Numerical solution of this equation for $q_{\mu}$ given by equation (2.8) shows us that this transition is possible only if $J_\mu < J_\nu \leq 1$ (and corresponds to $|q_{\mu}| > |q_{\nu}|$). If we compare this equation with the stability condition for the $1q$ states, we can see, as expected, that this transition occurs within the unstable region of the $1q$-states. On the other hand, a similar Taylor expansion of equations (2.10) around the values $q_\mu \sim 0$, $q_\nu \sim 0$, shows that a direct transition from the $P$-phase to the $2q$-phase would only be possible if we had $J_\mu = J_\nu$.

In the limit $T \rightarrow 0$, the order parameters are found to be given by $(J_\mu \ll J_\nu)$

$$q_\mu \rightarrow \frac{1}{2} \left[ 1 - (J_\mu - J_\nu)/(J_\mu + J_\nu) \right],$$

$$q_\nu \rightarrow \frac{1}{2} \left[ 1 + (J_\mu - J_\nu)/(J_\mu + J_\nu) \right],$$

this state having a free energy (Eq. (2.6))

$$f_{\mu \nu} \rightarrow -\frac{1}{4} \left( J_\mu + J_\nu \right) - (J_\mu - J_\nu)^2/(J_\mu + J_\nu) < 0.$$  

However, one of the stability conditions (Eq. (2.11)) becomes

$$2T > J_1 \left[ 1 - (J_\mu - J_\nu)^2/(J_\mu + J_\nu)^2 \right]$$

which obviously cannot be satisfied for $T \rightarrow 0$. Therefore, it is demonstrated that as $T \rightarrow 0$, all $2q$-states are unstable.

At higher temperatures $T > 0$, and for a number
of pairs \((J_\mu, J_\nu)\), equations (2.10) were solved numerically, and the solutions were found not to satisfy equations (2.12), thus showing their instability.

4.2 3 q-states. — The simplest among the remaining \(p - 2\) families of solutions to equations (2.5) is the one with three \(q\)'s different from zero given by

\[
\begin{align*}
q_\mu &= \frac{1}{4} \left[ \tanh \{A\} + \tanh \{B\} + \tanh \{C\} + \tanh \{B + C - A\} \right] \\
q_\nu &= \frac{1}{4} \left[ \tanh \{A\} + \tanh \{B\} - \tanh \{C\} - \tanh \{B + C - A\} \right] \\
q_\rho &= \frac{1}{4} \left[ \tanh \{A\} - \tanh \{B\} + \tanh \{C\} - \tanh \{B + C - A\} \right]
\end{align*}
\]  

(2.13a)

with \(\mu \neq \nu \neq \rho\), and \(A\), \(B\) and \(C\) given by

\[
\begin{align*}
A &= (q_\mu J_\mu + q_\nu J_\nu + q_\rho J_\rho) / T \\
B &= (q_\mu J_\mu + q_\nu J_\nu - q_\rho J_\rho) / T \\
C &= (q_\mu J_\mu - q_\nu J_\nu + q_\rho J_\rho) / T 
\end{align*}
\]

(2.13b)

By making a Taylor expansion of equations (2.13) around the value \(q_\rho \sim 0\) for \(q_\mu\), \(q_\nu \neq 0\), it is found that the transition temperature from the 2 \(q\) to the 3 \(q\) phase \((q_\mu, q_\nu \rightarrow q_\mu, q_\nu, q_\rho)\), is given by the solution of

\[
T_{\mu \nu \rightarrow \mu \nu \rho} = J_\rho [1 - q_\mu^2 - q_\nu^2]
\]

with \(q_\mu, q_\nu\) given by equation (2.10) evaluated at this same \(T\). The values of the \(J_\mu\) critical for which a 2 \(q \rightarrow 3 q\) transition could occur were calculated for a number of cases where the 2 \(q\)-phase existed \((J_\mu < J_\nu\). We found that this transition could occur whenever \(J_\mu < J_\nu\) \((J_\nu\) is related to the second \(q\) which ceased to be zero).

