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Accurate critical exponents for Ising like systems in non-integer dimensions

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Résumé. — Dans un article récent nous avons montré qu'en appliquant des méthodes raffinées de resommation au développement en ε de Wilson-Fisher, nous pouvions obtenir, à partir des termes des séries disponibles actuellement, des valeurs précises pour les exposants critiques du modèle de Heisenberg classique avec symétrie $0(n)$: ces valeurs sont en excellent accord avec les résultats tirés de calculs de Groupe de Renormalisation à 3 dimensions, ainsi qu'avec les résultats exacts du modèle d'Ising à 2 dimensions. Récemment divers auteurs ont suggéré qu'il était possible d'utiliser des réseaux fractals pour interpoler les réseaux réguliers en dimension non entière. Des calculs numériques ont été faits pour le modèle d'Ising. De façon à permettre une comparaison directe avec les valeurs du Groupe de Renormalisation, nous présentons ici nos résultats pour les exposants critiques pour des dimensions d non entières $(1 < d \leqslant 4)$. En imposant les valeurs exactes du modèle d'Ising à $d = 2$, nous améliorons également les valeurs à $d = 3$. Finalement nous remarquons que les valeurs des exposants extrapolés pour $d < 2$ ne sont pas en désaccord avec les valeurs tirées du modèle d'interface presque plat.

Abstract. — In a recent article we have shown that, by applying sophisticated summation methods to Wilson-Fisher's ε-expansion, it is possible from the presently known terms of the series to obtain accurate values of critical exponents for the $0(n)$ symmetric n-vector model : these values are consistent with the best estimates obtained from three-dimensional Renormalization Group calculations and, in the case of Ising-like systems, with the exactly known two-dimensional values of the Ising model. The controversial conjecture has been recently formulated that some fractal lattices could interpolate regular lattices in non-integer dimensions. Numerical calculations have been done for the Ising model. To allow for direct comparison with Renormalization Group values, we present here estimates for exponents in non-integer dimensions $d\ (1 < d \leqslant 4)$. By imposing the exactly known 2 d values, we at the same time improve the previous 3 d estimates. Finally we find indications that for $1 < d < 2$ the Renormalization Group values are consistent with those obtained from the near planar interface model.

1. Introduction.

Calculation of critical exponents [1] for the n-vector model using Renormalization Group method [2] has first been done following a suggestion by Parisi [3] from perturbation series at fixed dimensions $(d = 2 \text{ or } d = 3)$ because many terms in the series had been calculated by Nickel [4] (6 consecutive terms for $d = 3$). However, recently several groups [5] have extended the Wilson-Fisher $\varepsilon = 4 - d$ expansion [6], and we have shown [7] that the same summation methods which had allowed us [1] to obtain accurate values for critical exponents from fixed dimension perturbative calculations, could also be used with the ε-expansion. The exponents obtained in this way are consistent in three dimensions with the standard Renormalization Group values : the apparent error is larger by a factor 2 typically, which is consistent with the smaller length of the series. These exponents are therefore also consistent with the best high temperature series results [8]. In two dimensions the values for the exponents agree remarkably well with the exactly known Ising model values (although the problem of the identification of the correction exponent ω remains).

The popularity of fractal lattices has lead to the controversial conjecture that some fractal lattices could interpolate standard regular lattices in non-integer dimensions, although it seems that fractal lattices cannot be characterized in general by only one dimension.

In particular, numerical calculations have been done for the Ising model [9-11]. To allow a direct comparison with Renormalization Group values, we have calculated from the ε-expansion critical exponents for arbitrary values of the dimension d for $1 < d \leqslant 4$ for Ising-like systems.

Since we had shown [7] elsewhere that the agreement between exact 2 d values and Renormalization Group estimates was remarkably good, we have imposed the exact 2 d values.

As a consequence, with of course the additional very weak conjecture that the ϕ^4 field theory and the Ising model belong to the same universality class, we have also obtained more accurate 3 d values.

Finally, we have been able to add some elements to another controversial issue : the relation between the near planar interface model of Wallace and Zia [12] and the bulk phase transition of the Ising model. We have compared our results with the expansion for the exponent ν in powers of $(d-1)$ of this near planar interface model. A further extension of this model, the droplet model [13], also yields an asymptotic form for the exponent β. This allows then a comparison with our results for all exponents.

