Criticality in non-linear transport properties of heterogeneous materials

S. Roux, A. Hansen, E. Guyon

► To cite this version:

S. Roux, A. Hansen, E. Guyon. Criticality in non-linear transport properties of heterogeneous materials. *Journal de Physique*, 1987, 48 (12), pp.2125-2130. 10.1051/jphys:0198700480120212500 . jpa-00210661

HAL Id: jpa-00210661

<https://hal.science/jpa-00210661>

Submitted on 1 Jan 1987

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Classification
Physics Abstracts
05.40 — 64.60

Criticality in non-linear transport properties of heterogeneous materials

S. Roux, A. Hansen (*) and E. Guyon

Laboratoire d'Hydrodynamique et Mécanique Physique, Unité Associée au CNRS 857, Ecole Supérieure de Physique et Chimie Industrielles, 10 rue Vauquelin, F-75231 Paris Cedex 05, France

(Reçu le 23 juin 1987, accepté le 2 septembre 1987)

Résumé. — Les matériaux hétérogènes constitués d'éléments simples non-linéaires dont la caractéristique est « affine par morceaux » avec des seuils aléatoires, peuvent montrer des régimes de transition où une loi de puissance relie flux et potentiels. Ces régimes apparaissent pour des seuils bien définis. Nous suggérons que ces seuils soient des points critiques voisins du phénomène de percolation. Nous étudions numériquement la valeur du seuil par différents moyens, dans le cas dirigé et isotrope, et obtenons des estimations asymptotiques par une analyse des effets de taille finie.

Abstract. — In heterogeneous materials consisting of individual simple non-linear elements whose transport characteristics are piecewise linear separated by random « thresholds », one may encounter transition regimes where a power law relates fluxes to potentials. There is a well-defined threshold for the onset of this transition regime. We suggest that this threshold is a critical point closely related to the percolation one. We study numerically the value of the threshold by different means, in a directed and a isotropic case, and obtain estimates through finite-size scaling.

Most transport properties can be described by a linear relation between generalized fluxes and potentials. This *linear* relationship is however generally of finite extent. Beyond it, non-linearity sets in. It is often practical to replace these non-linearities by piecing together different linear parts. In many simple cases, one of these linear regimes is even degenerate in the sense that the tangent « conductivity » (i.e. derivative of the flux with respect to the potential) is either zero or infinite. Let us mention a few examples. In *diodes*, the discontinuity in conductance occurs either a zero intensity, or, at a finite one in the case of Zener diodes. In mechanics, *contacts* have also discontinuous and asymmetric characteristics as they can resist compression but no extension. In rheology, an *elastic-perfectly plastic* behaviour is similar as the material flows whenever the stress exceeds some critical value. *Bingham fluids* are viscous when submitted to high enough shear, but do not flow for low shear.

Let us consider a disordered medium consisting of elements whose characteristics behave as described above, i.e. are piecewise linear. We assume that there are only two linear regimes separated by a single threshold. This threshold is assumed to be randomly distributed, i.e. it varies from element to element. It has been shown on several examples that the overall behaviour of the medium can exhibit much less simple non-linear behaviour as a result of this disorder, such as power-law dependence of fluxes on the potentials [1, 2, 3]. In this letter we show that there is a well-defined threshold value for the potential at which the power-law regime appears. We study numerically this threshold which seems to be a critical point. Finite-size scaling analysis suggests that this critical point is closely related to the percolation critical point.

For the sake of simplicity, we will in the rest of this paper refer to potential and intensity as electrical potential and current, although it is possible to transpose what is presented here to different cases such as those mentioned above.

