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Wavenumber restriction in systems with discontinuous nonlinearities and the buckling instability of plates

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Résumé. — On montre que l'on peut calculer de façon presque entièrement analytique beaucoup de propriétés de systèmes qui forment spontanément des structures spatiales, avec des nonlinéarités discontinues et rigides. Ces systèmes ne fournissent pas seulement de nouvelles connaissances sur les mécanismes de restriction du nombre d’onde, mais ils sont aussi accessibles expérimentalement. Nous suggérons des expériences sur l’instabilité de flambement d’une plaque rectangulaire entre deux butées rigides restreignant l’amplitude.

Abstract. — It is shown that many properties of spontaneous pattern-forming systems with discontinuous, sharply saturating nonlinearities can be calculated almost analytically even far above threshold. Besides providing insight into the mechanisms of wavenumber restriction such systems are experimentally accessible. We suggest experiments on the buckling instability of a rectangular plate with rigid, amplitude-restricting stoppers.

1. Introduction.

The theoretical investigation of the selection and restriction of the wavelength in systems that form spontaneous periodic patterns usually requires substantial numerical work, except in the weakly nonlinear region near threshold, where the periodic variations are nearly sinusoidal and expansions are possible. Sufficiently far above threshold interactions with higher harmonics can lead to interesting effects [1] and it would seem useful to find systems where this region is accessible to a global analysis, even in the simplest case where the patterns are extended in just one dimension.

In section 2 we study such a model, which is a special case of the general class

$$\partial_t u = [\varepsilon - (1 + \partial_x^2) u - h(u)]$$

where \(h(u)\) is an appropriate nonlinear function of \(u (= u(x,t))\). A model of this type may be used to describe the buckling of an elastically supported elastic beam under axial compression [2, 3]. A term proportional to \(\partial_x^2 u\) must then be added to the lhs, but this does not lead to changes for the properties we are interested in. The axial load is proportional to \(1/\sqrt{1 - \varepsilon}\). For positive values of the reduced control parameter \(\varepsilon\) the basic state \(u = 0\) is unstable with respect to spatially periodic perturbations. Models of the above type have also been studied in other contexts [4-7]. The simplest case is that where \(h(u) = u^2\).

We choose \(h(u)\) to describe a discontinuous, sharply saturating nonlinearity so that \(h(u) = 0\) for \(|u| \leq u_0\) and, roughly speaking, \(h(u) = -\infty\) for \(|u| > u_0\). Since the results are not affected by the choice of \(u_0\) we take \(u_0 = 1\). An attractive feature of this system is that it admits a realistic interpretation in terms of the above-mentioned buckling problem when rigid stoppers that limit the buckling amplitude are present (see Fig. 1). Models with sharply saturat-
ing nonlinearities have been studied before from the point of view of mathematical bifurcation theory [8]. The treatment in reference [8] is in particular aimed at some problems in statistical physics and biology. In the former the sharp-saturation case arises naturally as the zero-temperature limit. An important difference of our treatment from that of reference [8] is that we require \( u \) to be twice continuously differentiable as is appropriate for the case of a purely elastic beam.

An interesting result of section 2 is that the periodic solutions of the model are stable with respect to slow modulations of the wavenumber ("Eckhaus-stability") where they exist, i.e. inside the full neutral curve (see Fig. 2) except on the low-wavenumber side for \( \epsilon > 0.36 \), where the ordinary Eckhaus instability [9-13] sets in. For \( \epsilon > 0.64 \) instability sets in simultaneously with the solutions developing secondary extrema.

The peculiar behaviour of the Eckhaus-stable region can be confirmed by considering models with \( h(u) = u|u|^n \), \( n \neq 0 \). In the Appendix we show that then the ratio between the width of the Eckhaus-stable wavenumber region to that of the full region of existence is \( n/(n+2) \) near threshold [14]. For \( n = 1 \) the usual 1/3-criterion is obtained [9-13] whereas for \( n \to \infty \), which corresponds to the abrupt nonlinearity, the ratio indeed becomes 1.

