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HAL Id: jpa-00210631
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Submitted on 1 Jan 1987

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Phenomenology of hydrodynamic interactions in suspensions of weakly deformable particles

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(Reçu le 31 mars 1987, accepté le 2 juillet 1987)

Résumé. — Quelle est l'influence des interactions hydrodynamiques sur le mouvement d'une suspension de particules déformables ? On propose ici une réponse qualitative pour une suspension qui ne sédimente pas et dont les particules ne se déforment que faiblement. On montre que l'influence de particules voisines se manifeste à travers deux fonctions seulement de la concentration quand on néglige les termes quadratiques en déformation. Le rôle du rapport des viscosités est complètement déterminé en supposant que les particules se déforment de manière homogène. On montre pourquoi les équations de Hand ne constituent pas le meilleur point de départ pour l'étude des interactions hydrodynamiques.

Abstract. — What is the influence of hydrodynamic interactions on the equations of motion of a suspension of deformable particles ? We propose here a qualitative answer for a neutrally-buoyant suspension with weak particle deformations. We show that the influence of the neighbouring particles can be pictured by only two functions of the particle volume fraction when quadratic deformation terms are neglected. The influence of the viscosity ratio is completely determined by assuming that the particles deform homogeneously. It is suggested that Hand's equations are not the most convenient starting point for discussing hydrodynamic interactions.

1. Introduction.

The equations describing a dilute suspension of elastic spheres [1, 2] or a dilute emulsion [3, 4] were obtained more than fifteen years ago. If C stands for the (small) particle deformation and e for the suspension strain rate, it was established that both dC/dt and the suspension stress τ are linear functions of e, C and e·C (when terms of order C² are discarded). A suspension of weakly deformable particles is thus characterized by six scalar transport coefficients. Their dilute-limit values were determined in the above references. For more concentrated suspensions no exact result has been obtained because of the difficulty of dealing with deformability and hydrodynamic interactions simultaneously. We just know that the transport coefficients are functions of the surface tension (or shear modulus), of the viscosity ratio λ and of the particle volume fraction φ. In the present state of art, phenomenology is the only way to make any progress.

Here we propose a simple approach where the λ-dependence of the six transport coefficients is completely determined, while the φ-dependence is expressed in terms of two functions of φ only. This approach considers a neutrally-buoyant suspension of deformable particles and discards flocculation, coalescence and break-up. This means that i) the suspension is stabilized by short-range repulsive forces between particles and the long-range attractive forces are low enough for the deflocculation to occur at a very low shear rate εd, ii) the resistance to deformation is high enough for the break-up to occur at a very high shear rate εb. We thus consider suspensions in the range εd < e < εb; this range is particularly large for suspensions of elastic particles and for stabilized emulsions with high surface tension. Moreover, the particle size will be high enough for any Brownian effect to be absent. Lastly, we shall suppose that the particle inertia is negligible and that its internal deformation is homogeneous, i.e. the velocity gradient inside the particles is supposed to depend on time only. It must be stressed that the latter assumption is not rigorously tenable when surface tension plays a role [1]. A more
rigorous approach would distinguish between the mean velocity gradient inside the particles and its value at the interface. In what follows, we will assume that both quantities are identical. We are not able to appreciate the quantitative implications of such an assumption for dense suspensions, but our final results for dilute emulsions are remarkably similar to the exact ones [3, 4]. In fact this assumption must be considered as a first step to express in a manageable form the complex problem we want to handle.