The set up of this article is the following : in section 2 we recall briefly how we sum Renormalization Group series to extract values for physical quantities. In section 3 we present our new results and compare them with the other existing data mentioned above.

2. Numerical calculations.

Since the method has been described in detail elsewhere [1, 7], we shall here only recall the main points.

Starting from the ε-expansion for an exponent $E(\varepsilon)$:

$$E(\varepsilon) = \sum_k E_k \varepsilon^k, \qquad (1)$$

we introduce a Borel transform $B(t)$ of $E(\varepsilon)$, which depends on the free parameter ρ:

$$E(\varepsilon) = \int_0^\infty t^\rho \mathrm{e}^{-t} B(\varepsilon t)\, \mathrm{d}t\,. \qquad (2)$$

The series (1) for $E(\varepsilon)$ transforms into a series expansion for $B(t)$:

$$B(t) = \sum_k E_k t^k / \Gamma(k+\rho+1)\,. \qquad (3)$$

We know the large order behaviour of E_k [14, 15]:

$$E_k \sim_{k\to\infty} k!\, a^k k^b c\,, \qquad (4)$$

with :

$$a = -1/3,\ b = 7/2 \text{ for } \eta,\ b = 9/2 \text{ for } 1/\nu\,. \qquad (5)$$

This translate into a large order behavior for the series expansion of $B(t)$:

$$E_k / \Gamma(k+\rho+1) \sim_{k\to\infty} a^k k^{b-\rho} c\,. \qquad (6)$$

The Borel transform $B(t)$ is therefore analytic at least in a circle. The singularity closest to the origin is located at the point $-1/a$ and is of the form $(1+at)^{\rho-b-1}$, except when $(\rho-b)$ is an integer in which case the singularity is logarithmic.

We have consistently assumed that actually $B(t)$ is analytic in the maximal domain possible, i.e. a cut-plane, and therefore mapped the cut-plane onto a circle of radius 1 by :

$$t = (4/a)\, u/(1-u)^2\,. \qquad (7)$$

With this hypothesis, after mapping, the Taylor series is convergent on the whole domain of integration of the Borel transformation (2), except at infinity which corresponds to $u = 1$. To possibly weaken the singularity of $B(t(u))$ at $u = 1$, we have multiplied the function by $(1-u)^\sigma$ obtaining the expansion :

$$B(t) = (1-u(t))^{-\sigma} \times \sum_k A_k(\rho,\sigma)\, [u(t)]^k\,, \qquad (8)$$

and therefore :

$$E(\varepsilon) = \sum_k A_k(\rho,\sigma) \times \int_0^\infty t^\rho [u(t)]^k (1-u(t))^{-\sigma} \mathrm{e}^{-t}\, \mathrm{d}t\,. \qquad (9)$$

The conditions for the convergence of such an expression have been considered in references [1, 15]. In the absence of any rigorous proof, we study numerically the apparent convergence of this expansion with the five terms [5] of the ε-expansion available. Actually, and this is a difference [16] between the treatment of the fixed dimension perturbation series and the ε-expansion, we have in the latter case introduced an additional parameter λ and performed the homographic transformation in the ε-complex plane :

$$\varepsilon' = \lambda \varepsilon / (\lambda - \varepsilon) \qquad (10)$$

before Borel transformation to send away a possible singularity on the real positive ε axis. For the Ising system the interface model [12] for example predicts a singularity at $d = 1$, i.e. $\varepsilon = 3$. Since we do not know the location of the singularities of the function $E(\varepsilon)$ in the complex plane, we have kept λ as a free parameter.

All three parameters λ, ρ, σ have been moved freely in a reasonable range and chosen to improve the apparent convergence of expansion (9).

Finally, since we had previously [7] verified that the results for $d = 2$ were remarkably consistent with the exactly known 2 d Ising values (see Table I), we have imposed these exact values setting

$$E(\varepsilon) = E(2\,d\,\text{Ising}) + (2-\varepsilon)\, \tilde{E}(\varepsilon) \qquad (11)$$

and performed all the manipulations described above on $\tilde{E}(\varepsilon)$.