We also consider solutions that match to non-periodic boundary conditions. The limits of their existence can be obtained from transcendental equations. The additional restriction produced by large classes of boundary conditions [15, 16, 3, 7] is thereby calculated. In the extremely nonperiodic case one limit of existence coincides with the band centre. The presentation in section 2 is such that it can easily be applied to one-dimensional models with different linear parts.

In section 3 we present a similar investigation for the post-buckling behaviour of a long, thin rectangular elastic plate held laterally and compressed longitudinally. In the context of wavenumber selection this problem was first considered by Pomeau [17] and subsequently by others [18, 19]. We here assume again the presence of sharply saturating nonlinearities that restrict the amplitude to values which are so small that intrinsic nonlinearities of the plate are negligible. Figure 1 may again serve as an illustration of the situation near the centre of the plate. Experiments on this system can be done similarly to those that have been performed without the rigid stoppers [20]. We only consider simply supported ends which corresponds to the case of periodic boundary conditions.

Since the system is two dimensional simple analysis is possible only near the neutral curve, where it can be mapped onto the model of section 2, and where the results essentially carry over. Outside this region we find numerically transitions from solutions where the amplitude restriction is reached at isolated points to solutions where it is reached along line segments that lie either parallel or perpendicular to the long direction, and to solutions where the amplitude restriction is reached on finite areas. The Eckhaus-stability limit behaves similar to that for the one-dimensional model.

2. One-dimensional model.

The problem we wish to consider can be formulated in terms of the potential

\[
V = \frac{1}{2} \int dx \left[ -2 (\partial_x u)^2 + (\partial_x^2 u)^2 + (1 - \epsilon) u^2 \right].
\]

(2)

This is to be minimized subject to the subsidiary condition

\[
|u(x)| \leq 1 \quad \text{for all } x
\]

(3)

which acts as an abrupt and rigid nonlinearity. Appropriate boundary conditions will be introduced below. We restrict ourselves to \( \epsilon \leq 1 \).

We assume that the integrand of (2) is continuous. Then \( u^* (= \partial_x^2 u) \) is continuous and \( u \) satisfies an equation of the form

\[
[\epsilon - (1 + \partial_x^2)^2] u = \sum \alpha_i \delta(x - x_i) u(x)
\]

(4)
where the \( x_i \) are the points with \( u^2 = 1 \). The factor \( u(x) \) on the rhs of (4) takes the values \( \pm 1 \) and is introduced for later convenience. Since \( u' \) is continuous one has \( u'(x_i) = 0 \). Except for \( \varepsilon = 1 \) the function \( u = 1 \) is not a solution of equation (4) with the rhs set equal to zero. Thus for \( \varepsilon \neq 1 \) the points \( x_i \) are always isolated. The quantities \( \alpha_i \) are connected with discontinuities of the third derivative of \( u \\
\alpha_i = -u(x_i) \left[ u'''(x_i + 0) - u'''(x_i - 0) \right] \quad (5) \\
and can therefore not be chosen freely. The \( \alpha_i \) may be understood in terms of the forces acting between the rigid bars and the flexible beam and one therefore must have \( \alpha_i > 0 \). Otherwise the potential, which from (2) and (4) is 
\[
V = -\frac{1}{2} \sum_i \alpha_i \\
\]
would not be a minimum.

2.1 PERIODIC SOLUTIONS. — For \( \varepsilon > 0 \) the trivial solution \( u = 0 \) is unstable against small periodic disturbances with wavenumber \( p \) in the interval \( p_1 < p < p_2 \), where
\[
p_1 = \left( 1 - \sqrt{\varepsilon} \right)^{1/2}, \quad p_2 = \left( 1 + \sqrt{\varepsilon} \right)^{1/2} \\
(7)
give the neutral curve shown in figure 2 (curve 1). In this range we expect the existence of periodic solutions with the amplitude being restricted at points \( x_i \). In figure 2 also the 2-neutral curve as well as the 3- and 4-neutral curves are shown (broken curves), where e.g. the 2-neutral curve is the neutral curve with \( p \) replaced by \( p/2 \). In the intersecting regions one expects the possibility of wavenumber-doubling, -tripling, etc..

