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Kinetic equation of finite Hamiltonian systems with integrable mean field

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Résumé. — Nous proposons dans cet article une théorie cinétique des systèmes finis, ainsi qu’une nouvelle équation pour leur comportement « microscopique » moyen. Le mouvement est supposé intégrable dans le champ moyen, collectif et/ou imposé de l’extérieur. On suppose le temps de collision plus grand que les périodes caractéristiques du mouvement moyen, ce qui exclut une approche locale de type Fokker-Planck, ou Balescu-Lenard. On peut citer comme exemple un plasma de taille comparable à la longueur de Debye, ou un système gravitationnel auto-gravitant (amas globulaire), qui est un système dilué contenant un nombre fini d’étoiles. Nous discutons les conditions de validité d’une description statistique de ce type de système.

Abstract. — In this paper, we propose a kinetic theory of finite systems, and a new equation for their « microscopic » average behaviour. The equations of particle motion are supposed to be integrable in the mean collective field or/and in an outside given field. The collision time is assumed to be longer than the characteristic orbital time which rules out a local theory of the Fokker-Planck, or Balescu-Lenard type. Two applications of our equation are foreseen: a finite plasma with a Debye length comparable or larger than its size; a self confined gravitational system (i.e. a globular cluster) which is always a dilute system with a finite number of stars; we discuss the validity of a statistical description of such a physical system.

1. Introduction.

The initial motivation of this work is to be found in the study of globular cluster. Let us recall that globular clusters are « spherical » clusters of stars (typically a number of order 10^5) which have a great astrophysical interest: their age and characteristic evolution time are comparable to the estimated age of typical galaxies. The history of these systems is assumed to start with a violent relaxation on the Jeans time scale which leads to a virialized cluster of stars. Phase space mixing is supposed to damp the fluctuations around the average spherical equilibrium and simultaneously the collective particle correlations: this hypothesis is strong but will be assumed in the following. At the end of this process, the cluster is in quasi-equilibrium but on a slow time scale the binary interactions modify its subsequent evolution: a finite cluster cannot remain in Maxwellian equilibrium.

A standard kinetic theory is based on a hierarchy of time scales: \( \tau_C \ll \tau_R \ll \tau_D \), where \( \tau_C \) is the characteristic time of a collision free dynamic (for a plasma, the plasma frequency, for a gravitational system the inverse of the Jeans time); \( \tau_R \) describes the relaxation of particle correlations towards « thermodynamical » equilibrium; \( \tau_D \) is the diffusion time resulting from particle collisions. Our work is mainly relevant for long range interactions; in that case we already know from the plasma theory that \( \tau_R = (\tau_C^2 / \tau_D)^{1/3} \) which automatically ensures \( \tau_C \ll \tau_D \); this condition will be explicitly justified in this paper. \( \tau_R \) is directly related to the non-linear overlapping of the resonances between the different particle trajectories in the mean field; this overlapping is easily understood from the diffusive limit of a stochastic process. A kinetic description can be expected to be valid only for an averaged behaviour on a time scale \( \tau_D \gg \tau_R \).

Finally, for a globular cluster of \( N \) stars we know in advance that a local equation in the configuration space \( (x, v) \) is never valid: the mean free path \( \lambda = \langle v \rangle / \tau_D \) is of order of \( N \) times the cluster radius \( a \): if \( \langle v \rangle \) is the average star velocity, \( \tau_C = a/\langle v \rangle \); from the virial theorem \( \langle v \rangle^2 = GNm/a \) and the standard expression of \( \tau_D^1 = G^2(N/a^3) m^2 / \langle v \rangle^3 \) we deduce that \( \tau_D \approx N \tau_C ; m \) is the star mass and \( G \) the constant of gravitation.

This paper is organized as follows:

In part 2 we establish the kinetic theory; we start from the Klimontovitch formalism (2.1). We com-
pute the fluctuations of the system (2.2) ; we obtain the kinetic equation in 2.3 and its conservation properties. Some intermediate demonstrations are given in Appendix A.

In part 3 we apply our equation to spherical gravitational mean field configuration (i.e. a globular cluster) ; taking into account the property of plane trajectories we reduce our equation to a two-dimensional equation in phase space. Finally assuming a star distribution function only depending upon the star motion energy we obtain an equation which replaces the classical form of the Fokker-Planck equation given by Henon.

In part 4 we discuss the conditions of validity of our kinetic theory. The non-linear overlapping of resonances necessary for a kinetic description of a finite number of particle (i.e. stars in a globular cluster) is discussed in Appendix B.

2. The kinetic theory.

2.1 THE FORMALISM. — We start from the Klimontovitch formalism [1].

At \( t = 0 \) we assume a given statistic of the initial position \( x_i \) in phase space of each of the \( N \) particles of our system. We introduce the Klimontovitch distribution \( f(x, t) = \sum_i \delta(x - x_i) \) and the one particle distribution function \( \langle f(x, t) \rangle \) averaged on initial positions. Let us recall that this approach is closely related to the BBGKY formalism which involves the one point \( F(x, t) \) and the two-points \( F(x, x', t) \) distributions functions. We have for example:

\[
\langle f(x) \rangle = F(x)
\]

\[
\langle f(x) f(x') \rangle = F(x, x') + \delta(x - x') F(x).
\] (1)

At time \( t = 0 \) \( F(x, x') \) is taken as \( F(x) F(x') \) which implies that we have no collective correlations ; in « plasma language » the system is supposed to be kinetically stable and close to thermodynamical equilibrium ; for a globular cluster, the violent relaxation is achieved.