Table I. — *Estimates* [7] (A) *of Ising critical exponents for the dimension* $d = 2$ *from the* ε*-expansion, compared with the exactly known* 2d *Ising values* (B).

	γ	ν	β	η
(A)	1.73 ± 0.06	0.99 ± 0.04	0.120 ± 0.015	0.26 ± 0.05
(B)	1.75	1.	0.125	0.25

3. Results.

3.1 Improved estimates in $d = 3$ dimension. — Since we have imposed to the sums of the series to yield the values of the exponents of the Ising model for $d = 2$, we expect a decrease in the apparent error at $d = 3$ for the various exponents. This is exactly what happens as can be seen in table II where Renormalization Group values coming from fixed dimension perturbation series [1], ε-expansion without the $d = 2$ information [7], our present results, and some recent high temperature results [8], are compared.

The apparent error is approximately decreased by a factor two, and the new results are now as accurate as the standard Renormalization Group values. It is satisfactory that the results are still quite consistent. One observes however some small deviations which may be significant. The largest one concerns the exponent η for which the agreement is now marginal. Note also that these new values are even closer to the present best high temperature series estimates than the old Renormalization Group values.

3.2 Results for arbitrary dimensions and comparison with fractal estimates. — Between 4 and 2 dimensions we obtain very accurate results, since for example the apparent errors for γ and ν remain always smaller than 1 %. Below two dimensions the apparent errors increase very rapidly as expected. However, up to $d = 1.5$ γ and ν are estimated at better than about 10 %.

Table II. — *Estimates of Ising critical exponents for the dimension* $d = 3$: (A) *Renormalization Group values from fixed dimension perturbation series* [1]; (B) ε*-expansion without the* $d = 2$ *information* [7]; (C) *our present results*; (D) *some recent high temperature series results* [8]. *The notation* ± X *indicates the error* on the last digit.

	γ	ν	β	η
(A)	1.2410 ± 20	0.6300 ± 15	0.3250 ± 15	0.0310 ± 40
(B)	1.2390 ± 40	0.6305 ± 25	0.3265 ± 25	0.0370 ± 30
(C)	1.2390 ± 25	0.6310 ± 15	0.3270 ± 15	0.0375 ± 25
(D) [8a]	1.2385 ± 25	0.6305 ± 15		
(D) [8b]	1.2370 ± 30	0.6300 ± 30		0.0360 ± 20
(D) [8c]	1.2385 ± 15			
(D) [8d]	1.2395 ± 4	0.6320 ± 10		0.0390 ± 40
(D) [8e]	1.2370 ± 20	0.6295 ± 15		0.0350 ± 10

Table III and figures 1 and 2 present our results for various values of the dimension between 1 and 4.

In a recent paper, Bhanot *et al.* [10] have measured the critical exponent γ of the Ising model on a fractal lattice of the Sierpinsky carpet type. They interpreted their results in terms of a dimension d' defined from the average number of nearest neighbours of an active site, and different from the Hausdorf dimension d_H. Figure 3 reports both their results for $\gamma(d')$ and our present results for $\gamma(d)$, assuming that d' is the same as our dimension d. The agreement appears quite reasonable and compatible with their conclusion that such fractals can be used to interpolate between integer

Table III. — *Estimates of Ising critical exponents for various values of the dimension d between* 1 *and* 4. *The notation* ± X *indicates the error* on the last digit. *Values for* $d = 1$ *are those predicted by the near planar interface model* [12] *and the droplet model* [13].

d	γ	ν	β	η
4	1	0.5	0.5	0
3.75	1.046060 ± 30	0.523405 ± 15	0.458355 ± 20	0.001435 ± 10
3.5	1.10055 ± 35	0.55215 ± 20	0.41600 ± 20	0.00685 ± 20
3.25	1.1642 ± 15	0.5873 ± 10	0.3723 ± 10	0.0180 ± 10
3	1.2390 ± 25	0.6310 ± 15	0.3270 ± 15	0.0375 ± 25
2.75	1.328 ± 5	0.686 ± 3	0.2800 ± 30	0.067 ± 6
2.5	1.436 ± 8	0.758 ± 5	0.2305 ± 30	0.11 ± 1
2.25	1.571 ± 10	0.857 ± 6	0.1790 ± 25	0.17 ± 1
2	1.75	1	0.125	0.25
1.875	1.862 ± 15	1.10 ± 1	0.097 ± 3	0.30 ± 3
1.75	1.99 ± 4	1.23 ± 3	0.068 ± 6	0.35 ± 5
1.65	2.11 ± 8	1.37 ± 7	0.045 ± 10	0.40 ± 10
1.5	2.35 ± 20	1.65 ± 20	0.010 ± 15	0.50 ± 15
1.375	2.6 ± 4	2.1 ± 5	− 0.02 ± 3	0.55 ± 25
1.25	3.0 ± 10	3.0 ± 15	− 0.05 ± 5	0.65 ± 35
1	∞ ($\gamma \sim \nu$)	∞	0	1