First we consider simple periodic solutions with just one maximum and minimum per period (Type-A solutions). Choosing \( u(0) = 0 \) and 
\[
x_0 = \pi/2 \ p \\
(8)
\]
(wavelength = 4 \( x_0 \)) they are in the interval \( |x| < x_0 \) given by (see Eq. (4))
\[
u(x) = v(x)/v(x_0), \\
(9)
\]
\[
v(x) = p_1 \cos (p_1 x) \sin (p_2 x) - p_2 \cos (p_2 x) \sin (p_1 x). \\
(10)
\]
The function is continued by \( u(x_0 + x) = u(x_0 - x) \) and \( u(x + 4 x_0) = u(x) \) (for an example see Fig. 3, top). From equations (5) and (10) one finds
\[
\alpha_0 = 2 u'''(x_0 - 0) = 2 p_1 p_2 (p_1^2 - p_2^2) x \cos (p_1 x_0) \cos (p_2 x_0)/v(x_0). \\
(11)
\]
and the potential (energy) per unit length is
\[
V_p = -\alpha_0/4x_0. \\
(12)
\]

---

**Fig. 3.** — One period of a Type-A and a Type-B solution. The solid parts are given by equations (9), (10) and equations (14), (15) for the Type-A and the Type-B solutions, respectively.

\( V_p \) is zero at the neutral curve and negative inside it. The wavenumber where \( V_p \) is minimal \( \left( \partial_p V_p = 0 \right) \) is given by the dotted curve in figure 2. It deviates from \( p = 1 \) very slightly to the left. The maximum of the balancing force is given by the center of the band \( p = (p_1 + p_2)/2 \).

Actually not all the solutions (9) with \( p_1 < \pi/2 \ x_0 < p_2 \) may be allowed, because, when \( x_0 \) becomes too large, \( u''(x_0) \) can become positive so that (3) is not satisfied. From (9) the condition \( u''(x_0) = 0 \) is seen to be equivalent to
\[
p_1 \tan (p_1 x_0) = p_2 \tan (p_2 x_0). \\
(13)
\]
The solution of (13) is shown as curve 2 of figure 2. It starts at the intersection of the neutral and the 3-neutral curve (here \( \varepsilon = 0.64 \)) and ends at the right endpoint of the 2-neutral curve. For \( \varepsilon > 0.64 \) the Type-A solutions (9) exist to the right of curve 2. For \( \varepsilon < 0.64 \) they exist everywhere between the neutral curves.

To the left of curve 2 there exist periodic solutions with six extrema per period (Type-B solutions). They are obtained by matching onto (9) another segment
\[
u(x) = w(x)/w(x_0) \\
(14)
\]
\[
w(x) = p_1 \sin \left[ p_1(x_1 - x_0) \right] \cos \left[ p_2(x_1 - x) \right] - p_2 \sin \left[ p_2(x_1 - x_0) \right] \cos \left[ p_1(x_1 - x) \right] \\
(15)
\]
used in the interval \( x_0 < |x| < x_1 \). \( x_0 \) is chosen in such a way that \( u' \) is continuous there. This leads to the condition

\[
\tan \left( p_1 x_0 \right) \tan \left[ p_1 (x_1 - x_0) \right] = \tan \left( p_2 x_0 \right) \tan \left[ p_2 (x_1 - x_0) \right].
\]

(16)

The function \( u(x) \) is continued further as before. Now one has \( p = \pi / 2 x_1 \). An example is shown in figure 3, bottom. When \( x_0 \) is decreased, starting from \( x_0 = x_1 \) at curve 2 in figure 2, \( x_1 \) (= \( \pi / 2 p \)) increases and \( p \) decreases. Also, \( u(x_1) \) decreases, starting from 1. The smallest \( p \) which can be reached is given by the condition \( x_0 = x_1 / 2 \). From (16) one sees that this leads to \( (p_1 + p_2) x_1 = 2 \pi \), which is equivalent to

\[
p = \left( p_1 + p_2 \right) / 4.
\]

(17)

This relation is shown as curve 3 of figure 2.