One of the stability conditions of the 3 \(q\)-phases can be written as (see appendix):

\[
T > \frac{1}{4} J_\nu (\tanh^2 \{A\} + \tanh^2 \{B\} + \tanh^2 \{C\} + \tanh^2 \{B + C - A\})
\]

(2.14)

In the \(T \rightarrow 0\) limit we find that equations (2.13) have two different solutions. The first of these solutions is given by

\[
\begin{align*}
q_\mu &= \frac{1}{4} \left[ 3 - (3 J_\mu - J_\nu - J_\rho) / (J_\mu + J_\nu + J_\rho) \right] \\
q_\nu &= \frac{1}{4} \left[ 1 + (3 J_\mu - J_\nu - J_\rho) / (J_\mu + J_\nu + J_\rho) \right] = q_\rho
\end{align*}
\]

for \(J_\mu < J_\nu + J_\rho\), with a free energy

\[
f_{\mu \nu \rho} = -\frac{1}{2} \left[ J_\mu (J_\nu + J_\rho) / (J_\mu + J_\nu + J_\rho) \right].
\]

This solution is not valid if the index are permuted, as it implies \(|q_\mu| > \{ |q_\nu|, |q_\rho| \}\) for

\[
J_\nu > \{ J_\mu, J_\rho \}_{\text{max}},
\]

which is a requirement for the 2 \(q \rightarrow 3 q\) transition to occur. If we use equation (2.14) in this limit we find that this solution cannot satisfy this condition, therefore, this solution is found to be unstable for \(T \rightarrow 0\).

A second solution exists in the low \(T\) limit when each \(J\) is smaller than the sum of the two others. This solution is given by

\[
q_\mu, q_\nu, q_\rho \sim \frac{1}{2}
\]

with an energy

\[
f_{\mu \nu \rho} = -(1/8) [J_\mu + J_\nu + J_\rho] < 0.
\]

If we study this solution at higher temperatures, we find that it disappears suddenly at a point where the \(q\)'s are not equal to each other anymore but are still different from zero (see Fig. 3). At this point the determinant of the Hessian becomes zero thus indicating that two of the surfaces defined by the order parameters become tangent.
number of $q$'s different from zero increases. We considered several solutions involving 4 $q$'s for several sets $\{J_\mu\}$ and found numerically, as a function of the temperature, the values of the order parameters and the elements of the Hessian. In all the cases we always found at least one negative eigenvalue of the Hessian, this showing the unstability of these solutions.

Figure 4 shows a typical set of solutions involving four different $J$'s whose values are included inside the parenthesis in the order in which the order parameters related to them ceased to be zero. The solution on the right hand side corresponds to 4 $q$'s for $T \leq 0.8$.

5. Discussion.

By comparing our results to those of Amit et al. [9], we can add the following remarks:

For memories stored with non uniform weights there is no symmetry between stored patterns and as a consequence not all memories are equally stable. Solutions related to memories with smaller (bigger) weights in the partition function appear at lower (higher) temperatures and have to satisfy stronger (weaker) conditions in order to be stable. We can compare figure 1 showing the pure solutions for several values of $J_{\mu}$ with the case with uniform weights were all pure solutions appear at $T = 1$ and are stable anywhere below that temperature.

Something similar happens for the mixed solutions : if all memories have the same acquisition strength there exist symmetric and asymmetric mixed solutions having and equal or unequal overlap of several order parameters. Among them the only solutions found to be stable in some temperature range are the symmetric solutions involving an equal overlap of an odd number of memories. In contrast, we find that for non uniform (strictly different) weights, symmetric solutions do not exist, since all solutions involve an unequal overlap of several memories. None of the mixed solutions we obtained
by the methods previously described was found to be stable even at low temperatures. It would be convenient to perform numerical simulations in order to reinforce or discard the possibility of this model not having stable mixed states.

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Appendix.