2. The four basic relations.

2.1 EVOLUTION IN TIME OF THE PARTICLE SHAPE. — Consider deformable particles with a spherical equilibrium shape (radius $r_0$). When submitted to a homogeneous deformation described by a symmetric tensor $\mathbf{C}^*$, the spheres transform into ellipsoids defined as

$$(r_0/r_s)^2 = 1 - 2 \mathbf{n} \cdot \mathbf{C}^* \cdot \mathbf{n}$$

where $r_s = r_0 \mathbf{n}$ is a point of the deformed surface. For incompressible particles, the volume of the ellipsoid is a constant and this implies some relations between the components of $\mathbf{C}^*$. This is a constraint which is not easy to take into account, but for small deformations ($C_{ij} C_{ji} \ll 1$), a convenient way to circumvent this difficulty is to represent the deformation by a symmetric and traceless tensor $\mathbf{C}$ related to the particle shape by

$$(r_0/r_s)^2 = 1 - 2 \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} + \frac{2}{3} \mathbf{C} : \mathbf{C} + O(C^3). \quad (1)$$

The particle is still transformed into an ellipsoid but its volume $V_p$ is now a constant and more precisely

$$V_p = \frac{4}{3} \pi r_0^3 (1 + O(C^3)).$$

From now on, we speak of $\mathbf{C}$ as the particle deformation tensor. If the particles deform homogeneously, their velocity at any point $\mathbf{r}$ relative to their centre-of-mass is given by

$$\frac{d\mathbf{r}}{dt} = \mathbf{\omega}_p \times \mathbf{r} + \mathbf{\phi}_p \cdot \mathbf{r} \quad (2)$$

where $\mathbf{\omega}_p(t)$ is the particles' rotation rate while $\mathbf{\phi}_p(t)$ (another symmetric and traceless tensor) is their homogeneous deformation rate. The evolution in time of $\mathbf{C}$ is related to $\mathbf{\omega}_p$ and $\mathbf{\phi}_p$; taking the time-derivative of (1) and substituting $d\mathbf{r}/dt$ from (2) one gets

$$\frac{d\mathbf{C}}{dt} + \mathbf{\omega}_p \cdot \mathbf{C} - \mathbf{C} \cdot \mathbf{\omega}_p + 2 \mathbf{Sd}(\mathbf{\phi}_p, \mathbf{C}) = \mathbf{\phi}_p + O(C^2) \quad (3)$$

where $\mathbf{\omega}_p$ is the antisymmetric tensor associated with $\mathbf{\omega}$ and $\mathbf{Sd}$ is an operator which selects the symmetric and traceless part of a tensor

$$\mathbf{Sd}(A_{ij}) = \frac{1}{2} \left( A_{ij} + A_{ji} - \frac{2}{3} A_{kk} \delta_{ij} \right).$$

2.2 CONSERVATION OF ANGULAR MOMENTUM. — Since the particle inertia is neglected and no external torque is applied, the torque exerted by the fluid on a particle must vanish. For a single particle in an infinite fluid, this implies [5, 1]

$$\mathbf{\tau}_p = \mathbf{\phi} + (\mathbf{e}_p - \mathbf{\phi}_p) \cdot \mathbf{C} - \mathbf{C} \cdot (\mathbf{e}_p - \mathbf{\phi}_p) + O(C^2), \quad (4)$$

where $\mathbf{\phi}$ is the antisymmetric part of the suspension velocity gradient

$$\mathbf{\phi} = \mathbf{\phi} = \mathbf{V} \times \mathbf{V}.$$

No result is yet available for a non-dilute suspension. However, if one remembers that for spherical particles $\mathbf{\omega}_p$ is always equal to $\mathbf{\omega}$ whatever the concentration, one is led to write

$$\mathbf{\tau}_p = \mathbf{\phi} + f(\mathbf{e}_p, \mathbf{\phi}_p, \mathbf{C}, \phi) \quad (5)$$

where $f$ is of first order in $C$. Consequently, one can always replace $\mathbf{\omega}_p$ by $\mathbf{\omega}$ in (3).