We can decrease \( x_0 \) further, but then \( x_1 \) also decreases and \( p \) increases again. \( u(x_1) \) continues to decrease and for \( p = p_2 / 3 \) we have \( u(x_1) = -1 \). At this point the solution has become 3-periodic, i.e. it is the same as that at the neutral curve \( p = p_2 \). Although the solutions on this branch look similar to the Type-B solutions we call them Type-C. The potential \( V_p \) can be calculated similarly as for the Type-A solutions. In figure 4 \( V_p \) as a function of \( p \) is plotted for \( \varepsilon = 0.9 \). The inset shows the region where the Type-C solutions exist in greater detail. The Type-C solutions have a higher potential and are clearly unstable. In fact they represent the saddle points that separate the Type-B solutions, which are stable with respect to wavenumber tripling, from the (Type-A) solutions with tripled wavenumber. For comparison the potential for the latter solutions is also plotted in the inset of figure 4 (« Type-A »).

The above discussion appears to exhaust all different periodic solutions. Thus to the left of curve 3, figure 2, there are no solutions with basic wavenumber \( p \). Since a solution with basic wavenumber \( p \) also has the wavenumbers \( p/n \) with \( n = 2, 3, \ldots, \) one can, however, assign periodic solutions from the above set to that region.

To test the Type A and -B solutions for stability with respect to slow modulations of the wavenumber (Eckhaus-stability) we have to check if \( \delta^2 V_p \) is positive [13, 18]. For Type-A solutions one obtains from equations (11) and (12),

\[
\delta^2 V_p = -\frac{4(p_2^2 - p_1^2)(p_1 p_2 x_0)^2}{\pi^2[p_1 \tan (p_2 x_0) - p_2 \tan (p_1 x_0)]^2} \times \left[ \frac{\tan (p_2 x_0) - p_2 \tan (p_1 x_0)}{\cos^2 (p_1 x_0) - p_1 \cos^2 (p_2 x_0)} \right] \times \left[ \frac{\tan (p_2 x_0) - p_1 \tan (p_2 x_0)}{p_1} \right].
\]

(18)

The condition for the Eckhaus limit \( \delta^2 V_p = 0 \) requires one of the terms in square brackets in the numerator of equation (18) to be zero. The second one is zero on curve 2 of figure 2 (see Eq. (13)). The first one is zero on the dash-dotted curve 4. Thus Type-A solutions are stable all the way up to their limit of existence except in the small region between curve 4 and the neutral curve, where they exist unstably. The Eckhaus-unstable solutions have the property that the curvature at point \( x_0 \) becomes larger when the wavelength becomes larger, in contrast to the stable solutions. Thus curve 4 can also be characterized by the fact that the curvature at \( x_0 \) has a minimum there.

A similar but considerably more complex formula can be obtained for \( \delta^2 V_p \) for Type-B solutions. We find that \( \delta^2 V_p \) is slightly negative so all Type-B solutions are very weakly Eckhaus-unstable. In an actual experiment the solutions might, however, be stabilized by friction.

2.2 NONPERIODIC BOUNDARY CONDITIONS. — The solutions considered in the last subsection are applicable for finite systems only if the boundary conditions are consistent with periodicity. Now we consider a semi-infinite system \( 0 < x < \infty \) and study the influence of more general boundary conditions

\[
\lambda \partial_x u(0) + \mu \partial^2_x u(0) = 0, \quad |\lambda| + |\mu| \neq 0
\]

(19)

on solutions which become periodic with wavenumber \( p \) for \( x \to \infty \). The solutions may also be used for finite systems with nonperiodic boundary conditions on both sides. For \( \lambda = 0 \) we have \( \partial^2_x u(0) = 0 \). This corresponds to the periodic case.

