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Period distribution for Kauffman cellular automata

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Résumé. — Les réseaux booléens aléatoires à deux dimensions, introduits par Kauffman, présentent une transition vers le chaos. Sur un réseau sous-jacent carré, nous trouvons par la méthode de Monte Carlo les périodes des cycles limites, pour chaque site aussi bien que pour le réseau entier, au point de transition et en dehors de celui-ci. Le réseau triangulaire semble conduire aux mêmes dimensions fractales que le réseau carré.

Abstract. — The two-dimensional random Boolean networks suggested by Kauffman have a transition to chaos. We find by Monte Carlo simulation the periods of the limit cycles, both for each site as well as for the lattice as a whole, at the transition point and away from it, for the square lattice. The triangular lattice seems to have the same fractal dimensions as the square lattice.

1. Introduction.

Kauffman models [1] for genetic systems are particularly disordered cellular automata [2]. Each site depends through a fixed random function on its K neighbours in the lattice. Depending on parameters one finds either chaotic behaviour (class 3 cellular automaton in Wolfram's classification) or frozen behaviour (class 2). Finite systems are forced into limit cycles by the deterministic nature of the automata. On a square lattice with nearest-neighbour interactions (K = 4), the period of these limit cycles seems to increase exponentially with lattice size [3] in the chaotic phase and weaker in the frozen phase. References [4] and [5] reviewed recent analytical and numerical work on Kauffman models. The present work tries to quantify the concept of local periods [3] and their distribution [6]. Fractal dimensions at the threshold to chaotic behaviour were already found for the spreading of a single « error » in two [5] and three [7] dimensions. We check here if the triangular lattice has the same fractal dimensions as the square lattice.

Thus we now go through our square lattice (K = 4) like a typewriter. Each site first gets a « spin » (Boolean variable) which initially points up or down (is true or false) randomly. For each site we then go through all $2^K = 16$ configurations which the four neighbours can form. For each such configuration we determine the outcome of the function for that site to be « up » if and only if a random number drawn for that site and that configuration happens to be smaller than a parameter $p < 1/2$; otherwise the function value is « down ». If in this way we have gone through the whole $L \times L$ lattice, we have initialized the system (time $t = 0$). From then on, each site at time $t + 1$ is determined by the value of the function constructed for that site, with the function depending on the four neighbour spins at time $t$. In this sense the Kauffman model is a random mixture of all possible rules for deterministic cellular automata. Most of our simulations were made at the threshold to chaos which we took [5] as $p_c = 0.28$ in the $K = 4$ square lattice. For the triangular lattice no previous simulations are known to us at all, and thus we determined there the threshold $p_c$ on a vector computer.

2. Techniques.

Some technical aspects of simulations were already given in reference [3]. In our case we have to store the history of the Kauffman model in order to detect periodic motions of the spins. To find a period $\tau$ we need to observe the spin over at least 2 $\tau$ time steps. Moreover the Kauffman cellular automata are irreversible; thus they relax into their limit cycle only...
after some transient time has elapsed. We used half of the time steps for this « equilibration » and half for checking the periods. Thus the total observation time is 4 times the maximum observable period.

To store all the configurations produced this way in the second half of the simulation, we stored 60 consecutive time steps in one 60-bit CDC computer word for each lattice site. Nevertheless only very small lattices could be simulated because of memory limitations. To find a period $\tau$ for one site, we selected a trial value for the period, starting with period $\tau = 1$, then shifted the computer words by the corresponding number of bits, and checked if they agreed. (Of course, we had to combine and mask two shifted words to compare them with one unshifted word since $\tau$ in general is not a multiple or divisor of 60). We required that over the whole second part of the simulation the stored unshifted computer words agreed with the shifted ones, i.e. that the spin flipped in a periodic sequence over the whole observable time interval. If that was not the case, the trial value for the period was increased by one, and the search for the period was repeated for this trial value, until the maximum observable period was reached. Similar methods were already used for reversible Q2R automata [9] where, however, one does not need an initial transient time. In this way a site which never moves in equilibrium has period 1; a site with period larger than observable gets period $\infty$. If all sites have finite « local » periods, the lattice as a whole has a « global » period equal to the smallest common multiple of all local periods.

The global period of the lattice as a whole can be determined more easily: we have to check whether the current spin configuration is identical with that stored at the end of the equilibration time. Thus somewhat larger lattices can be investigated; however, since the periods increase rapidly with increasing lattice size, computer time still allows only rather small sizes. Also, each simulation gives only one global period but $L^2$ local periods.

Because of these size limitations we restricted this work to two dimensions where we know [8, 10] that at least for voltage distributions in random resistor networks already very small lattices give a good picture. Typically, one run over 20 000 time steps (maximum observable local period = 5 000) on a $20 \times 20$ lattice took about 20 s on a Cyber 176, most of it for the production of the new configurations.

In the triangular lattice we need to know $p_c$ since no previous work on that lattice is known to us for the Kauffman model. The periods are not a practical tool to determine the threshold, and thus we used the method of reference [11]: two lattices were simulated at the same time, with the same set of rules and nearly the same initial configurations. Only a small fraction of spins differs initially in the two lattices. One then sees how this « damage » spreads, i.e. if the Hamming distance [1] or the difference picture [12] remains finite ($0 < p < p_c$) or spreads over the whole lattice.

