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Renormalization by substitution on 2-D Potts models

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1. Introduction.
In recent years, the Potts model on hierarchical lattices has been extensively studied [1-7], the reason for this interest being that the bond moving approximation introduced by Migdal and Kadanoff [8-9] (M.K.) becomes exact for such self-similar networks, which are perfectly adapted to the application of the Position Space Renormalization Group (PSRG). For a realistic Euclidean two-dimensional lattice, the M.K. method becomes a very crude approximation: recently it has been shown [10-12] that bond moving transforms the initial Euclidean lattice into a self-similar one; this is the origin of the poor agreement between exact and calculated critical temperatures and exponents. The great interest of the M.K. approach being its facility of implementation, it is tempting to search for a new approximate method which gives an improvement over the M.K. method, whilst retaining the simplicity of the algorithm. A first attempt was proposed in [10] and called « Renormalization by Substitution » (R.S.): it is further developed here on Kagomé and triangular lattices in the framework of a Potts model. In comparison with the results given by the M.K. method, the improvement in the determination of unstable fixed points and critical exponents is noticeable.

2. Kagomé lattice.
The Kagomé two-dimensional lattice can be seen as David stars assembled in a triangular array on a plane (Fig. 1a). Each site can be found in q states (Potts model) and interacts only with its four first neighbours. The interaction parameter is \(-J\) \((J > 0)\) for an antiferromagnetic interaction, \(J > 0\) for a ferromagnetic one) and the parameter \(z = e^{\beta J}\) will be employed in order to simplify the notation. The two steps of the R.S., where the linear contraction ratio introduced by one renormalization step is \(b = \sqrt{3}\), are presented in figures 1b and 1c.

i) In the first step the « closed triangle » ABC (three bonds, each characterized by the parameter \(z\)) is replaced by the « open triangle » abc (two bonds, the parameter being \(\tilde{z}\)). The substitution rule \((z \rightarrow \tilde{z})\) is necessarily an approximation. As in [10] we will apply the following rule:

Rule a) The probability of finding three parallel spins on the sites ABC and abc is equal for the two triangles, considered as disconnected from the remainder of the lattice.
This hypothesis gives the following equation:

\[ z^3 / \left[ z^3 + (3q - 1)z + (q - 1)(q - 2) \right] = \tilde{z}^2 / \left[ \tilde{z}^2 + 2(q - 1)\tilde{z} + (q - 1)^2 \right] \]  

(2.1a)

or, with \( w = 1/z \):

\[ w^2(3 + (q - 2)w) = \tilde{w} \left( 2 + (q - 1)\tilde{w} \right). \]  

(2.1b)

ii) The second step eliminates the intermediate sites; the corresponding transformation \( \tilde{z} \rightarrow z' \) is exact and well known:

\[ z' = (\tilde{z}^2 + q - 1) / (2\tilde{z} + q - 2). \]  

(2.2)

The procedure \( z \rightarrow \tilde{z} \rightarrow z' \) will be called « Falling Renormalization » (F.R.) and \( z' \rightarrow \tilde{z} \rightarrow z \) « Climbing Renormalization » (C.R.). The F.R. is the usual renormalization procedure which reduces the number of degrees of freedom. Starting from an initial arbitrary value \( z \) (real and positive) the iterative application of the F.R. expression gives successive images flowing towards a stable fixed point, \( Z_1 = 1 \) (infinite temperature fixed point) or \( Z_\infty = \infty \) (0 temperature fixed point), according to the initial value of \( z \) (respectively smaller or greater than the unstable fixed point value, \( Z^* \)).

However in the present example a difficulty appears: equation (2.1) being of the second order in \( \tilde{z} \), one has to choose between two values of \( z \) (and \( z' \)), corresponding respectively to the two roots:

\[ \tilde{w}_\pm = \left( -1 \pm \sqrt{1 + (q - 1) \left[ 3w^2 + (q - 2)w^3 \right]} \right) / (q - 1). \]  

(2.3)

The validity of this rule (which will be called physical choice) is advocated by the following results. We have computed the (real) fixed points \( Z^* \) of the transformation for all values of the Potts parameter \( q \). They are represented in figure 2: the full lines correspond to attractive fixed points in the F.R. case, the dashed lines to repulsive fixed points. The thick lines are obtained with the plus sign in equation (2.3), the thin lines with the minus sign (this diagram can be compared with the corresponding one given by Itzykson and Luck [7] for the hierarchical diamond lattice).