Let \( x_0 \) again be the point where \( u \) reaches for the first time the amplitude restriction (3). The solution of (4) with the boundary conditions (19) and \( u(x_0) = 1, \partial_x u(x_0) = 0 \) is for \( 0 \leq x < x_0 \) given by

\[
u(x) = \frac{y(x)}{y(x_0)}.
\]

(20)
As it stands $|u|$ may have extrema in the range $0 < x < x_0$ which are larger than 1. These cases have to be eliminated. For $x > x_0$ the solution can only be one of the periodic solutions with wavenumber $p$. At $x_0$ the second derivatives $\partial_x^2 u$ have to be matched, which leads to a relation between $x_0$ and $p$. Moreover one has to restrict oneself to cases where the discontinuity in the third derivatives is negative so that $\alpha_0 > 0$ (see Eq. (5)). All these conditions lead to transcendental relations (one matching condition and two inequalities) which may easily be implemented numerically.

For the case $\mu / \lambda = 0$ we find solutions inside a restricted wavenumber band. They have $\alpha_0 < \alpha_1$ ($\alpha_1$ now pertains to the periodic part of the solution). Clearly these solutions are the analogs of the Type-I solutions introduced for models with gradual non-linearities previously [7, 21]. There the amplitude near the boundary falls below its bulk value. For sufficiently small $\epsilon$ one has extrema of $u(x)$ in the range $x < x_0$ where $|u| < 1$. For $\mu = 0$ the band restriction is strongest. This case is shown in figure 5 (boundaries designated by the diamonds). For this case the transcendental relations lead to the explicit analytical expression for the left boundary $p = (p_1 + p_2)/2$, which coincides with the centre of the band. Also shown in figure 5 are the case $\mu / \lambda = -100$ which may be termed « nearly periodic » and an intermediate case $\mu / \lambda = -5$. We note that in the almost periodic case almost the full stable band is accessible (for $\mu / \lambda \to -\infty$ it becomes fully accessible). Moreover the left boundary actually crosses into the (Eckhaus-unstable) Type-B region (not shown in Fig. 5). There now also exist nearly periodic solutions in the small region where Type-A solutions are unstable. They are, however, completely separate from the stable ones. We have not investigated the unstable nonperiodic solutions in detail. Their existence raises questions connected with structural stability which are briefly discussed in section 4.

For the case $\mu / \lambda > 0$ one has Type-I solutions which exist in approximately the same regions as those with $\mu$ replaced by $-\mu$. In addition one has solutions with $\alpha_0 > \alpha_1$ which exist in the full band of existence of the periodic solutions. These solutions correspond to the Type-II solutions where the amplitude near the boundary rizes above its bulk value [7, 21]. As in the earlier work the Type-I solutions are expected to be unstable in this case.

For $\epsilon \to 0$ one may expand the neutral curve as follows

$$p_{1,2} = 1 \mp a \sqrt{\epsilon} + b \epsilon \mp \ldots$$

(for our model one has from equation (7) $a = 1/2$ and $b = -1/8$). Carrying through this expansion in the transcendental relations leads to the following expression for the band restriction to order $\epsilon$ for Type-I solutions

$$|p - 1 - \epsilon (a^2/2 + b)| \leq \epsilon a^2(1 + 4 \mu^2/\lambda^2)^{1/2}/2.$$  \hfill (23)

The band restriction is symmetric to the line $p = 1 = \epsilon (a^2/2 + b)$, which gives the minimum of $V_{\text{p}}$ to order $\epsilon$. In figure 5 the result of this analysis is also plotted (solid lines). This asymptotic analysis is analogous (although much simpler) to that first introduced by Cross et al. [15] and used subsequently [16, 3, 7, 21, 19]. The band given by (23) is twice as wide as that found for the model with the gradual nonlinearity proportional to $u^3$ [15, 16, 3, 7].

