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Lateral instability modes of fronts in the atmosphere and in the oceans

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Résumé.— La stabilité latérale d’un front, séparant deux milieux de densités différentes, est étudiée pour des mouvements bidimensionnels, dans le cadre de l’approximation du plan f de la météorologie et de l’océanographie. Les écoulements considérés sont des courants gradients, de part et d’autre d’un front à trace courbe. La relation de dispersion des modes d’instabilité bidimensionnelle est calculée, et ses implications sur les phénomènes observés dans l’atmosphère et les océans sont discutées.

Abstract.— The lateral stability of a frontal surface, separating two media with different densities, is studied for two-dimensional flows, within the f-plane approximation of meteorology and oceanography. The flows we consider are gradient currents, on both sides of a curved frontal trace. The dispersion relation for the two-dimensional instability modes is derived, and its implications on observed phenomena in the atmosphere and in the oceans are discussed.

Nature often provides scientists with phenomena which are quite difficult, nay impossible to reproduce in the laboratory. One instance of such a situation happens in the atmosphere and in the oceans, where sloping fronts separate two fluids with different physical properties. These fronts are characterized by very rapid variations of temperature and density over short distances, which indeed allow their modelization as discontinuity surfaces. The atmospheric Polar front separates cold polar air from milder middle latitude air [1]. The Gulf Stream runs along a front separating the cold Slope water from the subtropical Sargasso water [2]. One can observe quite severe lateral instabilities of these frontal surfaces, and for the last forty years, meteorologists and oceanologists have been led to suspect that they had a common physical origin.

Furthermore, long and narrow streams swiftly propagate at the surface of, and within, colder media with very different physical properties, and last permanently with almost no variation of their geographical location. Some of the best known examples of such streams are the North Atlantic and the North Pacific Currents.
The observed instabilities of frontal surfaces on the one hand, the propagation of streams on the other hand, display a marked two-dimensional behaviour. If for instance, one observes the atmospheric polar front at 500 mb, air currents on both sides of the front circulate at a constant altitude, and vertical displacements can be neglected within the first order approximation. By the same token, oceanic currents flow along stream lines at constant depth.

This paper is devoted to studying the lateral stability of a frontal surface separating two fluids with different densities, by following the movements of its trace on a horizontal plane. We are not concerned with aspects of the stability of the frontal slope which involve the acceleration of gravity. We are interested only in the possible lateral displacements of the frontal surface as a whole, without any vertical deformation. The obtained results will enable us to put forward an explanation for both the observed instabilities of atmospheric and oceanic fronts, and the origin of the permanent currents.

In a previous publication [3], we have performed such a study in the case of a linear West-East frontal trace in the $\beta$-plane, separating a denser fluid located to the North of a lighter fluid. Geostrophic currents, with shear in the North-South direction, were considered and the dispersion relation for the two-dimensional lateral instability modes was derived. No $\beta$-effect was brought out, the modes depending on the sole value of the Coriolis parameter $f$ at the latitude of the frontal trace. Here we will take advantage of this finding and neglect any possible variation of $f$ in the plane where we study the stability of the frontal trace.

The frame of reference of the present study is the $f$-plane of meteorology and oceanography, which takes account of the Coriolis force for two-dimensional flows. At latitude $\phi$, the Coriolis parameter $f$ equals $2\Omega \sin \phi$, where $\Omega$ is the rate of the Earth's rotation. We will consider gradient flows on both sides of a curved frontal trace in the $f$-plane, separating two incompressible fluids with different densities. We will implicitly assume the front to be located in the mid-latitudes of the Northern Hemisphere, with the denser fluid on the Northern side of the trace and the lighter fluid on its Southern side. The frontal curve will be assumed to be a portion of a circle of radius $R$, the centre of which will be chosen as the origin of polar coordinates.

