The dynamics of a four-level three-mode system. Operator solution

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The dynamics of a four-level three-mode system. Operator solution

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Résumé. — Dans l'approximation dipolaire et dans l'approximation du champ tournant, on donne les solutions pour les opérateurs correspondant aux populations des niveaux et aux nombres de photons. Les résultats numériques pour l'évolution dans le temps des populations des niveaux, dans le cas de modes de pompage 1 et 3 initialement cohérents (le mode 2 étant initialement vide), sont donnés et comparés aux résultats pour un atome à trois niveaux dans les configurations en échelle et en lambda.

Abstract. — In the dipole and rotating wave approximation the operator solutions for the level populations and photon numbers are found. The numerical results for the time-evolution of the level populations for the case of initially coherent pumping modes 1 and 3 (mode 2 initially in vacuum) are reported and compared with those for a three-level atom in the ladder and lambda configurations.

1. Introduction.

The exact operator solution for the Jaynes-Cummings (J-C) model [1] of a two-level atom interacting with a quantized single-mode radiation field was presented by Ackerhalt [2] in 1974.

A number of recent papers have been devoted to studies of the dynamics of a three-level atom interacting with two modes of classical [3-6] or quantized [7-12] electromagnetic field. In papers [7, 8], exact Schrödinger wave functions have been obtained for some special initial states. Li and Bei [9] have derived, in the interaction picture, the explicit expression of the evolution operator. The exact operator solutions for the level populations and photon numbers have been found as well for both lambda [10] and ladder [11] level structure. Moreover, the strict operator solutions for lambda configuration of the levels in the case when the two lower levels are coupled to the upper one by multiphoton transitions have been obtained [12]. An interesting review of the dynamical theory of J-C-type models has been given by Yoo and Eberly [13].

In the present paper we study the dynamics of a four-level atom coupled in a lossless cavity to a three-mode resonant quantized field. The structure of the levels is given in figure 1. The assumed model contains, in fact, three three-level subsystems with a common fourth level ; one can distinguish here two subsystems in the ladder configuration (levels 1-4-3 and 2-4-3) and one subsystem in the lambda configuration (1-4-2).

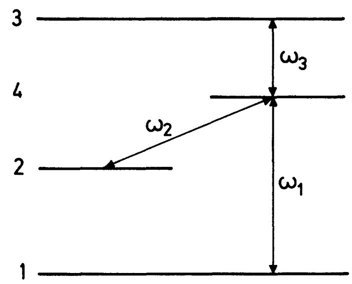

Fig. 1. — Energy-level scheme.

Some aspects of the dynamics of a four-level atom in other level configurations interacting with a classical field have been studied in references [14-17].

In this paper we show that the operator equations of motion for the model assumed can be solved explicitly. Applying our solutions, we further examine the dynamical behaviour of the populations of levels 1 and 2 and compare the results with those for the three-level atom in the ladder (1-4-3) and lambda (1-4-2) configurations, respectively, assuming in both cases the mode 2 as initially unexcited. In other words we study, in the first case, the influence of the depopulation rate enhanced by the possibility of « spontaneous » transitions of the atom to the additional adjoined level 2 on the dynamics of the fundamental level 1. In the second case we compare the dynamical properties of the level 2 « spontaneously » populated in the presence or absence of pumping of the atom to the upper level 3.

2. The Hamiltonian of the system.

The Hamiltonian for the model under consideration in the dipole and rotating wave approximation is given by :

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{F}} + \hat{H}_{\mathrm{I}} \tag{1}$$

where

$$\hat{H}_{\mathrm{A}} = \sum_{i=1}^{4} \hbar\omega_{0i}\, \hat{R}_{ii} \tag{2}$$

is the free atomic part ;

$$\hat{H}_{\mathrm{F}} = \sum_{\alpha=1}^{3} \hbar\omega_{\alpha} \left(\hat{a}_{\alpha}^{+}\, \hat{a}_{\alpha} + \frac{1}{2} \right) \tag{3}$$

represents the free field Hamiltonian, and :

$$\hat{H}_{\mathrm{I}} = \hbar g_1(\hat{a}_1\, \hat{R}_{41} + \hat{a}_1^{+}\, \hat{R}_{14}) + \hbar g_2(\hat{a}_2\, \hat{R}_{42} + \hat{a}_2^{+}\, \hat{R}_{24}) + \hbar g_3(\hat{a}_3\, \hat{R}_{34} + \hat{a}_3^{+}\, \hat{R}_{43}) \tag{4}$$

is the dipole interaction part ; the $g_{\alpha}(\alpha = 1, 2, 3)$ are atom-mode coupling constants.