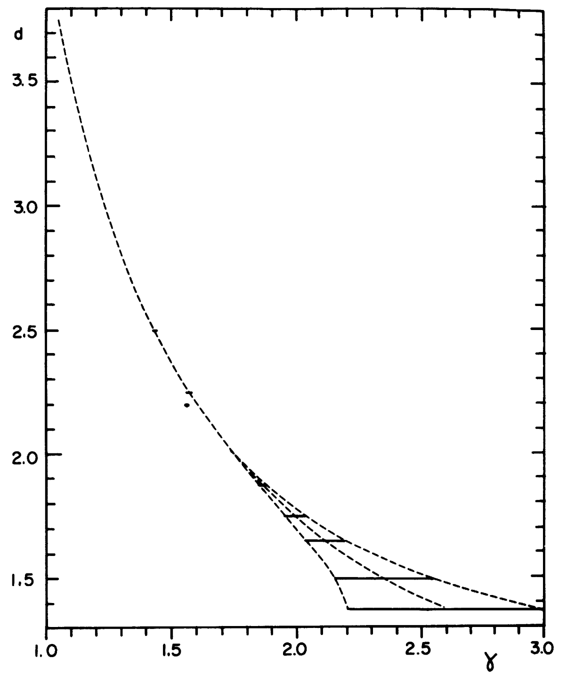

Fig. 1. — Our present results giving the dimension d *versus* the Ising critical exponent γ.

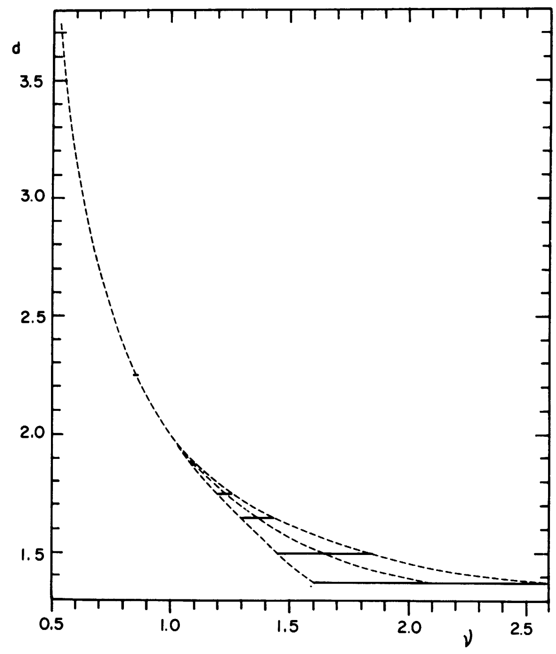

Fig. 2. — Our present results giving the dimension d *versus* the Ising critical exponent ν.