2.3 TWO-PHASE AVERAGING WITH SURFACE TENSION. — Since we neglected at the outset any inertia effect and any relative velocity between particles and fluid, the average stress of the suspension is equal to the sum of the average stresses in each phase (weighted by the corresponding volume fraction) and a contribution from the surface tension $\Gamma$ (see Appendix A)

$$\mathbf{\varphi} = \phi \langle \mathbf{\varepsilon}_p \rangle + (1 - \phi) \langle \mathbf{\varepsilon}_l \rangle + \frac{8 \Gamma}{5 \rho_0} \phi \mathbf{C}. \quad (6)$$

The fluid being a Newtonian one,

$$\langle \mathbf{\varepsilon}_l \rangle = 2 \mu_f \langle \mathbf{e}_l \rangle$$

where $\mu_f$ is the fluid viscosity and $\langle \mathbf{e}_l \rangle$ the average value of the fluid strain rate. Moreover, the velocity being continuous on the interfaces, there is no singular velocity gradient and consequently the average strain rate of the suspension is

$$\mathbf{e} = \phi \mathbf{e}_p + (1 - \phi) \langle \mathbf{e}_l \rangle.$$

Combining the above relations we get

$$\mathbf{\varphi} = 2 \mu_f \mathbf{e} + \phi \langle \mathbf{\varepsilon}_p \rangle - 2 \mu_f \mathbf{e}_p + \frac{8 \Gamma}{5 \rho_0} \phi \mathbf{C}, \quad (7)$$

which is a generalization of results previously obtained in [1, 2].
2.4 THE PARTICLE STRESS. — The particle stress $\tau_p$ has two expressions, the first one depending on internal parameters and the second one depending on the surface tension and the fluid velocity field at the particle surface.

2.4.1 « Internal » point of view. — The particle stress is the sum of two contributions coming from the particle elasticity on the one hand and from the particle viscosity on the other hand

$$\langle \tau_p \rangle = 2\nu\bar{C} + 2\mu_p\bar{\sigma}_p.$$ (8)

For fluid particles the shear modulus $\nu$ is zero, while for elastic particles the viscosity $\mu_p$ is usually neglected.

2.4.2 « External » point of view. — The neglect of the particle inertia implies that the average particle stress is the sum of two contributions from the surface tension and from the fluid stress at the interface (see Appendix A)

$$\langle \tau_p \rangle = \frac{1}{V_p} \int_{S_p} \mathbf{r} \cdot dS - \frac{8\Gamma}{5r_0}\bar{C}.$$ (9)

The fluid velocity outside the particle is the solution of a Stokes equation with the boundary condition

$$\mathbf{v}_f = \left(\bar{\sigma}_p + \bar{\sigma}_p\right) \cdot \mathbf{r}$$ (10)

on all the particles contained in the volume of averaging, and another boundary condition (on the external surface of this volume) which involves $\omega$, $e$, and $\phi$ in a self-consistent way. The particle elasticity or viscosity nowhere appears in the boundary conditions and (9) can be written quite generally as

$$\langle \tau_p \rangle = 2\mu_1\bar{\mathbf{g}}(e, e_p, \omega, \omega_p, C, \phi) - \frac{8\Gamma}{5r_0}\bar{C}.$$ (11)

Combining the two expressions (8) and (11) for $\tau_p$ is nothing but expressing some kind of « average boundary condition » on the stress.

2.5 SUMMARY OF THE RESULTS. — The above results can be presented as a set of four equations for the four unknowns $\omega_p$, $e_p$, $C$ and $r$:

$$\begin{align*}
\frac{d\bar{C}}{dr} + \bar{C} \cdot \bar{\omega} - \bar{\omega} \cdot \bar{C} &= \bar{\sigma}_p - 2Sd(\bar{\sigma}_p \cdot \bar{C}) + O(C^2) \\
\bar{\omega} &= 2\mu_1\bar{C} + 2(\mu_p - \mu_1)\phi\bar{\sigma}_p + \phi\bar{\sigma}_p(C) \\
\bar{\sigma}_p(C) &= 2\mu_p\bar{C} + 2\mu_1\bar{\mathbf{g}}(e, e_p, \omega, \omega_p, C, \phi) \\
\bar{\sigma}_p(C) &= \left(2\nu + \frac{8\Gamma}{5r_0}\right)\bar{C} + O(C^2).
\end{align*}$$

(A)

where

It is worth noting that the stress $\phi\sigma_p$ is nothing but the $C$-derivative of the elastic and surface tension energy of the suspension, i.e.