The photon annihilation $\hat{a}$ and creation $\hat{a}^{+}$ operators for the modes α and β satisfy the commutation rule :

$$[\hat{a}_{\alpha}, \hat{a}_{\beta}^{+}] = \delta_{\alpha\beta}\ , \tag{5}$$

while the atomic operators $\hat{R}_{ij} = |i\rangle\langle j|$, describing transition of the atom from the level j to the level i, obey the relations :

$$\begin{aligned} \hat{R}_{ij}\, \hat{R}_{kl} &= \hat{R}_{il}\, \delta_{jk}\ , \\ [\hat{R}_{ij}, \hat{R}_{kl}] &= \hat{R}_{il}\, \delta_{jk} - \hat{R}_{kj}\, \delta_{il}\ . \end{aligned} \tag{6}$$

In turn, the operator $\hat{R}_{ii}$ represents the population of the level i with energy $\hbar\omega_{0i}$; the following conservation law is fulfilled :

$$\sum_{i=1}^{4} \hat{R}_{ii} = 1\ . \tag{7}$$

3. Solution of the problem.

With respect to the commutation rules (5) and (6), the Heisenberg equations of motion for the level population and photon number $(\hat{n}_{\alpha} = \hat{a}_{\alpha}^{+}\, \hat{a}_{\alpha})$ operators have the form :

$$\begin{aligned} \dot{\hat{R}}_{11} &= i g_1(\hat{a}_1\, \hat{R}_{41} - \hat{a}_1^{+}\, \hat{R}_{14})\ , \\ \dot{\hat{R}}_{22} &= i g_2(\hat{a}_2\, \hat{R}_{42} - \hat{a}_2^{+}\, \hat{R}_{24})\ , \\ \dot{\hat{R}}_{33} &= -\, i g_3(\hat{a}_3\, \hat{R}_{34} - \hat{a}_3^{+}\, \hat{R}_{43})\ , \end{aligned} \tag{8}$$

$$\dot{\hat{n}}_1 = \dot{\hat{R}}_{11}\ ,\ \dot{\hat{n}}_2 = \dot{\hat{R}}_{22}\ ,\ \dot{\hat{n}}_3 = -\, \dot{\hat{R}}_{33}\ . \tag{9}$$

The relations (9) give the following constant excitation number operators $\hat{N}_{\alpha}$:

$$\begin{aligned} \hat{N}_1 &= \hat{n}_1 - \hat{R}_{11}\ , \\ \hat{N}_2 &= \hat{n}_2 - \hat{R}_{22}\ , \\ \hat{N}_3 &= \hat{n}_3 + \hat{R}_{33}\ . \end{aligned} \tag{10}$$

By the second differentiation of equations (8), one finds :

$$\ddot{\hat{R}}_{\alpha\alpha} = 2\, \hat{\Omega}_{\alpha}^2 (\hat{R}_{44} - \hat{R}_{\alpha\alpha}) - g_{\alpha} \sum_{\beta \neq \alpha}^{3} g_{\beta}\, \hat{T}_{\alpha\beta}\ , \quad (\alpha = 1, 2, 3) \tag{11}$$

where, for brevity, we have introduced the auxiliary operators

$$\begin{aligned} \hat{T}_{12} &= \hat{T}_{21} = \hat{a}_1\, \hat{a}_2^{+}\, \hat{R}_{21} + \hat{a}_1^{+}\, \hat{a}_2\, \hat{R}_{12}\ , \\ \hat{T}_{13} &= \hat{T}_{31} = \hat{a}_1\, \hat{a}_3\, \hat{R}_{31} + \hat{a}_1^{+}\, \hat{a}_3^{+}\, \hat{R}_{13}\ , \\ \hat{T}_{23} &= \hat{T}_{32} = \hat{a}_2\, \hat{a}_3\, \hat{R}_{32} + \hat{a}_2^{+}\, \hat{a}_3^{+}\, \hat{R}_{23}\ . \end{aligned} \tag{12}$$

They describe two-photon transitions between the levels α and β by the common fourth level.