Since each site in the triangular lattice has $K = 6$ neighbours, there are $2^K = 64$ different neighbour configurations, and the rules take up a lot of memory. Thus each function value is now stored in one bit, which requires 64 bits for the whole rule of one site. Thus we worked on the 64-bit CDC Cyber 205 vector computer for this triangular lattice. Each Monte Carlo step took about 0.2 $\mu$s, faster than the Cray-XMP of reference [11] on the simpler square lattice. One reason for this speed-up is the fact that a shift by a variable number of bits is used here, and can be vectorized on the Cyber 205 but not on the Cray-XMP.

In all calculations, periodic or helical boundary conditions were employed.

3. Results.

3.1 Global Periods. — Reference [3] has already shown that there are large fluctuations in the global periods. At the threshold, $p = p_c$, we sometimes found the global periods to be spread over three orders of magnitude. Thus figure 1 shows the « median » global period [3] determined such that half of the observed periods are smaller and half are

![Fig. 1. — Semilog plot of the median global period (dots) and the average global period (squares) versus $L$ in a $L \times L$ square lattice. We analyse about 2 000 configurations for $L$ from 6 to 12, 250 for $L$ from 15 to 20, and 100 configurations for larger $L$. In the insert, the average natural logarithm of the global period ($\times$) is shown in a log-log plot versus $L$.](image)
larger than that period. This period seems to increase exponentially with system size, even though we are only at the onset of the chaotic phase:

\[ \text{median period} = \exp \left( L / 2.66 \right) \]

at the critical point \( p = 0.28 \) of a \( L \times L \) square lattice. This law differs from the variation with \( \exp(\text{Const.} \, L^2) \) observed \[3\] far above \( p_c \) and the proportionality to \( \sqrt{L} \) found in the frozen phase below \( p_c \) \[5\]. Thus it seems that the logarithm of the period is the most suitable quantity for statistical analysis: it is found to vary as \( L^2 \) above, as \( L \) at and as \( \log(L) \) below \( p_c \).

If instead we average over those periods which were found within the finite observation time, then both \( \langle \text{period} \rangle \) and \( \langle \ln\text{ period} \rangle \) will for large periods have systematic errors due to finite observation times, and the curves flatten for larger lattice sizes, as can be seen in figure 1 and its insert. The median time, on the other hand, can be determined as long as at least half of the periods were found, and is therefore more reliable.

3.2 LOCAL PERIODS. — The local period \( T \) of one site gives the time after which this site repeats the same pattern of bits in its orientation \[3\]. The global period of the whole lattice is the lowest common multiple of all local periods in the lattice and thus in general much larger. We call \( n(T) \) the number of sites with local period \( T \), and define the moments

\[ M_q = \langle T^q \rangle = \left( \sum n(T) \, T^q \right) / \left( \sum n(T) \right) \]

through averages over many realizations at a given lattice size, ignoring the sites with period longer than observable in our runs. These moments have, as a function of \( p \), a maximum at or near \( p_c \), if \( q > 0 \). However, the positive moments also fluctuate strongly, and our data are more reliable for negative \( q \). Our results, in the form of \( M_q^{-1/q} \), are shown in figure 2a. Figure 2b gives analogous results with the local period \( T \) replaced by its logarithm in the definition of \( M_q \).

In these results, our averages necessarily ignore those local periods which were too long to be observed in our observation time of, typically, 4 500 time steps. Thus in the fully chaotic phase near \( p = 0.5 \), the moments are dominated by those sites with local period \( T = 1 \) or very small; most sites have a period far longer than can be observed. \( (T = 1 \) refers to sites which never change.) Thus for \( p \rightarrow 0.5 \), the period moments go to unity, their logarithms to zero, as seen in figure 2. If we remove also the sites with \( T = 1 \) from our averages, then the moments of \( T \) change drastically near \( p = 0.5 \), whereas the behaviour near \( p_c \) remains about them same, as seen in figure 2a.

Fig. 2a. — \( Q \)-th root of the moments \( M_q = \langle T^q \rangle \) of the local periods (dots) are shown as function of \( p \) for \( q = -1, -2, -3, \) and \( -4 \) (averages over 150 configurations). Typical error bars are shown for \( p = 0.3 \) and 0.4. The \( q = -4 \) results without the contribution from the spins with \( T = 1 \) go to zero as \( p \rightarrow 1/2 \) (triangles; typical error bars are also shown) \((L = 15)\). The lines are guides to the eye.