Let us consider a regular lattice. Its bonds are elements whose non-linear characteristic is shown in

(*) Groupe de Physique des Solides de l'Ecole Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France.

figure 1a. The element behaves as an insulator if subjected to a difference of potential less than a threshold v_g , or has a tangent conductivity di/dv which is constant, say one, if the potential difference is larger than v_g . The threshold voltage, v_g , is a random variable uniformly distributed between zero and one. Suppose now that a voltage [4] V is imposed across the network. The behaviour of the network can be described by three different regimes (Fig. 1b) :

i) for voltages V below a threshold voltage V_g , the lattice will be an insulator ;

ii) for $V = V_g$, there will exist a first connected path of conducting bonds spanning the system. For still higher voltages, the proportion of conducting bonds will increase. Therefore, the tangent conductivity of the lattice will increase as a function of V . It has been observed both numerically [1] and exper-

imentally [4] that this leads to the following power-law relation :

$$I \propto (V - V_g)^\alpha \quad (1)$$

where I is the current, and with $\alpha \cong 2.0$;

iii) finally, when the last bond becomes conductive at a voltage $V = V_s$, the tangent conductivity will become a constant. From then on I , is an affine function of V .

A « symmetric » problem can be considered through a rescaling of the currents. Suppose that the conductivity of the bonds is constant and finite until the voltage difference across them reaches a value v_g , whereafter the tangent resistance dv/di becomes zero. A similar lattice to the one we have just discussed, but made from these elements, also with random thresholds, will show an overall characteristic consisting of three domains :

i) If the voltage drop is smaller than a V_r , corresponding to V_s above, the current I is proportional to V ;

ii) for $V = V_r$, a first bond in the lattice saturates. As V is increased further, the number of bonds that saturate increases, resulting in a non-linear relation between I and V . It has been suggested that this regime could be described by a power law

$$I - I_g \propto (V - V_g)^\beta \quad (2)$$

where V_g is the potential difference across the network for which there exists a first connected path of elements with a zero tangent resistance, and I_g is the minimum current flowing through the lattice at this voltage. Using the fact that this problem is just a rescaling of the currents in the previous problem [6] one obtains $V_g = V_g$;

iii) once the voltage V_g has been reached it cannot be increased further. The characteristic therefore reduces to $V = V_g$ for $I \geq I_g$.

The value of the threshold V_g , which plays a crucial part in these problems, may be obtained directly in a purely geometrical way. Let us consider a path C joining the two ends of our lattice (where the potential drop is applied), and let us call $V(C)$ the sum of the thresholds v_g of the bonds along this path. Thus, for any path we have the property $V(C) \geq V_g$ since in the second problem discussed, the voltage drop across a bond cannot be larger than v_g , and V_g is $V(C)$ along the *first* path going superconducting as the voltage is increased. This leads immediately to the expression

$$V_g = \min_C V(C) \quad (3)$$

where the minimum is taken over all connected paths. We note that this purely geometric definition is very close to the determination of an effective percolation threshold in a finite-size geometry [7].

Fig. 1. — a) The non-linear characteristic of one of the elements that constitute the networks we study in this paper. The element is an insulator if the voltage drop across it is less than some random v_g , and has a tangent conductivity di/dv which is constant ($= 1$) otherwise ; b) The overall characteristic of the network consisting of elements as the one displayed in figure 1a. The details of this characteristic are explained in the text.

Let us briefly recall this geometrical method to obtain the effective percolation threshold : we assign to each bond i of a lattice a random number x_i uniformly distributed between 0 and 1. We decide to cut all bonds having an x_i larger than a given value P . The percolation threshold of the lattice is the lowest value of the parameter P , P_c , for which there is a continuous path through the lattice. To any connected path from one end of the lattice to the other end, C , we associate the value $X(C)$ which is the *largest* x_i occurring along this particular path. Then the effective threshold P_c is given by the minimum of $X(C)$ over all paths,

$$P_c = \min_C X(C). \quad (4)$$

The voltage thresholds v_g and the random numbers x_i play identical roles in these problems. $V(C)$ and $X(C)$ are actually just two different *norms* (« measures ») on the path C . To make this point clearer, we generalize equation (4) as follows :

$$\|C\|_\gamma = \left[\left(\sum_{i \in C} x_i^\gamma \right) / L \right]^{1/\gamma} \quad (5)$$

where the sum $\sum_{i \in C}$ runs over all bonds i along the path C , and where L is the size of the lattice. In particular $V(C)/L = \|C\|_1$ and $X(C) = \|C\|_\infty$. The value

$$\min_C \|C\|_\gamma = A_\gamma(L) \quad (6)$$

averaged over a large number of lattices, should tend to a well-defined limit when the lattice size L goes to infinity, in the same way as the averaged effective percolation threshold P_c tends to a specific limit determined only by the lattice type.