Both concavities will be studied, a Northward located centre corresponding to cyclonic flow, a Southward located centre corresponding to anticyclonic flow. The fluid in the inner part of the curve will be labelled 1, the fluid in the outer part, 2. In the case of cyclonic flow, fluid 1 will thus be the denser, whereas for anticyclonic flow, fluid 1 will be the lighter. For script lightening, indices will be omitted where unnecessary. The equations of motion of each fluid in polar coordinates are [4]:

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho} \frac{\partial \rho}{\partial \theta} - fu_r$$

(1)

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial \rho}{\partial r} + fu_\theta$$

(2)

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0$$

(3)

For two-dimensional flows, total vorticity ($\omega + f$) is a constant of the motion [5]. As $f$ is here assumed to be constant, so will be $\omega$. For each fluid, we will consider steady flow with the general characteristics:

$$u_\theta = U(r) \text{ zonal current } , u_r = 0 ;$$

vorticity $\omega = \frac{1}{r} \frac{\partial}{\partial r} (rU(r)) ; p = p^\circ (r)$.

More precisely, the stationary solution for each fluid will satisfy the gradient current relation, with total constant vorticity:

$$\frac{1}{r} \frac{\partial (rU(r))}{\partial r} + f = C$$

(4)

$$\frac{U^2(r)}{r} = -\frac{1}{\rho} \frac{\partial p^\circ (r)}{\partial r} + fU(r)$$

(5)

The stability of the frontal trace will be studied using the standard techniques of hydrodynamic stability analysis [6]. One assumes that the front with radius $R$ separating the two fluids is bent out of its initial shape with a displacement : $\xi(\theta, t) = \xi e^{i\omega t + \alpha}$. For each fluid one considers the induced perturbed flow:

$$u_\theta(r, \theta, t) = U(r) + u^\circ_\theta (r, \theta, t) ;$$

$$u_r(r, \theta, t) = u^\circ_r (r, \theta, t) ;$$

$$p(r, \theta, t) = p^\circ (r) + p^\prime (r, \theta, t)$$
After linearization, the equations of motion for each fluid are:

\[
\begin{align*}
\frac{\partial u'_\theta}{\partial t} + u'_r \frac{\partial U(r)}{\partial r} + u'_\theta \frac{\partial U(r)}{\partial \theta} + \frac{u'_r U(r)}{r} &= - \frac{1}{\rho} \frac{\partial p'}{\partial \theta} - f u'_r,
\frac{\partial u'_r}{\partial t} + \frac{U(r) u'_\theta}{r} - 2 U(r) \frac{u'_\theta}{r} &= - \frac{1}{\rho} \frac{\partial p'}{\partial r} + f u'_\theta,
\frac{1}{r} \frac{\partial}{\partial r} (r u'_r) + \frac{1}{r} \frac{\partial u'_\theta}{\partial \theta} &= 0
\end{align*}
\]

The last relation, which expresses mass conservation for an incompressible fluid, is identically satisfied by introducing a stream function \(\psi'(r, \theta, t)\) for each fluid, such that:

\[
\begin{align*}
u'_\theta &= - \frac{\partial \psi'}{\partial r} \quad \text{and} \quad u'_r = \frac{1}{r} \frac{\partial \psi'}{\partial \theta}
\end{align*}
\]

One looks for a solution of the general form:

\[
\begin{align*}
\psi'(r, \theta, t) &= \Phi(r) e^{i n \theta + s t},
\frac{1}{\rho} p'(r, \theta, t) &= \Pi(r) e^{i n \theta + s t}
\end{align*}
\]

The expression of \(\Pi(r)\) in terms of \(\Phi(r)\) can be derived, for each fluid, from the first equation of motion:

\[
\begin{align*}
in \Pi(r) &= \frac{d \Phi(r)}{dr} + \frac{i n U(r)}{r} \phi'(r) - \frac{\omega^*}{r} \Phi(r)
\end{align*}
\]

Moreover, the vorticity of each fluid \(\omega'(r, \theta, t) = - \Delta \psi'(r, \theta, t)\) can be shown to be a constant of the motion, verifying \(d(\Delta \psi'(r, \theta, t))/dt = 0\). We thus look for a solution with zero vorticity in each fluid, satisfying obvious boundary conditions:

\[
\begin{align*}
\Phi_1(r) &= A_1 \left(\frac{r}{R}\right)^n, \quad r < R
\Phi_2(r) &= A_2 \left(\frac{r}{R}\right)^n, \quad r > R
\end{align*}
\]