3. Buckling of the plate.

We now consider the post-buckling behaviour of a long, thin, elastic plate that is compressed longitudinally. The buckling amplitude is limited to a small value by rigid, flat stoppers. The problem may again be formulated most generally in terms of the minimizing functional [22, 18]

$$V = \iint dF \left[(\Delta w)^2 - \lambda (\partial_x w)^2\right]. \quad (\Delta = \partial_x^2 + \partial_y^2)$$

\hfill (24)
for the transverse displacement $w(x, y)$ with subsidiary condition

$$|w| = 1 \quad \text{for all } x, y .$$

(25)

We use dimensionless variables so that lengths are measured in units $d^{2}/\pi$ ($d$ = width of plate) and the control parameter

$$\lambda = 4(1 + \varepsilon)$$

(26)

is proportional to the longitudinal load [18]. For simplicity we will here consider the case of simply supported boundaries all around, which implies periodicity with period $2\pi$ in the $y$-direction and with period $2\pi/p$ in the $x$-direction.

Clearly $w$ satisfies the linear version of the Von-Karman equation [22, 18],

$$\Delta^2 w + \lambda \partial_x^2 w = 0$$

(27)

wherever $|w| < 1$. As in section 2 one has to find the places where $|w| = 1$.

To obtain the neutral curve and the behaviour in its vicinity we make the ansatz $w(x, y) = v(x) \cos(y)$. Equation (24) then goes over into

$$V = \frac{1}{2} \int dx \left[ -2(1 + 2\varepsilon) (\partial_x v)^2 + (\partial_x^2 v)^2 + v^2 \right].$$

(28)

Clearly this can be mapped onto the problem considered in section 2 by the transformation (see Eq. (2)).

$$v(x) = u(\sqrt{a}x), \quad a = (1 - \varepsilon)^{-1/2}, \quad \varepsilon = (a - 1)/2$$

(29)

where the $\varepsilon$ of section 2 was renamed $\bar{\varepsilon}$. From (7) and (29) we obtain for the neutral curve of (24)

$$p_1 = \sqrt{1 + \varepsilon - \sqrt{\varepsilon}}, \quad p_2 = \sqrt{1 + \varepsilon + \sqrt{\varepsilon}}$$

(30)

which is shown in figure 6 (solid curve 1). Also shown in figure 6 are the 2- and the 3-neutral curves (broken). This result can of course easily be derived directly from (24) or (27). The analogy also suggests that the periodic solutions are stable up to the neutral curve wherever this is the case in the system of section 2, i.e. to the right of the 2-neutral curve. The following investigations show that this is correct.

To investigate the behaviour in the nonlinear regime we next look for rigorous solutions where $w$ reaches the amplitude restriction only at the periodic points $x = x_j, y = 0$ (centre of the plate). Making a Fourier expansion and minimizing $V$ from equation (24) with the subsidiary condition $|w| = 1$ at the points $(x_j, 0)$ gives

$$w = \left[ \sum_{j,k} W_{jk} \right]^{-1} \sum_{j,k} (-1)^j \sin((2j + 1)px) \times$$

$$\times \cos((2k + 1)y)/W_{jk},$$

where the sums extend from 0 to $\infty$. A sufficient number of terms is easily summed by computer.

The representation (31) does not guaranty that $|w| < 1$ everywhere. To insure this one has to require that the curvatures of $|w|$ at $(x_j, 0)$ are negative, similar to the condition for the Type-A solutions in section 2. Numerical testing shows that the region of validity of the above solutions is restricted from above by curve 2 in figure 6. On curve 2 either $\partial_x^2 w$ becomes zero (right of point A, large $p$) or $\partial_y^2 w$ becomes zero (left of point A).

What type of solutions exist above curve 2? Since $w = \pm 1$ is a solution, we cannot conclude that the points with $|w| = 1$ are isolated. We expect that above curve 2 of figure 6 the plate touches the restriction on line segments or finite planes with continuous first and second derivatives everywhere. Direct numerical minimization of (24) with (25) on a discrete mesh confirms this expectation. Up to 120 points per period were used. Half a period for one such solution is shown in figure 7.

We tentatively conclude from the numerical evidence that there are three types of solutions which are separated approximately by the dotted curves in figure 6. For large $p$ the solutions touch the amplitude restriction on line segments in the transverse $y$-direction (region III), whereas for low $p$ (region I) the line segments are in the longitudinal $x$-direction. For intermediate $p$ the amplitude restriction is reached on finite planes (region II). The transitions at curve 2 and at the dotted curves appear to be continuous.