Then we introduce an integrable hamiltonian for the motion in the mean field: \( H_0(I, \phi) \) where \( (I, \phi) \) are the action variable associated to this Hamiltonian. To complete the formalism, we call \( \tilde{A} \) the fluctuation of a physical variable \( A \), \( \tilde{A} \) being defined by \( \tilde{A} = A - \langle A \rangle \). The equation for \( \langle f(x) \rangle \) then reads:

\[
\partial_t \langle f(x) \rangle + [H_0 + \partial_S, \langle f \rangle] + \langle [\tilde{H}, f] \rangle = 0
\] (2)

in this equation \([ , ]\) stands for the Poisson brackets ; \( \partial_S \) comes from the fact that the conjugate variables of an individual star \( (I, \phi) \) are not in general constants of the average motion (this will be different in part 3) ; \( S \) is the genatrix function of the canonical motion. Taking now profit of the time scales separation (i.e. \( \tau_C/\tau_d = N^{-1} \) for globular clusters) we define our averaging scheme by integrating upon \( d\phi (d\phi = d^d \phi)/(2 \pi)^d \) where \( d \) is the dimension of the configuration space. This approach is coherent with the mean field kinetic description of the statistical system and corresponds in plasma language to the quasi linear approximation [2]. Then our starting equation will be:

\[
\partial_t \int d\phi \langle f(x) \rangle = - \lim_{t \to \infty} \int d\phi \langle [\tilde{H}, f] \rangle = \\
= \partial_{\tilde{f}} \int d\phi \left( \partial_{\omega} \tilde{H} \right) f = C(f).
\] (3)

To complete our task, we have to compute self consistently the fluctuations of \( \tilde{f} \) and \( \tilde{H} \) explicitly from \( H \) and \( \langle F(x) \rangle \) within some assumptions which will be given.

2.2 COMPUTATION OF THE SELF CONSISTENT FLUCTUATIONS OF THE HAMILTONIAN AND THE DISTRIBUTION FUNCTIONS. — It is convenient to define the local potential of interaction by:

\[
V(x) = \int \rho(x, x') f(x') dx'
\] (4)

the kernel \( \rho(x, x') \) is symmetrical ; \( \rho(x, x') = \rho(x', x) \). In the case of gravitational interaction of particles of mass \( m \rho = Gm [r(x) - r(x')]^{-1} \) and for particle of charge \( q \), \( \rho = \frac{q^2}{4 \pi \epsilon_0} [r(x) - r(x')]^{-4} \).

We first compute the fluctuations in the linearized approximation, i.e. we neglect the effect of the fluctuating potential on the particle trajectories. Their effect will be taken into account in appendix B to justify the continuous limit of the kinetic equation. Within this approximation, and with \( H = \tilde{V} \), the equation for \( \tilde{f} \) reads:

\[
\partial_t \tilde{f} + \Omega_i \partial_{\phi_i} \tilde{f} - \partial_{\phi_i} \tilde{V} \partial_{\phi_i} \langle f \rangle = 0
\] (5)

with \( \Omega_i = \partial_{\phi_i} H_0 \), the frequencies of the average motion and following (4),

\[
\tilde{V}(x, t) = \int \rho(x, x') \tilde{f}(x') dx'.
\]

Let us call \( G(x, x', t) \) the solution of equation (5) with the given initial condition : \( G(x, x', 0) = \delta(x - x') \); \( V(x, x', t) \) is the collective response to the field of the particle \( x' \) seen by a particle \( x \) at time \( t \). From (5) we obtain:
This Green function formalism allows to compute formally $f(x, t)$ and $V(x, t)$ from initial conditions:

$$G(x, x', t) = \sum \delta(x - x' - \Omega t)$$

$$V(x, x', t) = \int \rho(x, x') G(x', x', t) \, dx'$$

(6)

To simplify the notations, we have set $(x + d^2 \Omega t) = (x_0 + \Omega t)$; our formalism allows to separate in a natural way the statistical averaging upon initial conditions; then the second member of (3) reads:

$$C(f) = \lim_{t \to \infty} \partial_t \int d\Phi \, \partial_q \rho \, G(x, t) \, f(x') \, \delta(x - x' - \Omega t) \, f(x') \, dx' \, dx$$

(7)

and after averaging:

$$C(f) = \lim_{t \to \infty} \partial_t \int d\Phi \, \partial_q \rho \, G(x, t) \, f(x') \, \delta(x - x' - \Omega t) \, f(x') \, dx' \, dx$$

(8)

substituting (6) in (7) we obtain the first expression of the collisional operator $C(f)$:

$$C(f) = \lim_{t \to \infty} \partial_t \int d\Phi \, \partial_q \rho \, G(x, t) \, f(x') \, \delta(x - x' - \Omega t) \, f(x') \, dx' \, dx$$

(9)

In $C(f)$ the first term gives the diffusion contribution and the second one, the friction term. After averaging, we will neglect in the following the difference between $\left< f_0(x') \right>$ and $\left< f(x) \right>$. This standard approximation for kinetic equation is justified in our case by the separation of time scales. The diffusion term is formally identical with the quasilinear result known in plasma physics [2]. The friction term will now be written in a more symmetrical form. Integrating by part on $\partial_q$ and permuting $x$ and $x'$, we obtain for the friction term:

$$- \partial_t \int d\Phi \, dx' \, \partial_q \rho \, G(x', x, t) \, \delta(x' - x - \Omega t) \, f(x')$$