This kind of very general property of lattices can be extended to the case where the paths are restricted to be directed, i.e. when the projection of the path onto some chosen direction in the plane of the lattice is unique [8]. Besides the existence of a well-defined limit, the formal parallel drawn between the non-linear conductivity problem and the theory of percolation suggests that this threshold may be a critical point closely related to the percolation critical point.

Now, phrasing into the language of percolation the exponents α and β introduced in equation (1) and (2), appear to play a role similar to critical exponents [9]. The relation between the two « symmetric » problems introduced at the beginning is similar to the relation between a percolation lattice whose bonds either are resistors or insulators (case A) and a lattice whose bonds are either superconductors or resistors (case B). The conductivity Σ in these two problems is known to vary as

$$\Sigma \approx (P - P_c)^\nu \quad (\text{case A}) \quad (7)$$

or

$$\Sigma \approx (P_c - P)^{-s} \quad (\text{case B}) \quad (7')$$

while we deduce from equation (1)

$$\Sigma = dI/dV \approx (V - V_g)^{\alpha-1} \quad (8)$$

and from equation (2)

$$\Sigma \approx (V_g - V)^{\beta-1}. \quad (8')$$

There is however a very important point to make here. If α and β are critical exponents, they will only be found for V in the vicinity of V_g , as the exponents s and t only are obtained for P close to P_c . The previously quoted value for α however, $\alpha \approx 2.0$ [1], was the one observed for the entire range $V_s \rightarrow V_g$, and not only for V close to V_g . Thus, this value for α is the one that should be well described by mean-field theory, which predicts $\alpha = 2$ on a square lattice, and not be related to the exponents describing the criticality at V_g . This is analogous to the conductivity of the percolation lattice for P far from P_c being well described by the mean-field result $\Sigma = 2(P - 1/2)$.

Equation (3) may easily be adapted to numerical simulation by Monte Carlo methods, and particularly in the directed case.

We have studied the distribution of thresholds V_g in the directed case on square $L \times L$ lattices. The paths were restricted to be directed, that is to say each path could be followed by moving only in the direction of increasing x and y directions where x and y refer to the two axis of the square lattice. We used an algorithm very close to the one presented in reference [7]. We studied both the mean value $\langle V_g(L) \rangle$ of these thresholds of lattice of size $L \times L$ (1 000 samples varying from size 3 to 200), and the mean square deviation $\sigma_g^2(L) = \langle V_g(L)^2 \rangle - \langle V_g(L) \rangle^2$. We assumed the following form of the dependence of $\langle V_g(L) \rangle$ on the lattice size L :

$$\langle V_g(L) \rangle = V_{g\infty} + AL^{-1/\nu_\parallel} + BL^{-1/\nu_\perp} \quad (9)$$

as already encountered in the standard directed percolation case [7]. The exponents $\nu_\parallel$ and $\nu_\perp$ refer to the parallel and perpendicular correlation length exponents, and we observed that the values $\nu_\parallel = 1.7334$ and $\nu_\perp = 1.0972$ corresponding to the standard directed percolation case [10] gave very good fits for the whole range from $L = 3$ to 200, as can be seen in figure 2. The minimum χ^2 was found for $A = -0.242$, $B = 0.308$ and $V_{g\infty} = 0.231 \pm 0.002$. Moreover, we found that $\sigma_g(L)$ vary as a power law, see figure 3 :

$$\sigma_g(L) \approx L^{-z} \quad (10)$$

where $z = 0.707 \pm 0.005$. This is very close to the

Fig. 2. — The effective directed threshold voltage $\langle V_g(L) \rangle$ as a function of L for an ensemble of square ($L \times L$) lattices. the best-fit curve is shown.