The component of the velocity normal to the interface in each fluid should be equated to the velocity of the interface and this condition provides the following relation for \(A_i\) in terms of \( \xi \):

\[
in A_i / R = (s + i n U_i / R) \xi, \quad \text{where} \quad U_i = U_i(r = R)
\]

Finally the continuity of pressure on both sides of the displaced interface, taking due account of the displacement of the interface in the zero order pressure gradient given in (5), leads to the following dispersion relation:

\[
s = - i \frac{n}{R} \left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}\right) \frac{\rho_2 C_2 - \rho_1 C_1}{2 (\rho_1 + \rho_2)} \pm \frac{n}{R} \left[ n \left(\frac{\rho_1 \rho_2 (U_2 - U_1)^2}{(\rho_1 + \rho_2)^2} + f \frac{\rho_1 U_1 + \rho_2 U_2}{(\rho_1 + \rho_2)} \frac{\rho_1 - \rho_2}{R} \left(\frac{\rho_1 U_1}{R} - \frac{\rho_2 U_2}{R}\right)\right]^{1/2}
\]

where \(C_i = \omega^* + f\).

This dispersion relation displays three different instabilities, respectively associated with the first three terms within the brackets. The first is a Kelvin-Helmoltz instability [6], damped by the last two terms in the brackets. The second and the third correspond to a novel and natural demonstration of instabilities of the Rayleigh-Taylor type [7], which, in the case of an interface between two layered fluids with different densities, accelerated in an acceleration field \(\gamma\), leads to the dispersion relation:

\[
s = \pm \left[\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \gamma k\right]^{1/2}, \quad \text{fluid 2 lying over fluid 1, } \gamma \text{ being counted positive upwards and } k \text{ being the wave number.}
\]

Here, the accelerations are respectively
Coriolis and centrifugal. Signs are to be carefully examined. In the case of cyclonic flow, \( p_1 > p_2 \) and \( U_1 > 0 \). Both terms should contribute to the instability, but in real situations the zonal circulation of the lighter fluid is much more intense than that of the denser fluid. Consequently, \( U_2 \gg U_1 \) and the centrifugal term is altogether negative. In the case of anticyclonic flow, signs are all reversed, i.e. \( p_1 < p_2 \), \( U_1 < 0 \) and \( |U_1| \gg |U_2| \). Both terms contribute to the instability. This actually means that anticyclonic flow at the front should be fundamentally unstable at the wavelength where the Kelvin-Helmholtz terms are non negative, while cyclonic flow at the front should be stable as long as the centrifugal term dominates the Coriolis term. Obviously all these considerations, including an assessment of the relative weights of the Kelvin-Helmholtz and the Rayleigh-Taylor instabilities, have to be checked with real figures from observations both in the atmosphere and in the oceans. A rough assessment of the Coriolis term along a linear front, for a perturbation of 100 km wavelength and a density contrast factor of \( 10^{-3} \), leads to a quite realistic growth factor \( s \) of the order of \( \frac{[U]}{10^{-6}} \), \( [U] \) (m/s) being the velocity of the swiftest flow at the front.

Moreover, in both Rayleigh-Taylor and Saffman-Taylor instabilities, one of the fingers will, if geometry allows and if it is permanently fueled, rapidly dominate the others and stretch out over long distances ([8, 9]). We propose that the North Atlantic and North Pacific Currents are natural demonstrations of such a behaviour, in a diagonal "5-point" geometry [12], westerlies merely supporting these permanent currents initially developed through the instability of the Subtropical front.

In this paper, we have presented a derivation of the dispersion relation for the two-dimensional lateral instability of a front separating two media with different densities. Both this result and its implications have to be related to, and compared with, the observed phenomena in the atmosphere and in the oceans.

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