(10)

we substitute in (10) the $\delta$ function from (6), with $x \leftrightarrow x'$. The contribution from $G(x', x, t)$ vanishes, as follows

$$\partial_t \int d\Phi \, dx' \, \partial_q \rho \, G(x', x, t) \, f(x') = 0$$

(11)

where we have used the definition of $V$ and integrated by part. The second member of (11) is antisymmetrical in $(x', x')$ and vanishes by integrating upon $d\Phi_q$. This allows to obtain the final form of the collisional operator $C(f)$:

$$C(f) = \partial_t \int d\Phi \, \left\{ \bar{D}_i(l, I') \left< f(l') \right> \partial_t \left< f(l) \right> - \bar{A}_i(l, I') \partial_t \left< f(l') \right> \left< f(l) \right> \right\}$$

(12)

with the following respective definition of microscopic diffusion, $\bar{D}_{ij}$, and friction, $\bar{A}_{ij}$, coefficients:

$$\bar{D}_{ij} = \lim_{t \to \infty} \int d\Phi \, d\Phi' \, \partial_q \rho \, G(x, x', t) \times$$

$$\times \int_0^\infty \partial_q \rho \, G(x - \Omega t, x', t) \, dt'$$

$$\bar{A}_{ij} = \lim_{t \to \infty} \int d\Phi \, d\Phi' \, \partial_q \rho \, G(x, x', t) \times$$

$$\times \int_0^\infty \partial_q \rho \, G(x, x', t) \, dt'$$

(13)

The final step of our computation deals with the formal properties of $\bar{D}_{ij}$ and $\bar{A}_{ij}$, taking first, the limit $t \to \infty$ based on the kinetic stability of the mean behaviour and, subsequently, profit of the periodicities of the motion. Taking into account the assumed damped contribution of the collective modes, when $t/\tau_0 \to \infty$ we keep only the balistic contribution, i.e.:

$$\bar{G}(x, x', t) \to \bar{G}(x, x' + \Omega t)$$

and (6) now reads:

$$\bar{G}(x, x', t) = \int_0^\infty d\tau \, \partial_q \rho \, G(x - \Omega t, x' - \Omega t) \times$$

(14)

with the corresponding definition of $\bar{V}$. At this stage of our computation it is interesting to deduce from (12) the Balescu-Lenard equation for a plasma invariant by translation [3]. In that case the appropriate set of canonical variables is $[v, x - vt/L]$ where $L$ is the length of the system $\rho = (e^2/m4\pi\epsilon_0(r - r'))$ if we limit ourselves for sim-
plicity to electron electron collisions (ions are taken with an infinite mass). Then after Fourier transform upon \( x \) one gets for \( \tilde{V} \):
\[
\tilde{V} = \left( e^2/m \right) \int d^3k \exp ik(x - x' - vt)/k^2 \varepsilon(k, k \cdot v) 
\]
(15)
and after averaging upon phase variables \( (\Phi \to X/L) \), we recover the classical result for the friction and diffusion coefficients:
\[
\tilde{D}_{ij} = \tilde{F}_{ij} = \pi \frac{e^4(4\pi e_0^2 m^2)}{\varepsilon^2(k \cdot v - k \cdot v')}/|\varepsilon|^2 k^4 
\]
(16)
where \( \varepsilon(k, -k \cdot v) \) is the dielectric response to the « balistic frequency » \( k \cdot v \). Similar results, within geometrical approximations, could be obtained for dilute mirror type plasma configurations.

Let us now come back to the general case when the Green function and the associated potential cannot be explicitly computed. If in a first step we neglect the collective effect in (14), i.e. \( G = G(x-x') \) we easily verify the properties given in the following paragraph. In appendix A we give the needed demonstrations for the corresponding properties of the complete solution.

2.3 SYMMETRY CONSERVATION PROPERTIES AND DIVERGENCES OF \( C(f) \). — Due to the periodicities of the test and field particle mean motions we may expand \( \tilde{V}(x, x') \) in action and angle variables:
\[
\tilde{V}(x, x') = \sum_{m,n} e^{im\Phi + in\Phi} V_{m,n}(l, l') . 
\]
(17)
Keeping the resonances with the balistic motion as explained before, we get in a first step:
\[
\tilde{D}_{ij}(l, l') = \pi \sum_{m,n} m_j |V_{m,n}(l, l')|^2 \delta(m\Omega + n\Omega') 
\]
\[\tilde{A}_{ij}(l, l') = -\pi \sum_{m,n} n_j |V_{m,n}(l', l)|^2 \delta(n\Omega + m\Omega') \]
(18)
we note the asymmetry between the two coefficients. \( C(f) \) conserves the energy and also the momentum
\[
\int f l_i dl_i \text{ if } m_i |V_{m,n}|^2 = -n_i |V_{m,n}|^2, \text{ i.e. the matrix elements of } V \text{ are diagonal, which is satisfied if } \rho(x, x') \text{ is invariant by translation upon } \phi_i \text{ (the case of the Balescu-Lenard equation). An important property of the general collective case holds in general (Appendix A), the symmetry of matrix elements at resonance:}
\[
|V_{m,n}(l, l')|^2 \delta(m\Omega + n\Omega') = |V_{m,n}(l', l)|^2 \delta(m\Omega + n\Omega') 
\]
which allows to get our final expression for the friction coefficient:
\[
\tilde{A}_{ij}(l, l') = -\pi \sum_{m,n} m_j |V_{m,n}(l, l')|^2 \times \delta(m\Omega + n\Omega') . 
\]
(19)
We finally from this expression establish easily the standard properties of a kinetic equation.