The Hessian of the problem is the \((p \times p)\) matrix of second derivatives \(H\), whose components are given by

\[
H_{ij} = \frac{\delta^2 f}{\delta q_i \delta q_j} = \beta J_p \left( \delta_{\mu \nu} - \beta J_\nu \times \begin{pmatrix} \xi^\mu & \xi^\nu \left( 1 - \tanh^2 \left( \beta \sum_p q_p J_p \xi^p \right) \right) \end{pmatrix} \right) \]  

(A.1)

where the average over the quenched variables \(\langle \xi^p \rangle\) given by equation (2.3) has still to be taken. This matrix is related to the fluctuations of the system around the stable mean field solution, and therefore, in order to have stability it is necessary that all its eigenvalues \(\lambda_i\) should be positive. The eigenvalues of \(H\) are given by the secular equation

\[
\det |H - \lambda I| = 0 .  
\]

(A.2)

where \(I\) is the unit matrix. The solution of equations (A.1-A.2) for the different phases will set the stability conditions for them.

1. 1 q SOLUTIONS. — In the case for the pure or 1 q-solutions only the diagonal elements of the Hessian are different to zero. Therefore equation (A.4) gives

\[
\prod_\mu (H_{\mu \mu} - \lambda) = 0 .  
\]

(A.3)

This equation yields eigenvalues \(\lambda = H_{\mu \mu}\), for all \(\mu\). Therefore, the strongest stability condition for the pure solution \(q_\mu\) is given by

\[
\lambda_1 = H_{11} = (J_1 / T) \left[ 1 - (J_1 / T) \times \left( 1 - \tanh^2 (q_1 J_\mu / T) \right) \right] > 0 .  
\]

(A.4)

2. MIXED SOLUTIONS. — The Hessian for the 2 q solutions \((q_\mu, q_\nu)\) has \(p-2\) rows and columns where the only element different from zero is the diagonal one \(H_{\mu \mu}\), and the determinant is given by:

\[
\det |H - \lambda I| = \prod_\mu (H_{\mu \mu} - \lambda) \times \left[ (H_{\mu \mu} - \lambda) \times (H_{\nu \nu} - \lambda) - H_{\mu \nu} H_{\nu \mu} \right] = 0 \]  

(A.5)

where the prime indicates that the terms \(\rho = \mu, \rho = \nu\) are excluded from the product. This equation yields eigenvalues

\[
\lambda_{\mu, \nu} = \frac{1}{2} \left( H_{\mu \mu} + H_{\nu \nu} \pm \sqrt{\left( H_{\mu \mu} - H_{\nu \nu} \right)^2 + 4 H_{\mu \nu} H_{\nu \mu}} \right) .  
\]

(A.6)

The condition \(\lambda_{\sigma} > 0\) for all \(\sigma\), is equivalent to the requirements

\[
H_{\sigma \sigma} > 0 \quad \text{all} \sigma \quad H_{\mu \mu} H_{\nu \nu} > H_{\mu \nu} H_{\nu \mu} .  
\]

For mixed solutions with a number \(t > 2\) of q's different from zero, equation (A.5) can be easily generalized. In this case we have that \(n\) lines of the Hessian have \(t\) elements different to zero. The remaining \(p-t\) lines have all but the diagonal elements \(H_{\sigma \sigma}\) equal to zero. In this way we have that

\[
\det |H - \lambda I| = \prod_\sigma (H_{\sigma \sigma} - \lambda) \times \det |H - \lambda I|_{\min} = 0 \]

(A.7)

where \(\det |H - \lambda I|_{\min}\) is the minor of \(\det |H - \lambda I|\), that is, the determinant of the submatrix formed by the elements outside the diagonal which corresponds to the q's different from zero

\[
\begin{vmatrix} (H_{\mu \mu} - \lambda) & H_{\mu \nu} & \cdots & H_{\mu \eta} \\ H_{\nu \mu} & (H_{\nu \nu} - \lambda) & \cdots & H_{\nu \eta} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\eta \mu} & H_{\eta \nu} & \cdots & (H_{\eta \eta} - \lambda) \end{vmatrix} 
\]

where \(\mu, \nu, \ldots, \eta\) are the index of the q's different from zero. Therefore, the solution of equation (A.7) gives the stability conditions for the mixed solutions.
References