The $\hat{\Omega}_{\alpha}$ are the operators of the one-photon Rabi frequency, and :

$$\begin{aligned} \hat{\Omega}_1^2 &= g_1^2(\hat{N}_1 + 1)\ , \\ \hat{\Omega}_2^2 &= g_2^2(\hat{N}_2 + 1)\ , \\ \hat{\Omega}_3^2 &= g_3^2\, \hat{N}_3\ . \end{aligned} \tag{13}$$

Differentiation of the operators (12) leads to the following three integrals of motion :

$$\hat{C}_{\alpha\beta} = -\, g_{\alpha}\, g_{\beta}\, \hat{T}_{\alpha\beta} + \hat{\Omega}_{\alpha}^2\, \hat{R}_{\beta\beta} + \hat{\Omega}_{\beta}^2\, \hat{R}_{\alpha\alpha}\ , \quad (\alpha \neq \beta = 1, 2, 3) \tag{14}$$

where, obviously, $\hat{C}_{\alpha\beta} = \hat{C}_{\beta\alpha}$. One easily checks that these operators commute with the operators $\hat{\Omega}_{\alpha}^2$.

With respect to equations (14) and the conservation law (7), we finally get the following closed set of differential equations for the level population operators :

$$\ddot{\hat{R}}_{\alpha\alpha} = -\, (3\, \hat{\Omega}_{\alpha}^2 + \hat{\Omega}^2)\, \hat{R}_{\alpha\alpha} - 3\, \hat{\Omega}_{\alpha}^2 \sum_{\beta \neq \alpha}^{3} \hat{R}_{\beta\beta} + \sum_{\beta \neq \alpha}^{3} \hat{C}_{\alpha\beta} + 2\, \hat{\Omega}_{\alpha}^2\ , \quad (\alpha = 1, 2, 3) \tag{15}$$

where

$$\hat{\Omega}^2 = \sum_{\beta=1}^{3} \hat{\Omega}_\beta^2 . \tag{16}$$

is the operator of the effective three-photon Rabi frequency.

The solution of the above equations can be found using the Laplace transform technique. To solve the problem one should note the commutativity of the operators $\hat{\Omega}_\alpha^2$ and $\hat{\Omega}^2$ with the operator $\dot{\hat{R}}_{\alpha\alpha}$.

After some lengthy algebra we finally find :

$$\hat{R}_{\alpha\alpha}(t) = -2\,\hat{\alpha}_\alpha \sin^2 \frac{\hat{\Omega} t}{2} + \hat{\beta}_\alpha \sin \hat{\Omega} t + \hat{\Omega}_\alpha^2\, \hat{P}(t) + \hat{R}_{\alpha\alpha}^0 , \quad (\alpha = 1, 2, 3) \tag{17}$$

and, owing to the conservation law (7),

$$\hat{R}_{44}(t) = -\hat{\Omega}^2\, \hat{P}(t) + \hat{R}_{44}^0 \tag{18}$$

where the superscript (^{0}) denotes the operators at $t = 0$.