A Maxwellian distribution function in energy: \( f = \exp -\beta E(l) \) is solution of the kinetic equation. A bi-Maxwellian distribution function in energy and action: \( f \sim \exp -[\lambda I_1 + \beta E(l)] \) is also solution if \( C(f) \) conserves \( I_1 \) (as follows from the invariance of \( \rho \) with respect with \( \phi_i \)). The kinetic equation has an H-theorem. By successive integration by part with respect to \( l \) and \( l' \) and adding the symmetrized contribution (\( f \leftrightarrow f' \)) one obtains:
\[
\int \ln f C(f) df = -\frac{\pi}{2} \sum_{m,n} \int (m_i \partial_{l_i} \ln f + n_i \partial_{l_i} \ln f')^2 \times f f'|V_{m,n}|^2 \delta(m\Omega + n\Omega') = 0 . 
\]
One expects from the begining these properties to hold in our case ; but we stress again that they have been obtained with less assumptions than usual : they result only from the existence of an integrable Hamiltonian of the averaged motion.

In the standard Fokker-Planck approach it is necessary to introduce a cut off at long distance which depending upon the authors is taken at the minimum size of the system or at the Jeans length. In the theory given in the paper, a natural cut off is introduced : in the discrete summation in \( m \), the \( m = 0 \) term vanishes ! the first term (\( m = 1, n = 1 \)) corresponding roughly to the size of the system which is also of order of the Jeans length. Simultaneously for these low \( m \) numbers, the Fourier components computed self consistently from (14) are different from the matrix components \( \rho_{mn} \) of the free interaction. In the limit \( m, n \to \infty \) one expects the usual Coulombian divergence which can be removed by the standard Landau cut off. Nevertheless this divergence in the action phase space cannot be expressed by a multiplicative factor of the \( \ln A \) type. For example in the gravitational case we expect convergence of the summation over \( m, n \) when \( l, l' \) correspond to trajectories in non intersecting volumes of ordinary space, since in that case the potential \( V(x, x') \) has obviously no more singularity in the corresponding range of \( (x, x') \).

In the same way, two separated parts of the phase space experience a small collisional effect between themselves : this effect is not described by the Fokker-Planck theory. As a consequence, we also expect from a solution of our equation (apart from numbers) some effect of heterogeneity in phase
space; this investigation is left for further work, in the case of globular clusters.

3. The gravitational case.

Our general equation includes the case of gravitational systems with a mean spherical symmetry, the best astrophysical example being globular clusters. It is convenient to choose an action integral for the motion in the mean field compatible with our formalism, which explicitly displays the bidimensional character of the star trajectories [4] and of the kinetic equation:

\[
S = \int_{r_1} V, \, d\tau + J, \, \Phi + J\psi - E(I, J) \, t
\]

\(r_1\) is the periastre radius, \(E\) the total energy, \(J,\) the orbital momentum and \(J_z\) its projection on \(e_z\) axis: \(J_z = J \cdot e_z = E \cos \mu.\) Our canonical set of variable \((\alpha, \beta)\) are respectively defined by: \(\alpha_r = \oint V, \, d\tau,\)[4] \(\alpha_{\phi} = J, \alpha_{\psi} = J, \) and \(\beta = -\partial_{\phi} S;\) \(I, J, J_z\) and \(J,\) are the three « adiabatic invariants » of the motion; \(\Phi, \psi, \mu\) are the Euler angles of a star trajectory in the mean central gravitational field. The two frequencies of a plasma star motion are \(f_2 = \frac{a}{J E} \) for the radial motion and \(f_z = \frac{a J_z}{\Omega E} \) for the azimutal motion.

From the invariance of the system by rotation, it is clear that \(\langle f \rangle\) will only be dependent upon \((I, J, t).\) \(C(f)\) is, in fact, the divergence of a current: \(a_1 + b_1 + c_1;\) in the two dimensional gravitational case \(a\) and \(b\) depends only upon \((I, J)\) and \(c = (J_z / J) b\) as follows from its definition and \(\partial_{J_z} \langle f \rangle = 0.\) Consequently \(C(f)\) may be written:

\[
C(f) = \partial_f (a) + J^{-1} \partial_J (Jb).
\]

A step further is achieved if \(\langle f \rangle\) is assumed to be only function of \(E,\) as assumed generally [5].

