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On the nonlinear thermal diffusive theory of curved flames

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Abstract. — A strongly nonlinear, spatially invariant equation for the dynamics of a cellular flame is derived, on the assumption that the curvature of the flame front is small. The equation generalizes the corresponding weakly nonlinear equation, obtained previously near the stability threshold.

1. Introduction.

The weakly nonlinear equation

\[ f_t + \frac{1}{2} f_x^2 + (\alpha - 1) f_{xx} + 4 f_{xxxx} = 0 \quad (1.1) \]

describing diffusion-thermal instability of a plane flame front and the formation of cellular structure was derived on the assumption that the system is near the stability threshold [1], i.e.,

\[ \alpha = \frac{1}{2} N (1 - \sigma)(1 - Le) = 1. \quad (1.2) \]

Here \( N = E/R^0 T_b \) is the nondimensional activation energy; \( \sigma = T_0/T_b \) is the ratio of the temperature of the fresh mixture to that of the combustion products; \( Le \) is the Lewis number. The function \( y = f(x,t) \) is a perturbation of the plane flame front \( (y = -t) \), where the latter is propagating at unit velocity in the direction of the negative \( y \)-axis (Fig. 1). Thus, equation (1.1) is associated with a highly specialized coordinate system.

In this paper, on the assumption that the curvature of the flame front is small, but with no restrictions on the parameter \( \alpha \), we shall derive a dynamic equation for the flame front, in a form invariant to the choice of coordinate system. This equation is capable of describing complex geometrical flame configurations, such as those frequently observed in strongly turbulent gas flow.

2. Fundamental equations.

Our point of departure is the «constant density» flame model, which neglects thermal expansion of the gas. In this model, a flame propagating relative to a motionless gas is described by a system consisting of the heat equation for a gaseous mixture and the diffusion equation for the limiting reactant, which is completely consumed in the course of the reaction. In terms of suitably chosen nondimensional magnitudes, these equations may be written as follows:

\[ T_t = \nabla^2 T + (1 - \sigma) Q \delta_t \quad (2.1) \]

\[ C_t = \frac{1}{Le} \nabla^2 C - Q \delta_t \quad (2.2) \]
Here $T$ is the nondimensional temperature, in units of $T_b$; $C$ is the nondimensional concentration of the limiting reactant, in units of its initial value $C_0$; $f$ and $x$ are expressed in units of the thermal thickness of the flame $l_{th} = D_{th}/v_b$ where $D_{th}$ is the thermal diffusivity of the gaseous mixture and $V_b$ the propagation velocity of the plane flame; $t$ is nondimensional time, in units of $l_{th}/v_b$.

In the context of the model (2.1), (2.2), the reaction zone is assumed to be infinitely narrow and concentrated on the surface $y = f(x,t)$ of the flame front. This corresponds to strong temperature-dependence of the reaction rate ($N = E/RT_b \gg 1$). To simplify the mathematical presentation, we shall confine attention through this paper to the two-dimensional version of the problem.

Far ahead of the front $(y \to -\infty$, see Fig. 1), the temperature is equal to the that of the fresh mixture ($\sigma$). Far behind the front $(y \to +\infty)$, the temperature is equal to the adiabatic temperature of the combustion products, i.e., unity. The concentration is unity far ahead of the front and zero just behind the front $(y > f(x,t))$, corresponding to complete consumption for the reactant in the reaction zone:

$$T(x,-\infty,t) = \sigma; \quad T(x,\infty,t) = 1$$
$$C(x,-\infty,t) = 1; \quad C(x,y,t) = 0$$

for $y > f(x,t)$. (2.4)

For the sequel, it is convenient to consider a transformed version of equations (2.1), (2.2), involving the enthalpy, $H$ instead of the concentration $C$ (cf. [2, 3]),

$$H = T + (1 - \sigma) C.$$ (2.5)