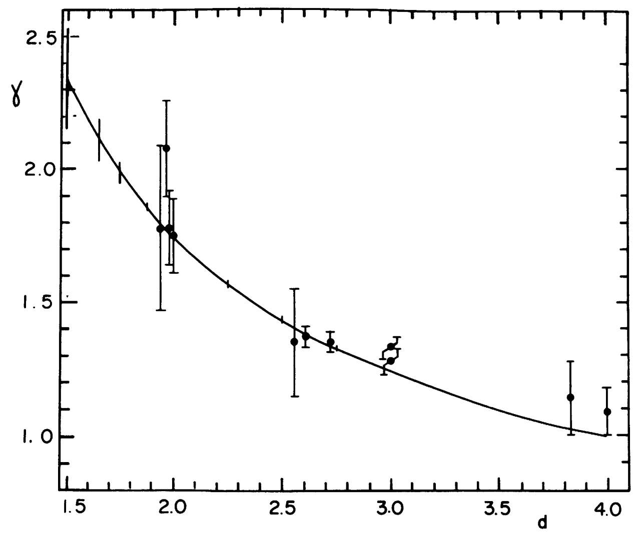

Fig. 3. — Comparison between our present results for $\gamma(d)$ [continuous line ; error bars indicated by vertical segments ———] and the results $\gamma(d')$ of Bhanot *et al.* [10] [vertical ⊢—•—⊣], assuming that d' is the same as our dimension d.

dimensions to study the critical behaviour of statistical systems.

In an earlier paper, Bhanot *et al.* [9] report a measure of γ and ν for a fractal of Hausdorff dimension $d_H = 1.86$:

$$\gamma = 1.90 \pm 0.05 \tag{12}$$
$$\nu = 1.28 \pm 0.05 . \tag{13}$$

Such values are consistent with our results with a dimension d of a regular lattice :

$$d = 1.75 \pm 0.10 . \tag{14}$$

On the other hand, Bhanot *et al.*, assuming that the system at criticality is governed by a single dimension d_c, get through hyperscaling relations :

$$d_c = 1.66 \pm 0.10 . \tag{15}$$

It seems then that such a scheme is compatible with $d_c = d$.

However, preliminary results of Bonnier *et al.* [11] seem to question this interpretation. They study the Ising model on two fractal lattices of the Sierpinsky carpet type : one (i) with $d_H \simeq 1.8$ and $d' = 1.5$, and the other (ii) with $d_H \simeq 1.9$ and $d' = 1.6$.

On the one hand, their results for γ :

$$2.30 \leqslant \gamma \leqslant 2.85 \quad \text{in case (i)} \tag{16}$$
$$2.15 \leqslant \gamma \leqslant 2.65 \quad \text{in case (ii)} \tag{17}$$

compared to our results give for the central value :

$$d \simeq 1.4 \quad \text{in case (i)} \tag{18}$$
$$d \simeq 1.5 \quad \text{in case (ii)} \tag{19}$$

and correspond to :

$$d < 1.60 \quad \text{in case (i)} \tag{20}$$
$$d < 1.67 \quad \text{in case (ii)} . \tag{21}$$

These results are compatible with the possibility that the critical behaviour on the fractal would be governed by the dimension d' being equal to d.

Their results for ν are less precise :

$$1.38 \leqslant \nu \leqslant 2.23 \quad \text{in case (i)} \tag{22}$$
$$1.24 \leqslant \nu \leqslant 1.85 \quad \text{in case (ii)} \tag{23}$$

and compared to our results give for the central value :

$$d \simeq 1.45 \quad \text{in case (i)} \tag{24}$$
$$d \simeq 1.55 \quad \text{in case (ii)} \tag{25}$$

and correspond to :

$$d < 1.68 \quad \text{in case (i)} \tag{26}$$
$$d < 1.76 \quad \text{in case (ii)}\,. \tag{27}$$

These results seem also to favour $d' \simeq d$.

On the other hand, however, Bonnier *et al.* [11] found, as a preliminary result for the fractal lattice (i), that the hyperscaling relation :

$$\gamma_4 - 2\,\gamma - \nu d_c = 0 \quad (\text{with } \gamma_4 = 3\,\gamma + 2\,\beta) \tag{28}$$

seems to involve on the contrary a dimension d_c such that :

$$1.75 \leqslant d_c \leqslant 1.85 \tag{29}$$

suggesting $d_c = d_H \simeq 1.8$.

It remains therefore unclear whether values for critical exponents on these fractal lattices can be interpreted as arising from regular lattices at a non-integer effective dimension.

3.3 THE NEAR PLANAR INTERFACE MODEL. — Wallace and Zia [12] have calculated the critical exponent ν of the near planar interface model in an $\varepsilon' = d - 1$ expansion. In addition they have argued that this model should have the same critical behaviour as the Ising model, at least in the sense of the ε'-expansion.

As can be seen in figures 4 and 5, our results seem to indicate that such an identification might be correct.