Fig. 3. — $\sigma_g(L)$ for the directed square ($L \times L$) lattices as a function of L . The best-line is shown.

expression $2/(\nu_{\parallel} + \nu_{\perp}) = 0.7066$ obtained with the above estimate of $\nu_{\parallel}$ and $\nu_{\perp}$. This expression has been suggested in reference 11 as a possible exponent to accommodate the occurrence of two length scales in the scaling functions. However, we do not understand why this exponent should show up in the scaling of $\sigma_g(L)$ and not simultaneously in the scaling of $\langle V_g(L) \rangle$.

Using a norm $\|\cdot\|_{\gamma}$ with finite γ , defined in equation (5), and here chosen to be 1, allows one to estimate the threshold with a transfer-matrix technique whose efficiency in the field of percolation (and other topics) has been demonstrated in many examples [12]. The algorithm used is close in spirit to the one described in reference 7, but the lattice geometry used here is very long and of moderate width. One should note by the way that the percolation case where γ is infinite, one cannot use this algorithm : only one bond contributes to the norm

on the paths and, therefore, the averaging on the strip-like geometries is inappropriate.

We generated strips of width L , ranging from 2 to 100 and length N between 5×10^4 and 10^5 , after having omitted the beginning of the strip over a length of 1000 in order to decrease the systematic errors resulting from the edge effects. The variation of the ratio $V_g(L)/N$ is shown in figure 4. The expected dependence of $V_g(L)/N$ is of the form

$$V_g(L)/N = V_{g\infty} + DL^{-\gamma}. \quad (11)$$

Fig. 4. — The effective directed threshold voltage $\langle V_g(L) \rangle$ as a function of $1/L$ for a long strip as obtained by the transfer-matrix method.

Through linear regression we obtained a best fit of the data for $\gamma = 1.0$, $D = 0.0975$ and $V_{g\infty} = 0.231 \pm 0.002$ [13]. This is in excellent agreement with our previous estimate of the threshold voltage. However, we do not understand which symmetry is responsible for suppressing the two leading exponents $\nu_{\parallel}$ and $\nu_{\perp}$ that appeared in equation (9).

Let us now consider the general case where the paths are no longer restricted to a directed conformation. Both the previous numerical approaches to the problem are applicable. As this non-directed problem is much more time demanding, we applied the two previous methods on a much smaller scale, and with fewer samples.

The direct study of square lattices of different sizes gave estimates of the average $\langle V_g(L) \rangle$ and of the mean square deviation $\sigma_g(L)$. The results are displayed in figures 5 and 6, and are based on averages over 500 samples of sizes $L = 2$ to 32. We proceed with a similar analysis as before, but noting that ordinary, non-directed percolation has only one correlation length. Therefore, the two exponents $\nu_{\parallel}$ and $\nu_{\perp}$ are in this case equal, and are given by the single exponent ν ($= 4/3$ in two dimensions), and corresponding to equation (9) is

$$\langle V_g(L) \rangle = V_{g\infty} + AL^{-1/\nu} + BL^{-1}. \quad (12)$$

Fig. 5. — Same as figure 2, but for the non-directed case.

Fig. 7. — Same as figure 4, but for the non-directed case.

Fig. 6. — Same as figure 3, but for the non-directed case.

The second correction-to-scaling exponent, 1 is just a guess, but gives good fits. These corresponded to $V_{g\infty} = 0.227 \pm 0.010$, $A = -0.582$, and $B = 1.026$.

We also studied this non-directed case by a transfer-matrix method similar to the one used in the directed case. The width of the trips generated ranged from 2 to 32 and the lengths were 5×10^4 . The results are displayed in figure 7. Again we assumed a finite scaling form as in equation (11), and linear regression gave the following values for the best fit : $V_{g\infty} = 0.227 \pm 0.002$, $D = 0.107$ and $y = 1.0$. Again we notice that the dominant scaling exponent is different in the two geometries, square or strip.