$$\phi\bar{\sigma}_p = \frac{\partial E_{el}}{\partial \bar{C}} + \frac{\partial E_f}{\partial \bar{C}}$$

with

$$E_{el} = \phi\nu C_{ij} C_{ji} + O(C^3),$$

and [6]

$$E_f = \frac{3\Gamma}{r_0} \phi \left(1 + \frac{4}{15} C_{ij} C_{ji} + O(C^3)\right).$$

$\bar{\sigma}_p$ is a known function of $C$ while $f$ and $g$ are unknown (except in the dilute limit) functions of their arguments. In principle, the values of $\bar{\sigma}_p$ and $\bar{\omega}$ can be deduced from equations (5) and (13).

When this operation is performed and the results introduced in (3) and (12), one arrives at a couple of equations involving $C$ and $r$, the well-known Hand’s equations [7]. But for this elimination to be possible one must know an explicit form for the tensorial functions $f$ and $g$. We now develop two very different kinds of arguments which both lead to the same results.

3. Phenomenological expressions for $f$ and $g$.

3.1 A GUESS FROM SINGLE PARTICLE RESULTS. — Jeffery [5] determined the pressure and velocity fields around a single rigid ellipsoid with the boundary condition

$$\mathbf{v}_f(r, t) = \bar{\sigma}_p(t) \cdot \mathbf{r}$$
when \( r \) belongs to the particle surface and
\[
v_t(r, t) = (\tilde{\omega}_p(t) + \tilde{\omega}(t)) \cdot r
\]
far from the particle. Goddard and Miller extended his results to a deformable ellipsoid for which (10) is the right boundary condition at the interface and proved that the velocity field can be written in the general form
\[
v_t(r, t) = \tilde{\omega}_p \cdot r + \mathbf{u}(e - e_p, \omega - \omega_p, C, r).
\]
Since \( \tau = 2 \mu \Psi \) one deduces from (9) and (11)
\[
g = \tilde{\omega}_p + \tilde{\omega}(e - e_p, \omega - \omega_p, C)
\]
while \( \omega - \omega_p \) is given by the no-torque condition (4).

Let us now assume that the only influence of hydrodynamic interactions is to change these single particle results into
\[
\bar{g} = \tilde{\omega}_p + \tilde{\omega}(e - e_p, \omega - \omega_p, C, \phi)
\]
and
\[
\bar{\omega}_p - \tilde{\omega}_p = \tilde{\tau}(e - e_p, C, \phi).
\]
These assumptions allow us to gather (5) and (13) into the single equation
\[
\bar{\sigma}_p(C) + 2 \mu_p \bar{\omega}_p = 2 \mu \tilde{\omega}_p +
+ 2 \mu \tilde{\tau}(e - e_p, C, \phi).
\]
Since \( h \) is expected to be a linear function of \( e - e_p \) (due to the linearity of Stokes equations and boundary conditions) then, developing \( h \) in powers of deformation one finally arrives at
\[
\bar{h} = \frac{F(\phi)}{\phi}(\bar{e} - \bar{e}_p) + \frac{G(\phi)}{\phi}Sd(\bar{e} - \bar{e}_p) \cdot \bar{C} +
+ O(C^2)
\]
where, for future convenience, the \( \phi \)-dependence was written in the form \( F/\phi \) and \( G/\phi \). Let us now develop a second argument which also leads to the same results but without the use of assumption (15).

3.2 THERMODYNAMIC ARGUMENT. — Discarding an irrelevant contribution from heat conduction, the entropy production of a suspension of deformable particles is found to be
\[
\frac{1}{T} [\bar{F} : \bar{e} - \phi \bar{\sigma}_p : \bar{e}_p].
\]
As seen above, the stress \( \phi \bar{\sigma}_p \) is nothing but the derivative of the internal energy of the suspension with respect to the particle deformation \( C \). The relevant energy production rate is
\[
\phi \bar{\sigma}_p : \frac{d \bar{C}}{dt}
\]
and taking (3) into account, the above form of entropy production is deduced. Note that \( \omega_p \) does not play any role in the energy production : this was expected since the deformation energy does not depend on the particle rotation.