To compare our equation with the standard result [6] based on the Fokker-Planck approach it is more convenient to replace \((I, J, t)\) by \((E, J)\); then \(\partial_I \rightarrow \Omega, \partial_{\Phi}, \partial_J \rightarrow \Omega, \partial \Omega, \partial_J,\) and after integrating (20) upon \(\Omega, J \, dJ,\) we first obtain:

\[
C(f) = \sum_{m, n} \Omega_n^{-1} J \, dJ \left( m \Omega \, \partial_{\Omega} + m J^{-1} \partial_J \right) m \Omega, \bar{a}_{m, n}
\]

with

\[
\bar{a}_{m, n} = \left\{ \delta E' \left( \langle f' \rangle - \langle f \rangle \right) \partial_{\Omega} \langle f \rangle \right\} \times \left\{ \int J, \, dJ' \, \Omega_n^{-1} \, d \cos \mu \left| V_{mn} \right|^2 \delta (m \Omega + n \Omega') \right\}
\]

To proceed further we explicit in (21), \(m \Omega = m_I \Omega_I, m_J \Omega_J,\) \(m \Omega,\) contributes as

\[
\partial_{E} \sum_{m, n} m_I \int J, dJ (m \Omega) \bar{a}_{m, n}
\]

The \(m_I \Omega_I\) contribution reads also:

\[
\sum_{m, n} m_I \left\{ \partial_{E} \int J, dJ (m \Omega) \Omega_n^{-1} \bar{a}_{m, n} + \right. \left\{ \int J, dJ (m \Omega) \Omega_n^{-1} \bar{a}_{m, n} \right\}
\]

where we have used the identities: \(\Omega_n^{-1} = - \partial_f, \Omega_n^{-1} = \partial_f.\) (21) now reads by adding all the contributions:

\[
C(f) = \sum_{m, n} \left\{ \partial_{E} \int J, dJ (m \Omega)^2 \Omega_n^{-1} \bar{a}_{m, n} + m_I \times \left\{ \int dJ (\partial f (m \Omega) \bar{a}_{m, n}) \right\}.\]

The last term vanishes by integration upon \(J\) and we get our final equation:

\[
\partial_f (f, E, t) = \partial_{E} \int J, dJ \Omega_n^{-1} \sum_{m, n} (m \Omega)^2 \bar{a}_{m, n}.
\]

Finally, our theory can be immediately generalized to bodies of different masses, by summation over the mass spectrum.

All along this paper, we have neglected the effect of direct inelastic collisions. They have a characteristic time scale \(\tau_{de}\) given by \(\tau_{de} = \tau_D (N / A)^{-2}\) (in \(\tau_D,\) the cross section \(\sigma_{mg} \approx m \sigma_{mg} / \Delta) - r_0\) is an average star radius. In order of magnitude, \(\tau_{de} \approx (10^{-7} N / A)^2 \tau_D\) for a dimension of a light year. We conclude that our approximation holds for most of the globular clusters at least outside the very central density peak which may be in the last stage of the gravothermal catastrophe [6]; this remark does not hold for the phase of accretion of planetesimals and, for the physics of planetary rings which both involve inelastic collisions.

4. Domain of existence of a kinetic theory.

In our demonstration of part 3, we have implicitly assumed that the microfields \(\tilde{V},\) the diffusion and friction coefficients would be computed in the continuous limit. This approximation has to be justified, especially for astrophysical situations (i.e. the globular clusters) when \(N\) may be a relatively small number, in contrary with familiar gas or plasma conditions. In this chapter, we successively justify the continuous limit and the \(\delta (m \Omega + n \Omega')\) approximation.

4.1 Renormalization of the linearized propagator. — Equations for computing the fluctuations \(\delta f\) and \(\tilde{V}\) are basically non-linear self-consis-
tent equations. We will not study explicitly all the non-linear contributions neglected in (5) but limit our analysis to the relevant ones in our case: our kinetic theory is only valid if the non-linear terms do not modify the result; simultaneously, some of the non-linear terms, those which are involved in the renormalization of the propagators, allow to take the continuous limit and the δ function approximation. In Appendix B, we give a brief account of the concept of renormalization and also of the kind of approximations we have made. Let us now summarize the main findings. The response of \((f, \bar{V})\) to a given source has been taken previously as given by the linearized equations:

\[
\partial_t \vec{f} + [H_0, \vec{f}] = \vec{S},
\]

or

\[
i(\omega - n\Omega)f_n(l, \omega) = \vec{S}_n(\omega)
\]

(23)

after Fourier transform upon time and angles. This equation is renormalized and takes in the vicinity of the resonances \((\omega \sim n\Omega)\) the following form:

\[
\partial_t \vec{f} + [H_0, \vec{f}] + \partial_i D_{ij} \partial_j \vec{f} = \vec{S}
\]

(24)

\(D_{ij}\) being the diffusion coefficient

\[
(D_{ij}) = \int dl' \bar{D}_{ij}(l', l') \langle f(l') \rangle.
\]

In Fourier space, we obtain for the Green function

\[
G_n^*(\omega) = [i(\omega - n\Omega) - \gamma_n(\Omega)]^{-1}
\]

with,

\[
\gamma_n(\Omega) = \left(n_k \partial_k \Omega_k \partial_i D_{ij} \partial_j \Omega_i\right)^{1/3}.
\]

An order of magnitude of \(\gamma\) is sufficient for the following:

\[
\gamma_n(\Omega) = \Omega (n^2 \Omega_d/\Omega_c)^{1/3}.
\]

Then, in the vicinity of resonances, one obtains a renormalized expression for the diffusion (and friction) coefficient as given by:

\[
D_{ij} = \int \bar{D}_{ij} dl' f(l') = \sum_{m,n} m_n \int dl' \left| V_{m,n} \right|^2 \langle f' \rangle \times
\]

\[
\times \left[+i(m\Omega + n\Omega) - \gamma_m - \gamma_n\right]^{-1}.
\]

(26)

(It is assumed that correlation functions are renormalized in the same way than propagators).