We summarize our results as follows : We studied the occurrence of a well-defined threshold in the voltage *versus* current characteristics across networks consisting of elements, each with a conductance which is zero if the voltage drop across the element is less than some threshold, and finite and constant if the voltage drop is larger. The threshold for each element is chosen from a uniform random distribu-

tion between zero and one. The well-defined threshold in the overall characteristic of the network at a voltage V_g , and the occurrence of power-law behaviour in its vicinity indicates that this is a critical point. For the directed square lattice we found the threshold to be at $V_g = 0.231 \pm 0.002$ and for the non-directed square lattice $V_g = 0.227 \pm 0.002$. These two thresholds are surprisingly close to each other when considering the difference between the two systems they describe. We found that the standard deviation of the distribution of the effective thresholds scale with non-trivial exponents compatible with the corresponding critical exponents occurring in the percolation problem. These fits were without free parameters. Moreover, the scaling of the effective thresholds can be very well fitted with combinations of these percolation exponents. This indicates that this problem is closely related to the directed or non-directed percolation problems, respectively. The transfer-matrix calculations gave the highest precision in the determination of the thresholds, but the scaling exponents were in this case very close to the integer 1.

The characteristics chosen for the elements constituting the networks studied are but one of many possible [5]. We mentioned in the introduction many examples of other systems where such behaviour of the characteristics may occur. However, the characteristics of the elementary constituents of each of these examples have in common that it is the *first derivative* of their characteristic, and not the characteristic itself, that is discontinuous. We mention here some examples of systems where it is the characteristics themselves that are discontinuous : networks consisting of *Josephson junctions* (that become superconducting below a critical current) ; *rupture*, and, symmetrically, *adhesion* are examples from mechanics. However, a recent paper by Bradley *et al.* [14] on networks of Josephson junctions with random critical currents indicates that sharp thresholds may also occur in this class of problems.

The authors thank H. J. Herrmann, J.-P. Nadal and J. Vannimenus for useful discussions. The numerical computations were done at the Ecole Normale Supérieure on an FPS-164 computer supported

by GRECO 70 (Expérimentation Numérique). S. R. is supported by the Ecole Nationale de Ponts et Chaussées and A. H. is supported by CEN-Saclay through a Joliot-Curie Fellowship.

References

- [1] ROUX, S., HERRMANN, H. J., preprint.
- [2] TRAVERS, T., BIDEAU, D., GERVOIS, A., TROADEC, J. P. and MESSEAGER, J. C., *J. Phys. A* **19** (1986) L1033.
- STAUFFER, D., HERRMANN, H. J., ROUX, S., *J. Physique* **48** (1987) 347.
- [3] GILABERT, A., ROUX, S. and GUYON, E., preprint.
- [4] We will denote quantities referring to the entire lattice by uppercase letters, and quantities referring to the components the lattice consists of by lowercase letters.
- [5] ROUX, S., HERRMANN, H. J., HANSEN, A. and GUYON, E., *C. R. Acad. Sci. Paris* **305**, série II (1987) 943.
- [6] This argument is identical to the one given to demonstrate that the percolation thresholds is the same in the random resistor network as in the corresponding superconducting network constructed by rescaling the currents; see e.g. STANLEY, H. E., in *On Growth and Form*, Eds H. E. Stanley and N. Ostrowsky (Martinus Nijhoff Publ. Comp. Boston) 1986.
- [7] HANSEN, A. and ROUX, S., *J. Phys. A* **20** (1987) L873.
- [8] This is just as in the directed percolation problem, see for example KINZEL, W., in *Percolation Structures and Processes*, Eds G. Deutcher, R. Zallen and J. Adler (Adam Hilger Publ. Comp. Bristol) 1982.
- [9] STAUFFER, D., *Introduction to Percolation Theory* (Taylor and Francis, London) 1985.
- [10] ESSAM, J. W., DE'BELL, K., ADLER, J. and BHATTI, F. M., *Phys. Rev. B* **33** (1986) 1982.
- [11] NADAL, J.-P., DERRIDA, B. and VANNIMENUS, J., *Phys. Rev. B* **30** (1984) 376.
- [12] VANNIMENUS, J. and NADAL, J.-P., *Phys. Rep.* **103** (1984) 47.
- [13] We tried possible values for y such as $1/\nu_{\perp}$ which gave a noticeable poorer fit. It is curious that y is so close to the integer 1.
- [14] BRADLEY, R. M., KUNG, D., DONIACH, S. and STRENSKI, N., preprint.