We compare in figure 4 our estimates for ν with the $\varepsilon' = d - 1$ expansion [12, 17] at first, second, and third leading order [18] :

$$\nu = \varepsilon'^{-1} - \frac{1}{2} + \frac{1}{2}\,\varepsilon'\,. \tag{30}$$

In figure 5 we present our estimates for $\left(\nu - \varepsilon'^{-1}\right)$, which is indeed negative and becomes compatible with $-\frac{1}{2}$ as d decreases towards 1.

Although our estimates become very poor near one dimension and the $\varepsilon' = d - 1$ expansion obviously cannot be trusted near two dimensions, there seems definitively to exist a region in which both agree in a significant way.

From an extension of this model, the droplet model [13], a second independent exponent, β, can be calculated as $\varepsilon' = d - 1$ tends to 0 :

$$\beta \sim 4\,\pi^{-1}\,\varepsilon'^{-\,(2+\varepsilon')/2}\,\mathrm{e}^{-\,\left(1+2\,C+2\,\varepsilon'^{-1}\right)\,\nu} \tag{31}$$

where $C = 0.577...$ is Euler' constant.

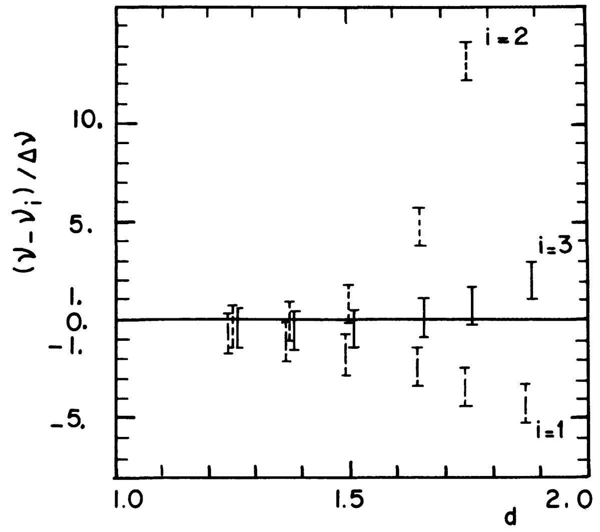

Fig. 4. — Comparison between our present results $\nu = \nu_0 \pm \Delta\nu$ and the successive predictions for ν of the $\varepsilon' = d - 1$ expansion [12, 17] for the near planar interface model : $\nu_1 = \varepsilon'^{-1}$; $\nu_2 = \varepsilon'^{-1} - \frac{1}{2}$; $\nu_3 = \varepsilon'^{-1} - \frac{1}{2} + \frac{1}{2}\,\varepsilon'$.

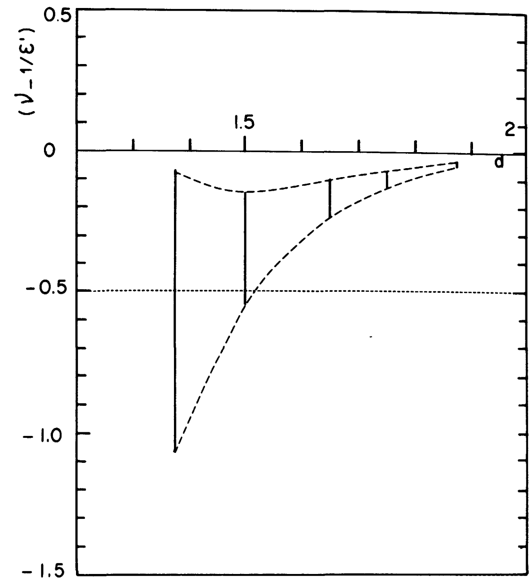

Fig. 5. — Our present estimates for $\left(\nu - \varepsilon'^{-1}\right)$ with $\varepsilon' = d - 1$, for various dimension d between 1 and 2.

Such an exponential form cannot possibly be reproduced by our approximants, which by construction yield only smooth or power law behaviours. Maybe this is related with the fact that the values we obtain for β have a tendency of becoming negative for d close to 1, which is clearly inconsistent, instead of dropping very rapidly to zero as predicted by expression (31).

The significant results are however that our β is small and our η increases to become compatible with 1 for d close to 1 as predicted.

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