The operator $\hat{P}(t)$ is :

$$\hat{P}(t) = -2\,\hat{\alpha} \sin^2 \hat{\Omega} t + \hat{\beta} \sin 2\,\hat{\Omega} t . \tag{19}$$

Formally, the solutions (17)-(18) resemble those for the three-level two-mode system [10, 11] ; due to the configuration of levels assumed, the four-level atom preserves the two Rabi frequency branches $\hat{\Omega}$ and $2\,\hat{\Omega}$ of the three-level atom [9-12]. However, the amplitude operators are different and their form is as follows :

$$\hat{\alpha} = \left(\sum_{\beta=1}^{3} \hat{\Omega}_\beta^2\, \hat{R}_{\beta\beta}^0 - \hat{\Omega}^2\, \hat{R}_{44}^0 + \sum_{\beta=1}^{3} \sum_{\gamma>\beta}^{3} g_\beta\, g_\gamma\, \hat{T}_{\beta\gamma}^0 \right) \Big/ 2\,\hat{\Omega}^4 ,$$

$$\hat{\beta} = \sum_{\beta=1}^{3} \dot{\hat{R}}_{\beta\beta}^0 / 2\,\hat{\Omega}^3 ,$$

$$\hat{\alpha}_\alpha = \left[2\,\hat{\Omega}_\alpha^2 \sum_{\beta=1}^{3} \hat{\Omega}_\beta^2 (\hat{R}_{\alpha\alpha}^0 - \hat{R}_{\beta\beta}^0) + g_\alpha\, \hat{\Omega}^2 \times \sum_{\beta \neq \alpha}^{3} g_\beta\, \hat{T}_{\alpha\beta}^0 - \hat{\Omega}_\alpha^2 \sum_{\beta=1}^{3} \sum_{\gamma \neq \beta}^{3} g_\beta\, g_\gamma\, \hat{T}_{\beta\gamma}^0 \right] \Big/ \hat{\Omega}^4 ,$$

$$\hat{\beta}_\alpha = \sum_{\beta=1}^{3} (\hat{\Omega}_\beta^2\, \dot{\hat{R}}_{\alpha\alpha}^0 - \hat{\Omega}_\alpha^2\, \dot{\hat{R}}_{\beta\beta}^0) / \hat{\Omega}^3 . \tag{20}$$

Moreover, with respect to the relations (10), we get :

$$\hat{n}_\alpha(t) = -2\,\hat{\alpha}_\alpha \sin^2 \frac{\hat{\Omega} t}{2} + \hat{\beta}_\alpha \sin \hat{\Omega} t + \hat{\Omega}_\alpha^2\, \hat{P}(t) + \hat{n}_\alpha^0 \quad (\alpha = 1, 2, 3) \tag{21}$$

4. Discussion.

To start with, we shall compare the dynamical behaviour of the fundamental level 1 of the atom under consideration and of that of the three-level atom in the ladder configuration (levels 1-4-3). Let us assume that at $t = 0$ the four- and three-level atom is in its lower state $|1\rangle$, i.e., that the expectation value $R_{11}^0 = \langle 1 | \hat{R}_{11}^0 | 1 \rangle\ \Gamma = 1$, and that the field mode 2 starts from vacuum whilst the field modes 1 and 3 are initially coherent with the mean photon numbers $\langle \hat{n}_1^0 \rangle = \overline{n_1^0}$ and $\langle \hat{n}_3^0 \rangle = \overline{n_3^0}$, respectively.

Under these conditions for the expectation values $R_{11}(t)$ in the case of $g_1 = g_2 = g_3 = g$, we find from (17) and (20) :

$$R_{11}(t) = 1 - \sum_{n_1^0=0}^{\infty} \sum_{n_3^0=0}^{\infty} \left\{ \frac{4\, n_1^0 (n_3^0 + x)}{(n_1^0 + n_3^0 + x)^2} \times \sin^2(\sqrt{n_1^0 + n_3^0 + x}\; gt/2) + \frac{n_1^{02}}{(n_1^0 + n_3^0 + x)^2} \times \sin^2(\sqrt{n_1^0 + n_3^0 + x}\; gt) \right\} P(n_1^0)\, P(n_3^0) , \tag{22}$$

where the statistical weights $P(n_i^0)$, $i = 1, 3$, are obviously given by the Poissonian distributions :

$$P(n_i^0) = \frac{\overline{n_i^0}^{\,n_i^0}}{n_i^0\,!}\, e^{-\overline{n_i^0}} . \tag{23}$$

In equation (22) we have introduced the parameter x in order to present in a single expression the formulas valid for both the four- and three-level atom. Namely, in the first case one should put $x = 1$ while in the latter case $x = 0$ (then $g_2 = 0$).