Fig. 2b. — \( Q \)-th root of the moments \( \langle \ln T \rangle^q \) as function of \( p \) for \( q = -1, -2, -3 \) and \( -4 \) (averages over 150 configurations). Typical error bars are shown for \( p = 0.3 \) and 0.4 \((L = 15)\).
We see no indication in figure 2 that the moments $M_q$ are critical, i.e. go to zero or infinity at $p = p_c$. One might argue that they remain finite since there are so many sites with rather small periods, dominating over the few sites with larger « critical » periods. Therefore figure 3 shows the maximum local period $T_{\text{max}}$ observed in the whole lattice, averaged logarithmically over many different lattices. Again this period $T_{\text{max}}$ has a maximum at or near $p_c$. But this maximum is finite, and changes only slightly if we vary the observation time by one order of magnitude, figure 3. Thus the local periods seem not to diverge at $p_c$ according to figures 2 and 3. Since the present system sizes are necessarily limited, this lack of criticality could be due to finite $L$. We therefore will investigate below the variation with $L$ at $p = p_c$.

![Graph](image)

**Fig. 3.** Average natural logarithm of the maximum local period (dots) versus $p$ for observation times of 4 500 steps (150 configurations). Typical error bars are given for $p = 0.3$ and 0.4. The dotted line shows the same quantity for 450 observed time steps ($L = 15$).

The fraction of sites with « infinite » period, i.e. period larger than observable and thus ignored in figures 2 and 3, is zero far below $p_c$ and close to unity far above $p_c$. Figure 4 shows it to vanish at $p = p_c$, within the accuracy of the data, as was found also in reference [3]. We also find that this « disorder » parameter goes to zero roughly as $(p - p_c)^{1/2}$, for $p$ near $p_c$.

At $p = p_c$, as a function of system size $L$, the moments of the local periods would increase to infinity with increasing $L$ if they would be critical. Figure 5 shows that this is not the case in the observed region: for both positive and negative $q$, the averages of $(\ln T)^q$ seem to approach a constant with increasing $L$, if the observation time is large enough.

Finally, figure 6 shows the distribution of observed local periods. We find for each configuration the quantity $\ln \langle T \rangle / \ln \langle T_{\text{max}} \rangle$ into 500 intervals between zero ($T = 1$) and one ($\ln T = \ln T_{\text{max}}$), and average over different lattices with different $T_{\text{max}}$. Such a plot is analogous to the distribution of currents in random resistor networks [10, 13] or distribution of damage probabilities in Kauffman models [14]. In contrast to these other examples, the present curve seems quite noisy.

### 3.3 TRIANGULAR LATTICE.

First we determined the onset of chaos in a $120 \times 120$ triangular lattice using the method of reference [11] as

$$p_c = 0.16$$

to be compared with about 0.28 for the square and 0.12 for the simple cubic lattice and in agreement with the empirical upper and lower bounds of reference [5], equations (11, 12).
Fig. 5. — Log-log plot of the $q$-th root of the moments of the logarithm of the local periods $T$, versus $L$, for $q = -1, -2, -3, -4$ as well as $q = 1$ and 4 (dots). The data are averaged over 5 000 configurations for $L = 6$, about 2 000 for $8 \leq L \leq 15$, and about 500 for larger $L$. Error bars are given for $L = 15$ and 25; where not shown they are comparable to the size of the dots. The crosses show the values for $q = 1$ and 4 for $L = 10$ and 25 with observation time 450 instead of 4 500. By extrapolating the value for each moment to infinite observation times for the larger values of $L$, the moments seem to go to a constant value as $L$ increases.

At $p = p_c$, we then looked at the fractal dimensions [3, 5] of damage spreading: after what time $t$ does the effect of a change of the central site affect any site at the upper boundary of the system. We call the damage $M_{\text{act}}$ the number of sites affected by that initial flip at the moment this « damage » touches the boundary. Thus we simulate two lattice with the same rules and nearly the same initial configurations, at $p = p_c$; only the orientation of the centre site is different. We then observe which sites differ at later times in a comparison of the two configurations, and call these sites damaged.

Figure 7 shows the average touching time $\langle t \rangle$ and the average damage $\langle M_{\text{act}} \rangle$ at the touching moment. Our data obey roughly a power law:

$$\langle t \rangle \sim L^{d_f} \quad \text{and} \quad \langle M_{\text{act}} \rangle \sim L^{d_{\text{act}}}$$

at $p = p_c$ for large $L$. We find near $L \sim 100$ the effective fractal dimensions $d_f = d_{\text{act}} = 1.5$, in reasonable agreement with earlier results ($d_f = 1.7, d_{\text{act}} = 1.5$) for the square lattice [3] but different from the simple cubic lattice ($d_f = 2.2, d_{\text{act}} = 1.8$) [7]. Thus the dimension is more important than the number $K$ of nearest neighbours, and the universality principle seems to hold for these fractal dimensions. Therefore no periods were investigated in the triangular lattice since we expect qualitatively the same behaviour there.
Conclusion.

This work presented a quantitative study of both global and local periods in the two-dimensional Kauffman model. In the square lattice at $p = p_c$ we found the logarithm of the median global period to vary linearly in $L$, whereas it varies as $L^2$ above $p_c$ [3] and as $\log (L)$ below $p_c$ [5]. The moments of the distribution of local periods are apparently not critical.

We also support universality for the phase transition in the nearest-neighbour Kauffman model: the fractal dimensions of damage spreading are about the same for the square and the triangular lattice.

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