If we limit ourselves to terms linear in deformation, the usual methods of irreversible thermodynamics [8] lead from (18) to the following expressions
\[
\bar{\tau} = 2 \eta_2 \bar{e} - 2 \eta_2 \bar{e}_p + \eta_3 Sd(\bar{e} \cdot \bar{C}) -
- \eta_4 Sd(\bar{e}_p \cdot \bar{C}) + O(C^2)
\]
\[
\phi \bar{\sigma}_p = 2 \eta_2 \bar{e} - 2 \eta_1 \bar{e}_p + \eta_4 Sd(\bar{e} \cdot \bar{C}) +
+ \eta_5 Sd(\bar{e}_p \cdot \bar{C}) + O(C^2).
\]
The presence of \( \eta_2 \) and \( \eta_4 \) in both \( \tau \) and \( \sigma_p \) is the expression of the Onsager symmetry. Compatibility of the above results with (12) implies four relations between the six coefficients \( \eta_1 ... \eta_3 \). Let us choose the two independant coefficients as \( \eta \) and \( \eta_3 \), and write them as
\[
\eta = \mu_1(1 + F),
\eta_3 = 2 \mu_1 G.
\]

The two functions \( F \) and \( G \) may depend on the three variables \( \lambda = \mu_p/\mu, \nu \) and \( \phi \). They allow us to rewrite (19) as
\[
\frac{\eta}{2} \mu_1 = (1 + F) \bar{e} - F \bar{e}_p +
+ GSD[(\bar{e} - \bar{e}_p) \cdot \bar{C}] + O(C^2)
\]
\[
\phi \bar{\sigma}_p/2 \mu_1 = F \bar{e} = [F + (\lambda - 1) \phi] \bar{e}_p +
+ GSD[(\bar{e} - \bar{e}_p) \cdot \bar{C}] + O(C^2).
\]
Result (22) must be compatible with (13) which states that the sum
\[
\bar{\sigma}_p
\]
is independent of \( \nu \) and \( \lambda \). We conclude that \( F \) and \( G \) are functions of \( \phi \) only and that (22) is identical with the former results (16) and (17). In other words, assumption (15) and thermodynamics both lead to the same result.

At this stage, we have completely eliminated \( \omega_p \) and the set (A) is transformed into

\[
\begin{align*}
\frac{d \bar{C}}{dt} + \bar{C} \cdot \bar{\omega} - \bar{\omega} \cdot \bar{C} &= \bar{e}_p - 2 Sd(\bar{e}_p \cdot \bar{C}) + O(C^2) \\
\bar{\tau} &= 2 \mu_1 \bar{e} + 2(\mu_p - \mu_1) \phi \bar{e}_p + \phi \bar{\sigma}_p \\
\phi \bar{\sigma}_p/2 \mu_1 + (\lambda - 1) \phi \bar{e}_p &= F(\bar{e} - \bar{e}_p) + GSD[(\bar{e} - \bar{e}_p) \cdot \bar{C}] + O(C^2)
\end{align*}
\]
where hydrodynamic interactions are represented by \( F(\phi) \) and \( G(\phi) \).
4. The non-dilute form of Hand’s equations.
It is easy to draw out $e_p$ from (22). The set (B) is then transformed into a couple of equations for $C$ and $\tau$. To present them in a simple form we now define the relaxation time by

$$\frac{\phi \bar{\theta}}{2 \mu_t [F + (\lambda - 1) \phi]} = \frac{\bar{C}}{\bar{\theta}} + O(C^2) \quad (23)$$

and two deformability coefficients $\alpha$ and $\beta$ by

$$\alpha(\lambda, \phi) = \frac{(\lambda - 1) \phi}{F(\phi) + (\lambda - 1) \phi} \quad (24)$$
$$\beta(\lambda, \phi) = \frac{G(\phi)}{F(\phi) + (\lambda - 1) \phi} \quad (25)$$