For great photon numbers $\overline{n_1^0}$ and $\overline{n_3^0} \gg 1$ the summations over n_1^0 and n_3^0 can be performed analytically by using the saddle — point method as has been done for the Jaynes-Cummings model [18]. Here, we are interested in the influence of the enhanced depopulation rate related to the existence of spontaneous transitions of the atom from the level 4 to the level 2 on the time-evolution of the fundamental level population. This influence will be appreciable at relatively small photon numbers $\overline{n_1^0}$ and $\overline{n_3^0}$. Then, however, we cannot use the saddle-point method and we have to perform numerical computations. The results of our numerical solutions are presented in figures 2 and 3.

Figure 2 shows the short-time evolution of the fundamental level population. It is readily seen that the initial oscillation period is shorter for the four-level atom. This is so because the Rabi frequencies in the sinusoidal factors in the sums (22) are greater in this case owing to the extra 1 related to the spontaneous transitions of the atom to the level 2.

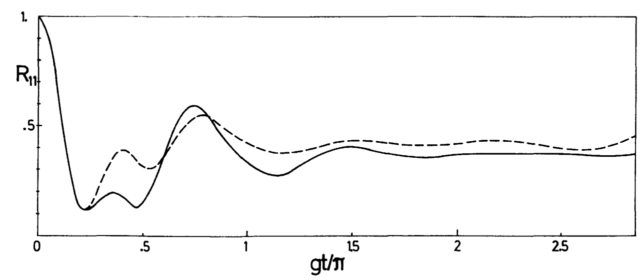

Fig. 2. — Short-time evolution of the fundamental level population $R_{11}(t)$ for the four-level atom (solid line, $g_1 = g_2 = g_3 = g$, $\overline{n_1^0} = 5$, $\overline{n_2^0} = 0$, $\overline{n_3^0} = 1$) and the three-level atom in the ladder configuration (broken line, $g_1 = g_3 = g$, $g_2 = 0$, $\overline{n_1^0} = 5$, $\overline{n_3^0} = 1$).

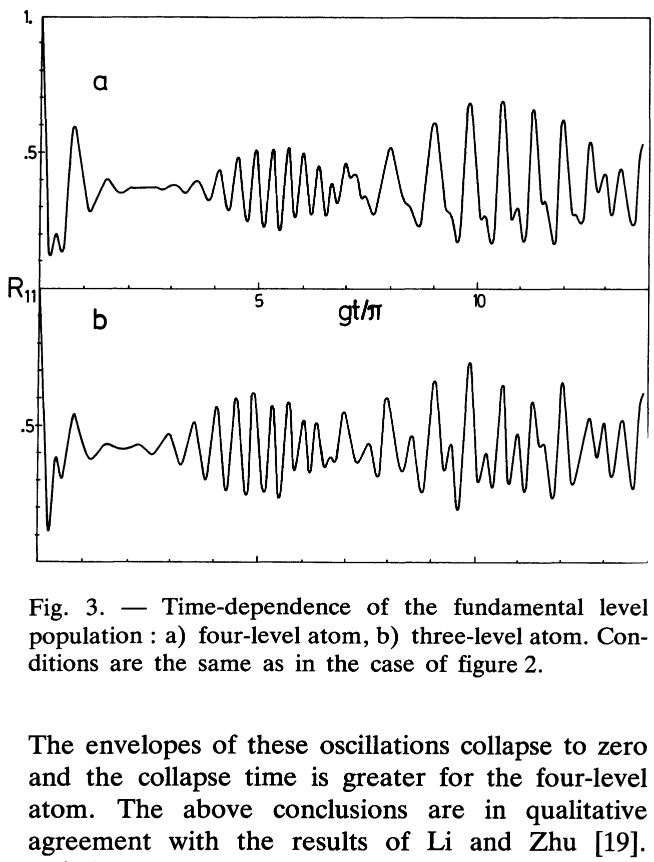

Fig. 3. — Time-dependence of the fundamental level population : a) four-level atom, b) three-level atom. Conditions are the same as in the case of figure 2.