4.2 OVERLAPPING OF RESONANCES. — Existence of a kinetic theory and intrinsic stochasticity are closely related: a particle moves in a fluctuating field with a « continuous » and « broad » spectrum. Leaving for a while the requirements on the spectrum width (see 4.3), we may state that, when integrating (26) upon \(dl'\), continuity follows if the resonances overlap after renormalization. In fact, as already said, we replaced implicitly, in part 2, the discreet summation upon the \(N\) actions by an integration upon \(dl'\):

\[
\int dl' \sum_I \delta(l' - l_i) K(l_i) \rightarrow \int dl' f(l') K(l').
\]

This is correct if the corresponding kernel \(K(l')\) is slowly varying, such that exact unknown positions in phase space are not needed; then, for a random set of the initial \(N\) positions, the result is correctly approximated by the integral upon \(dl'\), provided the relative fluctuations of the result remain negligible. To check this property, we compute the order of magnitude of the quadratic fluctuation of the diffusion coefficient:

\[
\sigma_{mn}^2 = \langle \Delta D_{mn}^2 \rangle / \langle D_{mn}^2 \rangle = n \Omega_c/N (\gamma_m + \gamma_n).
\]

Substituting expressions (25) for \(\gamma_m\) and \(\gamma_n\), with \(m = n\), we obtain:

\[
N \gg (n \Omega_c/\Omega_d)^{1/3}.
\]

(27)

For a self gravitating system of identical masses (an example of unequal masses is given in 4.4), \(\Omega_c = N \Omega_d\) and \(\sigma_m = (m/N^2)^{1/3}\). We should not conclude from this condition that for \(m > N^2\) \(\sigma_m\) becomes large and the kinetic theory breaks! For \(m\) large, another kind of overlapping of resonances with different \(m, n\) happens when \(\Delta m \Omega_c \ll \gamma_m\); for \(\Delta m = 1\) this condition reads:

\[
m^2 \Omega_d/\Omega_c = 1.
\]

The « fluctuation » \(\sigma\) of the diffusion coefficient due to this effect is now \(\Omega_c/\gamma_m \sim (N/m^3)^{1/3}\). The largest fluctuation is the maximum value for all \(m\) of \((N/m^2)^{1/3}\) and \((m/N^2)^{1/3}\); it is achieved for \(m = N\) and its value is \(N^{-1/3}\).

These results deserve a few comments:

— The increase of the fluctuation level \(\sigma\) is closely related to the possible existence of « islands » which break the diffusive character of the phase space motion. The KAM limit is expected for \(\sigma \gg 1\) as discussed in (4.4) for a simple example of a non self-consistent given mean field (i.e. stars orbiting around a central body).

— Our criterion of resonance overlapping (27) has been obtained by a diffusive approach (comparison between \(\gamma_m\) and \(\Omega_c/N\)). For one dimensional systems (for example the motion of a particle in a set of monochromatic waves) we may use the more precise criterion of Chirikov: this criterion is based upon a comparison between the trapping frequency \(\Omega_c\) in one wave and \(\Omega_c/N\). We are not able to use it in our general case but we observe that in the one dimensional case, the two criteria for transition to stochasticity are identical, i.e. \(\gamma = \Omega_c\).
These two remarks make us confident about our diffusive statistical approach as being also relevant for a dynamical one.

4.3 TRAPPING IN PHASE SPACE « ISLANDS ». — The previous discussion about the existence of a statistical limit is not sufficient to justify the kinetic equation. Non-linear effects are still possible in some domains of the phase space, effects which would invalidate the last step in the demonstration, i.e. when we replaced the renormalized propagator by its resonant contribution :

\[ [i (m\Omega + n\Omega') + \gamma_m + \gamma_n]^{-1} \rightarrow i M \delta (m\Omega + \Omega' n') . \]

This approximation is justified if the kernel in (26) is smooth. \( \Delta \Omega_n \), the width of \( \langle f' \rangle \) in phase space \( \left[ \Delta \Omega' = \left( \theta_{\Omega'} \log \langle f' \rangle \right)^{-1} \right] \), may be taken as the width of the fluctuation spectrum. We must verify \( n \Delta \Omega_n \gg \gamma_n + \gamma_m \) to justify simultaneously our quasilinear approach of the theory and the smallness of all the neglected non-linear contributions. This implies also implicitly that we have no degeneracy of the dynamics in the main field, i.e. the condition \( n\Omega = 0 \) cannot be satisfied for all \( l \). (The case of planar trajectories of part 3 is not degenerate in that sense.) The previous condition for a self gravitating system reads \( n \Delta \Omega / \Omega_c \gg (m^2 \Omega_c/\Omega_D)^{1/3} \) and reduces, for \( m = n = 1 \), to :

\[ \Delta \Omega / \Omega_c \gg N^{-1/3} . \] (28)

This inequality limits the initial packing of the phase space trajectories and for \( \Delta \Omega \sim \Omega_c \) is again equivalent to \( N \gg 1 \).

Let us summarize all our explicit findings for the gravitational case : \( N \) is the only free parameter and the condition \( N \gg 1 \) insures both the conditions for stochasticity and the absence of trapping effects. Consequently, our kinetic equation is valid for initial conditions sufficiently smooth in phase space. To conclude this subchapter, let us give two possible examples where trapping limitations are expected.