With these definitions, Hand’s equations appear as

$$\frac{d\bar{C}}{dt} + \bar{C} \cdot \bar{e}_0 - \bar{e}_0 \cdot \bar{C} = (1 - \alpha) \bar{e}_0 - \frac{\bar{C}}{\bar{\theta}} + \frac{\alpha \beta + 2 \alpha - 2}{Sd(\bar{e}_0 \cdot \bar{C})} \quad (26)$$

and

$$\frac{\tau}{2 \mu_t} = (1 + \alpha F) \bar{e}_0 + \frac{F}{\bar{\theta}} \bar{C} + \alpha^2 GSd(\bar{e}_0 \cdot \bar{C}) \quad (27)$$

The hydrodynamic interactions between particles enter these equations with the two functions $F$ and $G$ which appear either explicitly or implicitly in $\alpha$, $\beta$ and $\theta$. If $F(\phi)$ and $G(\phi)$ were known in the whole range of concentrations, then $\alpha$, $\beta$ and $\theta$ would be completely determined and Hand’s equations for non-dilute suspensions would contain no adjustable parameter. Unfortunately, our knowledge of $F$ and $G$ is rather fragmentary. Let us see what we do know about them.

The limit case of rigid particles is obtained in two different ways: either the viscosity ratio $\lambda$ is very large ($\alpha = 1$, $\beta = 0$ and $\theta = \infty$) or the surface tension (or shear modulus) is very large ($\theta = 0$ and $C/\theta = (1 - \alpha)e$). In both cases the suspension stress becomes

$$\tau = 2 \mu_t (1 + F) \bar{e}_0$$

Hence $F(\phi)$ is nothing but the $\phi$-dependence of the viscosity of a suspension of rigid particles, and $G(\phi)$ represents the first-order contribution of deformability. The dilute-limit of $F$ was obtained by Einstein [9] and Batchelor [10], and that of $G$ by Jeffery [5] and Goddard and Miller [1].

$$\lim_{\phi \to 0} F(\phi) = 2.5 \phi + 6.2 \phi^2 + \ldots$$
$$\lim_{\phi \to 0} G(\phi) = \frac{15}{7} \phi + \ldots$$

Various expressions for $F$ have been proposed for concentrated suspensions. Some of them diverge at the packing volume fraction [11], some others do not [12]. The situation is not yet clear. No expression for $G(\phi)$ outside the dilute regime has been proposed so far.

The relaxation time deduced from (23) and (14) is

$$\theta = \left( \frac{F}{\phi} + \lambda - 1 \right) \frac{5 \mu_t}{4 G}$$

for emulsions and

$$\theta = \left( \frac{F}{\phi} - 1 \right) \frac{\mu_t}{\nu}$$

for elastic particles (provided $\mu_0$ and $G$ are neglected). If $F(\phi)$ happens to diverge at the packing volume fraction, so does $\theta$.

In the dilute limit, equations (26) and (27) give back exactly the results of Goddard and Miller for elastic particles [1] and are a bit different from those obtained in [3, 4] for a dilute emulsion. This is due to a slightly different expression for the relaxation time $\theta$, itself presumably a consequence of assuming a homogeneous deformation of the drop.

In the very concentrated limit, equations (26) and (27) agree quite well with the result of Goddard [13]

$$\frac{\tau}{2 \mu_t} = 2 \mu_t (1 + F) \bar{e}_0$$

who found a divergence of $F$ like that proposed by Frankel and Acrivos [11].

The correct description of dilute and concentrated suspensions is no guarantee that intermediate concentrations are correctly described, but we have confidence that, provided the assumption of homogeneous deformation is not too far from reality, our equations (26) and (27) offer a good framework which only needs the numerical determination of $F(\phi)$ and $G(\phi)$ to be operational.