If the concentration and temperature fields are similar (i.e., $Le = 1$), $H$ is identically equal to unity. Below we shall consider the case in which $Le$ is near unity, $N$ is large, but $N (1 - Le)$ is finite. Then $T$ and $H$ can be expanded in powers of $1/N$:

$$T = T^{(0)} + (1/N) T^{(1)} + \cdots \quad (T^{(0)} \equiv 1)$$

for $y > f(x,t)$, (2.6)

$$H = H^{(0)} + (1/N) H^{(1)} + \cdots \quad (H^{(0)} \equiv 1).$$

Denoting

$$T^{(0)} = \sigma + (1 - \sigma) \theta$$
$$H^{(1)} = (1 - \sigma) S$$

we can write the principal (nontrivial) term of the asymptotic expansions in powers of $1/N$ of equations (2.1), (2.2) and the boundary conditions (2.3), (2.4), as follows:

$$\theta_\epsilon = \theta^2 \theta + \exp\left(\frac{1}{2} S\right) \delta_\epsilon$$

$$S_\epsilon = \theta^2 (S - \alpha \theta)$$

$$\theta(x,-\infty, +) = 0, \quad \theta(x,y,t) = 1$$

for $y > f(x,t)$, (2.7)

$$S(x, \pm, \infty, t) = 0.$$ (2.10)

In terms of the new variables, then, $\alpha$ will be the sole parameter of the problem.

We introduce coordinates attached to the curved flame front:

$$\eta = y - f(x,t)$$ (2.11)

and transform to scaled magnitudes:

$$\xi = \epsilon x, \quad t = \epsilon t, \quad \varphi = \epsilon f (\epsilon \ll 1)$$ (2.12)

which guarantee the smallness of the curvature ($\sim f_{xx}$) at finite velocities ($\sim f_t$) and inclinations ($\sim f_\xi$). In terms of these new variables. Equations (2.8), (2.9) and conditions (2.10) become

$$\epsilon \theta_\xi - \varphi_\xi \theta_\eta = \epsilon^2 \theta_{\xi\xi} - \epsilon \varphi_{\xi\xi} \theta_\eta + \left(1 + \varphi_\xi^2\right) \theta_{\xi\xi} - 2 \epsilon \varphi_\xi \theta_{\xi\eta} + \exp\left(\frac{1}{2} S\right) \sqrt{1 + \varphi_\xi^2} \delta(\eta)$$ (2.13)

$$\epsilon S_\xi - \varphi_\xi S_\eta = \epsilon^2 (S - \alpha \theta) \xi_\xi - \epsilon \varphi_{\xi\xi} (S - \alpha \theta)_\eta + \left(1 + \varphi_\xi^2\right) (S - \alpha \theta) \xi_\xi - 2 \epsilon \varphi_\xi (S - \alpha \theta) \xi_\eta$$ (2.14)

$$\theta = 1 \quad \text{for } \eta > 0, \quad \theta \to 0 \quad \text{as } \eta \to -\infty, \quad S \to 0 \quad \text{as } \eta \to \pm \infty.$$ (2.15)

3. Asymptotic solution.

To solve problem (2.13)-(2.15), we assume the solution to have the form of an asymptotic expansion in powers of the small parameter $\epsilon$, up to third order:

$$\varphi = \varphi_0 + \epsilon \varphi_1 + \epsilon^2 \varphi_2 + \epsilon^3 \varphi_3 + \cdots$$

$$\theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \cdots$$

$$S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 + \cdots.$$ (3.1)
Solving successively in the zeroth, first, second and third approximations and subjecting the results to straightforward but rather lengthy manipulations, we obtain

\[
\varphi_{0\tau} = -\lambda^{-1} \quad \text{where} \quad \lambda = \left(1 + \varphi_{0\xi}^2\right)^{-\frac{1}{2}} \quad (3.2)
\]

\[
\varphi_{1\tau} = \left(1 - \alpha\right) \lambda^2 \varphi_{0\xi \xi} - \lambda \varphi_{0\xi} \varphi_{1\xi} \quad (3.3)
\]

\[
\varphi_{2\tau} = -\left(\frac{\alpha^2}{2} + 1\right) \lambda^5 \varphi_{0\xi \xi} - 2 \left(1 - \alpha\right) \lambda^4 \varphi_{0\xi} \varphi_{1\xi} \varphi_{0\xi} \varphi_{0\xi} + \left(1 - \alpha\right) \varphi_{1\xi \xi} - \frac{1}{2} \lambda^3 \varphi_{1\xi}^2 - \lambda \varphi_{0\xi} \varphi_{2\xi}, \quad (3.4)
\]