When \( \Delta \Omega / \Omega = 1 \), but when \( N \) decreases, we first expect strong two body interaction (binary formation in globular clusters) and also three body collisions with a still possible but rather imprecise statistical description. If \( N \) decreases further, trapping prevents a mean description of the motion by the adiabatic theory. More generally, a statistical description breaks since the fluctuation \( \sigma \) of the diffusion (friction) coefficients becomes of order one.

In the opposite case of \( \Delta \Omega / \Omega \ll 1 \) and \( N \) large, if an important part of the particles is located in a spherical shell of width \( \Delta r \), the condition (28) reads :

\[ \Delta r / r \gg N^{-1/4} \] (29)

(to get (29), we have taken into account the fact that, in this case, \( \Omega_n / \Omega_c = r / \Delta r N \)). When (29) breaks, we expect a collective trapping and a negligible, anomalous, diffusion.

4.4 A HALO OF SMALL BODIES IN THE VICINITY OF A MASSIVE ONE. — For a self gravitating system of bodies with identical masses, the only relevant scaling parameter is the number \( N \) of bodies. The condition \( N \gg 1 \) implies simultaneously the stochastic limit and the absence of trapping, at least for a uniform occupation of phase space. This condition is modified if for example \( N \) bodies of small masses \( m \) are in orbit around a central one of much larger mass \( M \). Let us first assume \( \Delta r / r \sim 1 \), i.e. a, more or less, uniform occupation of phase space. Then, using, for \( \Omega_d \sim G^2 m^2 N / v^3 r^3 \) and for \( \Omega_c \sim r (GM/r) \), we obtain : \( \Omega_c / \Omega_d = M^2 / N m^2 \), instead of \( N^{-1} \) for \( m = M \). The conditions for stochastic behaviour (27) reads \( N^2 m/M \gg 1 \) and keeps simultaneously the fluctuations small, since \( \sigma \sim (M/mN^2)^{1/3} \). The only free parameter is now, instead of \( N \), \( N (m/M)^{1/2} \). Forgetting the fact that the solar system is coplanar and deserves a more detailed analysis, we nevertheless verify that its behaviour is expected to be non stochastic from our analysis : with \( M/m \sim 10^4 \), \( N \sim 10 \), \( \sigma \sim 10^{23} \).

We still have to discuss the trapping condition to complete the previous analysis. Let us recall that, in 4.3, we had excluded the case of degeneracy \( m\Omega \sim 0 \), which is obviously verified for Keplerian orbits if we take \( m \) as \( (l, -l) \). This degeneracy is removed by the presence of the \( N \) bodies which perturb the orbit of one of them ; in order of magnitude, \( \Delta \Omega \sim N m / M \). We then may obtain a condition for the absence of trapping (4.3) i.e. a diffusion behaviour, for \( l = 1 \) :

\[ \Delta \Omega / \Omega \gg (\Omega_c / \Omega_D)^{1/3} \]

or again \( N^2 m / M \gg 1 \) which implies also \( \sigma \ll 1 \). We may conclude that is this case the only relevant parameter is \( \sigma = (M/mN^2)^{1/3} \) and \( \sigma \ll 1 \) insures both stochasticity and absence of trapping. Obviously this analysis does not preclude local island formation like for the asteroid belts and deserves further work for the case of planetary rings.

5. Conclusion.

We have obtained a new kinetic equation for Hamiltonian systems, integrable in the mean field approximation. For Coulomb interactions of a periodic system of charged particles, we recover the Balescu-Lenard equation when the length goes to infinity. For a self gravitating system with an average spherical symmetry in the mean field (for example globular clusters), we give explicitly the two dimensional kinetic equation in phase space which replaces the Fokker-Planck approach ; we also find the equation for the one dimensional approximation in energy
space of Henon. Finally, we establish the conditions for the validity of a kinetic theory, respectively for a stochastic behaviour and the absence of island formation in phase space. For a smooth phase space occupation there two conditions are fulfilled in the limit of a « large » number of bodies: they imply also small fluctuations of both the diffusion and friction coefficient in the vicinity of their mean values given in the kinetic equation. One heuristic limitation is obtained for the average stability of a number of bodies orbiting around a central object. One of us (R. P.) thanks M. Duler for fruitful discussions.

Appendix A.

We demonstrate here the symmetry property of collective matrix components (17). \( V_{mn}(I, I') \), for a resonant case:

\[
V_{mn}(I, I') = V_{nm}(I', I) \quad \text{if} \quad m\Omega + n\Omega' = 0 \quad (A.1)
\]

This property, obvious for the non-collective case \( (V_{mn}(I, I') = \rho_{mn}(I, I') \) ensures the relaxation towards a Maxwellian distribution, as well as the \( H \)-theorem. It must be pointed out that the former property, in the case of a collective self-consistent problem, is not obvious.

We start from the « dielectric equation » (14), that reads, after Fourier transform:

\[
G_{mn}(I, I') = \frac{V_{mn}(I, I')}{m\Omega + n\Omega'} = \delta_{m-n} \delta(I, I') \quad (A.2)
\]

\[V_{mn}(I, I') = \int dI'' \rho_{ml}(I, I'') G_{ln}(I'', I')\]

(where \( \rho_{ml} \) are matrix components of the Kernel \( \rho(x, x') \), and \( dI = d^dI (2\pi)^d \)). Then one obtains:

\[
V_{mn}(I, I') + i \int dI'' \rho_{ml}(I, I'') \times 
\times V_{ln}(I'', I) \frac{l_i \partial_{l_i} \langle f(I'') \rangle}{n\Omega' - l_i \Omega} = \rho_{mn}(I, I') \quad (A.3)
\]

(summation over \( l \) is implicit).