The functions $F$ and $G$ can be deduced from experimental results on effective viscosity and normal stresses in a simple shear flow $\nu_x = \gamma_y$. The following relations are obtained from (26) and (27):

$$\frac{\tau_{xy}}{\mu_t \gamma} = 1 + F - F(\theta \gamma) \frac{\left(1 - \alpha\right) - \alpha \frac{\delta^2}{3}}{1 + (\theta \gamma)^2 \left(1 - \frac{\delta^2}{3}\right)} \quad \frac{\tau_{xx} - \tau_{yy}}{2 \mu_t \gamma} = \theta \gamma \frac{(1 - \alpha) F}{1 + (\theta \gamma)^2 \left(1 - \frac{\delta^2}{3}\right)}$$

$$\frac{\tau_{yy} - \tau_{yy}}{\tau_{xx} - \tau_{yy}} = \frac{\delta}{2(1 - \alpha)} - \frac{1 - \alpha}{2}$$

where $\delta$ is defined as

$$\delta = \frac{\alpha \beta}{2} + \alpha - 1.$$
It is worth noting that, according to Hand’s general analysis [7], equation (26) has a stationary solution for all \( \gamma \), provided \( |\delta| < 1 \). At variance, when \( |\delta| > 1 \), the stationary solution only exists for \( \gamma < \gamma_c \) with

\[
\theta \gamma_c = \frac{2}{\sqrt{\delta^2 - 1}} \quad \text{if} \quad 1 < |\delta| < 3
\]

and

\[
\theta \gamma_c = \frac{1}{\sqrt{\frac{\delta^2 - 1}{3}}} \quad \text{if} \quad |\delta| > 3.
\]

If it happens that \( F \) and \( G \) satisfy the conditions

\[
F > \phi \quad \text{and} \quad 0 < G < 2F.
\]

(fulfilled in the dilute regime) then one can show from (28) that

\[
|\delta| < 1 \quad \text{when} \quad \mu_p > \mu_f.
\]

Conversely, when the particle viscosity is smaller than the fluid viscosity and when the shear is high enough, no stationary solution for the particle strain \( \dot{\gamma} \) exists, possibly suggesting some kind of instability. Note that this would be the case for elastic particles supposed to have no viscosity!


The description of a suspension of deformable particles (with constant volume) is rather simple if the particles are supposed to deform slightly and homogeneously, and if all inertia effects may be neglected. The simultaneous use of these three assumptions leads to the rather simple set of equations (A). In a second step, thermodynamics or assumption (15) transform set (A) into set (B). Finally, \( e_p \) is eliminated and set (B) merges into Hand’s equations (26) and (27).

The main advantage of the phenomenological approach is to offer a simple scheme for deducing equations in the non-dilute regime. Needless to say that the problem is not completely solved since two unknown (except in the dilute regime) functions of the volume fraction are introduced. But the merit of this poor man’s approach is to show that we only need two such functions to model the effect of hydrodynamic interactions between deformable particles.

Hydrodynamic interactions are present in the right hand side of (22) but nowhere else in set (B) while they are present in the six coefficients of Hand’s equations (26) and (27). Hence a discussion of these interactions directly based on a Hand’s type of equations would have been much more difficult, if not impossible.

The use of thermodynamics arguments is debatable when applied to a two-phase mixture because of the averaging process and the difficulty of getting an “average equation of state”. Nevertheless, when the suspension is not far from equilibrium, the role of the fluctuations of the state variables is certainly not important. Here, the only departure from equilibrium was the particle deformation and it was supposed to be weak. There is thus no reason why (18) and (19) should be incorrect. Needless to say that the entropy production (18) would have been of little help without the relations (12) and (13) which transformed (19) into the much simpler results (21) and (22).

Appendix A: The role of surface tension in the suspension stress.