Synthesizing (3.2)-(3.5) in a single equation, we obtain

\[
\frac{\varphi_{\tau}}{\sqrt{1 + \varphi_{\xi}^2}} = -1 - (\alpha - 1) \kappa - \left(1 + \frac{1}{2} \alpha^2\right) \kappa^2 - \\
- \left(2 \alpha + 5 \alpha^2 - \frac{1}{3} \alpha^3\right) \kappa^3 - \alpha^2(\alpha + 3) \epsilon^2 \frac{1}{\sqrt{1 + \varphi_{\xi}^2}} \frac{\partial}{\partial \xi} \left(\frac{1}{\sqrt{1 + \varphi_{\xi}^2}} \frac{\partial \kappa}{\partial \xi}\right) \quad (3.6)
\]

where

\[
\kappa = \frac{\epsilon \varphi_{\xi \xi}}{(1 + \varphi_{\xi}^2)^{3/2}} = \frac{f_{xx}}{(1 + f_{\xi}^2)^{3/2}} \quad (3.7)
\]

is the curvature of the flame front. Transforming to the original (nonscaled) variables, we obtain from (3.6)

\[
\frac{f_{t}}{\sqrt{1 + f_{\xi}^2}} = -1 - (\alpha - 1) \kappa - \left(1 + \frac{1}{2} \alpha^2\right) \kappa^2 - \\
- \left(2 \alpha + 5 \alpha^2 - \frac{1}{3} \alpha^3\right) \kappa^3 - \alpha^2(\alpha + 3) \kappa_{ss} \quad (3.8)
\]

where

\[
ds = \sqrt{(dx)^2 + (df)^2} = \sqrt{1 + f_{\xi}^2} \quad \text{dx} \quad (3.9)
\]

is the element of arc-length along the curved front.


The left-hand side of equation (3.8) is the propagation velocity of the flame relative to the gas. The first term on the right corresponds to the propagation velocity of a plane flame front. The remaining terms represent the effect of the curvature and its variation on the propagation velocity of the flame. Expressing the flame-front equation \(y = f(x, t)\) as

\[
y - f(x, t) = F(x, y, t) = 0 \quad (4.1)
\]

we can write equation (3.8) in a form symmetric with respect to \(x\) and \(y\):

\[
V_n = 1 + (\alpha - 1) \kappa + \left(1 + \frac{1}{2} \alpha^2\right) \kappa^2 + \\
+ \left(2 \alpha + 5 \alpha^2 - \frac{1}{3} \alpha^3\right) \kappa^3 + \alpha^2(\alpha + 3) \kappa_{ss} \quad (4.2)
\]

where

\[
V_n = \frac{F_t}{\sqrt{\nabla F}}; \quad \kappa = \nabla \cdot \n; \quad \n = \frac{\nabla F}{|\nabla F|}. \quad (4.3)
\]

Near the stability limit (i.e., when \(\alpha - 1 \ll 1\)), equation (4.2) can be considerably simplified. Indeed, proceeding from the equation (1.1) for a weakly perturbed plane flame, we obtain in this case

\[
\kappa \sim f_{xx} \sim (\alpha - 1)^2, \\
\kappa_{ss} \sim f_{xxxx} \sim (\alpha - 1)^3. \quad (4.4)
\]

Hence, neglecting terms which are small compared with \((\alpha - 1)^3\), we deduce from equation (4.2) that

\[
V_n = 1 + (\alpha - 1) \kappa + 4 \kappa_{ss}. \quad (4.5)
\]

This simple relationship, together with (4.3), determines a spatially invariant equation for a cellular flame near the stability threshold. Thus, the geometrical nature of the second and fourth derivatives appearing in equation (1.1) has been fully clarified.
Regarding the three-dimensional version of equation (4.2), it may well include, for example, terms associated with the Gaussian curvature of the front. The form of these terms cannot be predicted on the sole basis of the structure of equation (4.2) and symmetry arguments. Thus, generalization of this equation to the three-dimensional case is no trivial matter, but demands special consideration.

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Reference