We will show the property (A.1) by expanding (A.3) in powers of \( \langle f(I'') \rangle \), that is, in powers of the collective effect, starting from:

\[
V_{mn}^{(0)}(I, I') = \rho_{mn}(I, I')
\]

we obtain, at the \( k \)-th order:

\[
V_{mn}^{(k)}(I, I') = (-i)^k \times 
\times \int dI, \ldots, dI_k \rho_{ml}(I, I_1) \frac{l_i \partial_{l_i} \langle f(I_1) \rangle}{n\Omega' - l_i \Omega} 
\times \rho_{-l_1 l_2}(I, I_2) \times \cdots \times \rho_{-l_k l_k}(I_k, I') \frac{l_k \partial_{l_k} \langle f(I_k) \rangle}{n\Omega' - l_k \Omega_k}
\]

then, if we exchange \( (l_1, \ldots, l_k), (l_1, \ldots, l_k) \) with \( (l_1, \ldots, l_k), (-l_k, \ldots, -l_1) \), using the symmetry of \( \rho_{mn} \), and the resonance condition \( m\Omega = n\Omega' \), we easily obtain the property (A.1).

Appendix B.

The aim of this appendix is to explain shortly the use of renormalized propagators (24); we will avoid explicit notations.

First, this concept is clear in the case of motion in an external stochastic field, when this field is quasi-Gaussian, that is, when the \( N \) point correlation function is reducible to two point correlations functions, plus « little » cumulants. In our case, this corresponds to the absence of non-equilibrium correlations. But, in the self-consistent field case, this concept is very unclear, because the statistics of fields is given by that of distribution functions, and the quasi-Gaussian hypothesis is generally ruled out [7]. In this case, one can introduce a DIA scheme [7], which is beyond the scope of this work.

In the DIA scheme, a different renormalization is obtained: for example, in (26), we would have, instead of the sum \( \gamma_m + \gamma_n \), a cross term. But our « naive » propagator renormalization is sufficient, in order of magnitude, to establish the conditions for the validity of our kinetic equation: in fact, when renormalization modifies the result, we can be sure that we omit contributions with the same order of magnitude, and that in fact we do not describe correctly the physics of trapping phenomena, adiabaticity breaking, and \( n \)-body processes. In that case, DIA or more refined schemes could only be checked numerically.

We must now point out technical differences between our case and the plasma case, arising from the different interaction kernel. Consider a stochastic equation (not self-consistant):

\[H_0 f + \tilde{H} f = S \quad (B.1)\]

where \( \tilde{H} \) is quasi-Gaussian.

The renormalized equation is then:

\[H_0 f - \langle \tilde{H} G_0 \tilde{H} \rangle f = S \quad (B.2)\]

where \( G_0 \) is the free propagator \( H_0^{-1} \).

In our case (B.2) leads to double Poisson brackets, and then, \( \partial_{l_i} \partial_{l_j} \), \( \partial_{l_i} \partial_{\phi} \), \( \partial_{\phi} \partial_{\phi} \) terms in the renormalized propagator, which will be (symbolically) written as:

\[[i(\omega + i\Omega \partial_{\phi}) + \partial_{l_i} D^1 \partial_{l_i} + \partial_{\phi} (D^2/1) \partial_{l_i} + \partial_{\phi} (D^2/1) \partial_{\phi}] G^*_n = 1 \quad (B.3)\]

For instance, the \( \partial_{l_i} D^1 \partial_{l_i} \) term is explicitly given by (B.4). We shall not write explicitly the two other contributions.
We set, near the resonance:

\[ G' = i(\omega - m\Omega + \gamma)^{-1} \]

with \( \gamma/\Omega \ll 1 \) one can then check that, in (B.3), the \( \partial_{I} \partial_{L} \) term does give the dominant contribution.

Within this approximation, one can write explicitly the equation (B.2), for a source \( S_m(\omega, I) \) in Fourier transform:

\[ i(\omega - m\Omega) f_m(\omega, I) + \partial_I \int dI' \frac{V_{g'}(I, I') V_{g''}(I, I') \langle f(I') \rangle}{i(\omega - m\Omega) + i(p\Omega' + n\Omega)} n_I I' \]

\( \times \partial_{I} \partial_{m-(n+1)} S_m(\omega, I) \). \hspace{1cm} (B.4)

(Summation over \( n, I, p \) is implicit). At a resonance \( \omega = m\Omega \), the "diagonal" part of (B.4), obtained with \( n + l = 0 \), is exactly given by the diffusion coefficient \( D_{ij} \) (18). But there are off diagonal contributions, neglected in the form (26), which couple simultaneous resonances, that is, \( f_m(\omega, I) \) and \( f_{m-(n+1)} \) when \( (n + l) \cdot \Omega = 0 \). Of course, at the quasi-linear level, these off-diagonal renormalizations disappear. But they are reminiscent of the fact that non-linear effects would particularly concern resonances in the mean field, i.e., actions \( I \) such that \( p \cdot \Omega = 0 \) for some \( p \) (here \( p = n + l \)). This effect is a consequence of the non invariance of the interaction with respect to a \( \phi \) translation. We have assumed that it does not modify the order of magnitude given by the diagonal part.

References