The envelopes of these oscillations collapse to zero and the collapse time is greater for the four-level atom. The above conclusions are in qualitative agreement with the results of Li and Zhu [19]. Strictly, they considered an N-level $(N-1)$-mode system with common upper level and examined the effect of spontaneous transitions to $N-2$ lower levels on the dynamical behaviour of the level populations and photon numbers.

With the mean photon numbers assumed, the maxima of the second revivals (Figs. 3) are noticeably greater than those of the first revivals. In the case of a two-level atom one has a sequence of revivals with monotonous decrease of their maxima [18]. Already in the case of the three-level atom one deals with different kinds of revivals related to the two branches of the Rabi frequencies in the sums (22). The maximum of the subsequent low-frequency branch revival is greater than that of the preceding high-frequency branch revival [11]. The greater maxima of the second revivals and the longer revival times (Figs. 3) suggest that they are related to the low-frequency oscillations ; albeit, their irregularities show that they are in fact compositions of the first revival of the low-frequency oscillations and the next successive revivals of the high-frequency oscillations. The maxima of the second revivals are practically comparable for both atoms, contrary to the maxima of the first revivals. We conclude directly that, for the fundamental level population, the role of the high-frequency branch of oscillations (at small photon numbers and upward of very short times) is diminished in the case of the four-level atom by comparison with the three-level atom (see also Eq. (22)). This conclusion coincides with that arising from the results for the model of Li and Zhu [19] ; namely, the participation of the high-frequency branch of oscillations in the fundamental level population decreases as the number of levels increases.

Let us now discuss the dynamical behaviour of the level 2 and compare the results with those for the three-level atom in the lambda configuration (levels 1-4-2). In both cases we assume that the level 2 is populated by spontaneous transitions. Here, for the three-level atom we have to consider one pumping field only, i.e. the field mode 1. For the four-level atom we must as previously take into account two pumping modes 1 and 3. Under the same conditions as assumed in the relation (22) we find for the four-level atom

$$R_{22}(t) = \sum_{n_2^0=0}^{\infty} \sum_{n_3^0=0}^{\infty} \frac{n_1^0}{(n_1^0+n_3^0+1)^2} \left\{4 \sin^2 \left(\sqrt{n_1^0+n_3^0+1}\, gt/2\right) - \sin^2 \left(\sqrt{n_1^0+n_3^0+1}\, gt\right)\right\} P(n_1^0)\, P(n_3^0) =$$

$$= \sum_{n_1^0=0}^{\infty} \sum_{n_3^0=0}^{\infty} \frac{4\, n_1^0}{(n_1^0+n_3^0+1)^2} \sin^4 \left(\sqrt{n_1^0+n_3^0+1}\, gt/2\right) P(n_1^0)\, P(n_3^0)\,, \qquad (24)$$

and for the three-level atom

$$R_{22}(t) = \sum_{n_1^0=0}^{\infty} \frac{4\, n_1^0}{(n_1^0+1)^2} \sin^4 \left(\sqrt{n_1^0+1}\, gt/2\right) P(n_1^0)\,, \qquad (25)$$

where the statistical weights for coherent pumping modes are given by (23). Moreover, for both atoms $R_{22}(t) = \overline{n}_2(t)$.

It is obvious from equations (24) and (25) that now the amplitudes of the high- and low-frequency branch of oscillations are reduced in the same degree for the four-level atom and hence, contrary to the formerly discussed case, the maxima of the first and the second revivals are less for this atom. The results of our numerical computations are plotted in figures 4.