The momentum balance of the whole suspension appears as

\[
\frac{\rho}{\partial r} \frac{dV}{dr} = F_p + F_f + \rho g + \\
+ \nabla \cdot \left( \phi \left( \frac{\partial \tilde{\Pi}}{\partial r} \right) + (1 - \phi) \left( \tilde{\Pi}_i \right) \right)
\]

where \( \langle \pi \rangle \) is the average stress of phase \( k \) and \( F_k \) is the force exerted on phase \( k \) through the interfaces. In the absence of surface tension

\[
F_p + F_f = 0.
\]

If the surface tension \( \Gamma \) is a constant (independent of temperature) one can show that

\[
F_p + F_f = -\frac{1}{dV} \int \Gamma \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right) dS \quad \text{(A.1)}
\]

where \( dS \) is the outward normal for the particles and \( R_1^{-1} + R_2^{-1} \) stands for the curvature of the interfaces while the integral is over all the interfaces contained in a small elementary volume \( dV \). For rigid spherical particles with radius \( r_0 \)

\[
F_p + F_f = -\frac{2}{r_0} \frac{\Gamma}{dV} \int dS = \nabla \left( \frac{2 \Gamma \phi}{r_0} \right)
\]

where \( \phi \) is the particle volume fraction. For particles deforming according to (1), a somewhat lengthy but straightforward calculation leads to

\[
F_p + F_f = \nabla \left( \frac{2 \Gamma}{r_0} \phi \right) + \nabla \cdot \left( \frac{8 \Gamma}{5 r_0} \phi \tilde{\nabla} \right) + 0(C^2).
\]
This result means that the overall stress tensor of the suspension is
\[ \mathbf{\bar{\tau}} = \phi \langle \mathbf{\bar{\tau}} \rangle + \phi (1 - \phi) \langle \mathbf{\bar{\tau}} \rangle + \frac{2 \Gamma}{r_0} \phi \mathbf{\vec{\epsilon}} + \frac{8}{5 r_0} \phi \mathbf{\bar{\epsilon}} + 0 (C^2). \]

If we separate the role of the mean phase pressure according to
\[ \langle \mathbf{\bar{\tau}} \rangle = -\langle p \rangle \mathbf{\bar{I}} + \langle \mathbf{\bar{\tau}} \rangle , \]
it is then possible to write
\[ \mathbf{\bar{\tau}} = -p \mathbf{\bar{I}} + \langle \mathbf{\bar{\tau}} \rangle \]
with
\[ p = \phi \langle p \rangle + (1 - \phi) \langle p \rangle - \frac{2 \Gamma}{r_0} \phi \]  \hspace{1cm} (A.2)
and
\[ \mathbf{\bar{\tau}} = \phi \langle \mathbf{\bar{\tau}} \rangle + (1 - \phi) \langle \mathbf{\bar{\tau}} \rangle + \frac{8}{5 r_0} \phi \mathbf{\bar{\epsilon}} , \]
a result reproduced in equation (6).

Starting from the Stokes equation, one can show that the average stress inside a particle can be written as
\[ \langle \pi \rangle_{ij} = \frac{1}{V_p} \int_{S_p} \tau_{ik} dS_k \]
where the integral is over the surface of the particle.

Using the boundary conditions for stress, one finally arrives at
\[ \langle \pi \rangle_{ij} = \frac{1}{V_p} \int_{S_p} \tau_{ik} dS_k + \frac{1}{V_p} \int_{S_p} \tau_{ik} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dS_j \]  \hspace{1cm} (A.3)

Again, some lengthy calculations based on (1) allow us to determine the contribution of surface tension with the result
\[ \langle p \rangle = \langle p \rangle + \frac{2 \Gamma}{r_0} \]  \hspace{1cm} (A.4)
\[ \langle \tau \rangle_{ij} = \frac{1}{V_p} \int_{S_p} \tau_{ik} dS_k - \frac{8}{5 r_0} C_{ij} + 0 (C^2) . \]

The last result is used in (9). Note that (A.2) and (A.4) imply
\[ p = \langle p \rangle \]
and
\[ \mathbf{\bar{\tau}} = (1 - \phi) \langle \mathbf{\bar{\tau}} \rangle + n_p \int_{S_p} \tau_{ik} dS_k \]
where \( n_p \) is the number of particles per unit volume of the suspension. Everything happens as if the surface tension was absent in \( p \) and \( \tau \). This occurs because the contribution of surface tension to both integrals (A.1) and (A.3) are identical to first order in \( C \). We were unable to perform the calculations up to second order terms to see if this coincidence still exists.

References