Eckhaus-stability was tested as in section 2. We
Fig. 7. — Half a period of a solution of equations (24), (25) with $\varepsilon = 1$, $p = 1$.

find that the solutions are unstable to the left of curve 3. For increasing $p$ it bends to the right so it should be measurable by increasing the load.


In section 2.2 it was found that nonperiodic solutions exist in the Eckhaus-unstable range. This is peculiar since it is in contrast to what one appears to find in systems with the usual gradual nonlinearities [7, 21]. Let us discuss these results in a more general context. Nonperiodic boundary conditions may be described as localized structural perturbations of the homogeneous system, i.e. of perturbations of the parameters. A periodic solution may be called structurally stable with respect to some localized structural perturbation if a solution exists in the presence of the perturbation that coincides with the periodic solution far away in space (phase shifts are of course irrelevant) [7]. Thus, whereas in systems with gradual nonlinearities ordinary (dynamic) instability appears to imply complete structural instability this is not the case for the abrupt nonlinearity. The reason why such a connection fails here is presumably the fact that the influence of the perturbation ends abruptly when the amplitude reaches the amplitude restriction for the first time, instead of falling off exponentially as in systems with gradual nonlinearities.

It is interesting to ask what happens to the buckling problem of section 3 when the distance between the rigid stoppers is increased so that the intrinsic (gradual) nonlinearity of the buckled plate becomes important. One then has an interval inside the neutral curve where the extrema of the buckled plate do not touch the stoppers and where the system therefore behaves according to the investigation of reference [18]. Then the neutral curve cannot be reached because of the Eckhaus-instability. After the plate touches the stoppers crossover to the behaviour investigated here occurs.

Appendix.

We here consider the one-dimensional model of section 2 with a term $|u|^{2(n+1)/(n+1)}$, $n > 0$, added to the potential (2) and condition (3) removed. Near threshold the solutions may be expanded in terms of a small amplitude

$$u = A \cos (px) + o(A). \quad (A.1)$$

To leading order this gives

$$V_p = -\frac{1}{4} \omega_p A^2 + \frac{a_n}{2(n+1)} |A|^{2(n+1)}. \quad (A.2)$$

where

$$\omega_p = \varepsilon - (p^2 - 1)^2. \quad (A.3)$$

Minimizing (A.2) with respect to $A$ yields

$$V_p = -\beta_n (\omega_p)^{1+1/n}, \quad A^2 = (\omega_p/4 \alpha_n)^{1/n} \quad (A.4)$$

(the actual values of $\alpha_n$ and $\beta_n$ are not needed). From this one deduces

$$\delta V_p = 4 \beta_n n + 1 \times \left[ \omega_p (3 p^2 - 1) - \frac{4}{n} p^2 (p^2 - 1)^2 \right]. \quad (A.5)$$

Next we write $p = 1 + \sqrt{\varepsilon Q}/2$ and expand (A.5) to leading order in $\varepsilon$. This gives

$$\delta V_p \sim 2 \varepsilon ^2 \beta_n \frac{n + 1}{n} (1 - Q^2) \left( 1 - \frac{n + 2}{n} Q^2 \right). \quad (A.6)$$

$Q = 1$ corresponds to the neutral curve (here $\omega_p \sim 0$) and $Q_E = \sqrt{n}/(n + 2)$ to the Eckhaus limit. The amplitude as given in (A.4) is maximal at the centre of the band ($p = 1$). Its value at the Eckhaus limit divided by its value at $p = 1$ is, to lowest order in $\varepsilon$, given by

$$(A_E/A_{\text{max}})^2 = (1 + n/2)^{-1/n}. \quad (A.7)$$

This is a monotonically increasing function of $n$ which tends to $\exp(-1/2)$ for $n \to 0$ and to 1 for $n \to \infty$. Thus for $n \to 0$ one has the maximal variation of the amplitudes in spite of sharp wavenumber selection. This is not surprising in view of the fact that in any linear problem (i.e. $n = 0$) the wavenumber is not affected at all by the amplitude.
References