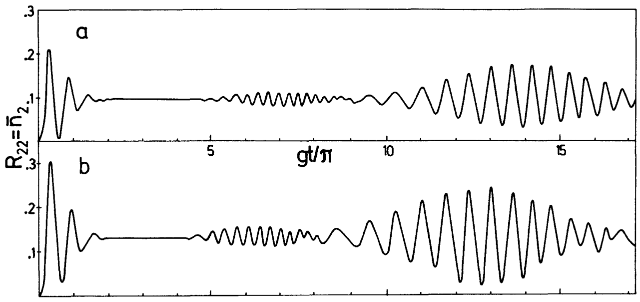

Fig. 4. — Time-dependence of the level 2 population: a) four-level atom ($g_1 = g_2 = g_3 = g$, $\overline{n_1^0} = 10$, $\overline{n_2^0} = 0$, $\overline{n_3^0} = 2$), b) three-level atom in the lambda configuration ($g_1 = g_2 = g$, $g_3 = 0$, $\overline{n_1^0} = 10$, $\overline{n_2^0} = 0$).

The maxima of the second revivals in the cases considered in figures 4 are remarkably larger than those of the first revivals. This feature, as evident from equation (24), is related with the factor 4 in the term representing the low-frequency branch of the Rabi oscillations (the factors containing photon numbers are the same for both branches).

To conclude briefly, we have solved the operator equations for the four-level atom explicitly for the case of three one-photon resonances. The quantum electrodynamical expression of the three-photon Rabi frequency has been found as well. We have shown that quantum collapse and revival are possible in the loss-free four-level three-mode system and, as in the case of the three-level atom, we deal with different kinds of revivals due to the existence of two branches of the Rabi frequency of oscillations. As for the fundamental level population, we note, at small pumping photon numbers, the diminished role of only the high-frequency branch of oscillations in comparison with the three-level atom in the ladder configurations as reflected in the less maximum of the first revival.

Acknowledgments.

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References

[1] Jaynes, E. T. and Cummings, F. W., *Proc. IEEE* **51** (1963) 89.

[2] Ackerhalt, J. R., Ph. D. Thesis, University of Rochester 1974, in: Ackerhalt, J. R. and Rzażewski, K., *Phys. Rev.* **A 12** (1975) 2549.

[3] Kancheva, L. and Pushkarov, D., *J. Phys.* **B 13** (1980) 427.

[4] Kancheva, L., Pushkarov, D. and Rashev, S., *J. Phys.* **B 14** (1981) 573.

[5] Elgin, J. N., *Phys. Lett.* **A 80** (1980) 140.

[6] Hioe, F. T. and Eberly, J. H., *Phys. Rev.* **A 25** (1982) 2168.

[7] Shi, Y. C. and Da, C. S., *Phys. Rev.* **A 25** (1982) 3169.

[8] Radmore, P. M. and Knight, P. L., *J. Phys.* **B 15** (1982) 561.

[9] Li, X. S. and Bei, N. Y., *Phys. Lett.* **A 101** (1984) 169.

[10] Bogolubov, N. N. Jr., Kien, Fam Le and Shumovsky, A. S., *Phys. Lett.* **A 101** (1984) 201; *ibid.* **107** (1985) 173.

[11] Bogolubov, N. N. Jr., Kien, Fam Le and Shumovsky, A. S., *J. Phys.* **A 19** (1986) 191.

[12] Shumovsky, A. S., Aliskenderov, E. J. and Kien, Fam Le, *J. Phys.* **A 18** (1985) L1031.

[13] Yoo, H. I. and Eberly, J. H., *Phys. Rep.* **118** (1985) 239.

[14] Białynicka-Birula, Z. and Białynicki-Birula, I., *Phys. Rev.* **A 16** (1977) 1318.

[15] Stettler, J. D., Bowden, C. M., Witrid, N. M. and Eberly, J. H., *Phys. Lett.* **73** (1979) 171.

[16] Deng, Z., *Opt. Commun.* **48** (1983) 284.

[17] Kancheva, L. and Rashev, S., *J. Phys.* **B 18** (1985) 3437.

[18] Narozhny, N. B., Sanchez-Mondragon, J. J. and Eberly, J. H., *Phys. Rev.* **A 23** (1981) 236.

[19] Li, X. S. and Zhu, S. Y., *Physica* **A 131** (1985) 575.