The physical region corresponds to \( Z^* \geq 0 \) and \( q \geq 0 \). For \( q \geq 0 \) and \( 0 < Z^* \leq 1 \) we find two full lines of stable fixed points: the upper line being obtained with the solution \( \tilde{w}_+ \), the lower line with \( \tilde{w}_- \). Probably one line corresponds to stable fixed points characteristic of the exotic phases describe by Berker et al. [13], the other line being unphysical. It is possible that \( \tilde{w}_- \) becomes the physical solution in this interval of \( q \) values; but we do not have any rigorous criterion of choice to propose and we will not study this domain. Let us consider the Ising case \( (q = 2) \): starting from any complex \( z \) and applying repetitively the F.R. process with the rule (b), defined above, the successive images flow towards one of the two stable fixed points \( Z_1 \) or \( Z_\infty \), according to the localization of the initial point in the complex plane. Conversely, with the opposite rule, one finds only the unphysical stable fixed point \( Z = -1.469316 \), independently of the initial point. Figure 3a is built with the physical rule (b): points belonging to dark (« cold ») regions flow towards \( Z\_\infty \), whereas points of the clear (« hot ») regions are attracted by \( Z_1 \). The set of dark and clear regions constitute the Fatou set [15] of the transformation.
Fig. 2. — Dependency of (real) fixed point positions with $q$. The thick lines are associated with the positive solution of $w^+$ $(w)$, the thin lines with the negative solution. The dashed lines correspond to unstable fixed points and the full lines to stable fixed points. For instance, for $q = 2$, we obtain a physical unstable fixed point at $Z^+ = 2.627...$ and an unphysical stable fixed point at $Z^- = -1.469...$ For $q < 1.207$ a very intricate structure is obtained which is not studied here.

The Julia set [14], which is the set of repulsive periodic points of the F.R. procedure [15], is complementary to the Fatou set [16] ($z' - z$); it can be obtained by applying $n$ times the C.R. procedure to an arbitrary initial point (theoretically, $n = \infty$; in practice, $n \gg 1$); if it is drawn without any precaution a shadowed figure $3b$ is obtained. To « clean up » this figure one must take rule (b) into account in the following way: starting from a given point $z_0$, the C.R. procedure gives six images $z_1, \ldots, z_6$, all these images being not necessarily valid. Each $z_i$ is checked by verifying that the application of the F.R. procedure (with the rule (b)) yields the original value $z_0$. In this way figure (3c) is obtained, presenting a beautiful self-similar structure.

Figure 4 gives the Julia sets for $q = 3$ (Fig. 4a) and $q = 6$ (Fig. 4b). The physical choice is the same as for the case $q = 2$. The structures of these Julia sets are also fractal but very different from those given by the hierarchical diamond lattice [6-7] or the triangular one (with M.K. approximation) [10], particularly in the case $q = 3$. For $q = 6$ we again find a fractal border limiting two homogeneous regions, as for the diamond and triangular M.K. lattices, where such a result is already obtained for $q = 4$.

Consider the case $q = 3$ with which it is easier to study some mathematical curiosities. The fractal structure on the diagram (Fig. 4a) is constituted as described below: the nearly central « dot » (which is in fact a small finite size region) has four images (by C.R.) which are the « dots » nearest to it. Each of these four dots has two images; further at each iteration by C.R. the number of images is twice as large. The distance between a point and its images decreases. Finally dots accumulate and define a sharp boundary.

We can interpret this map from a geographical perspective: a « hot » continent, or island, containing « cold » lakes (« dots » described above) is surrounded by a cold ocean. The Julia set can be seen as the whole of the shores. Starting from a point in a lake and applying iteratively the F.R., it flows from lake to lake until it reaches the ocean. The last visited lake (nearly central dot) contains the point on the real axis:

\[ z_1 = 3/(2-q). \quad (2.5) \]

Effectively from (2.3) the image by F.R. of $z_1$ is $Z_\infty$. The abscissa of this point is more and more negative for increasing $q$. Consider now a point close to $z_1$:

\[ z = z_1 + \varepsilon \quad (2.6) \]

where $|\varepsilon| \ll |z_1|$. By applying the F.R. transforma-
tion it is easy to find:

$$z' = \tilde{z} = e^{-1}(2(2-q)^3/3^4).$$ \hspace{1cm} (2.7)

Relation (2.7) shows that $z'$ is obtained from $z$ by an inversion around the centre, $z_1$ (plus a symmetry with respect to the real axis). Figure 5 represents the central lake for $q = 3$: its shore is deduced from the shore of the continents by the inversion given by the expression (2.7). The lake sizes are increasing with $q$. When $q = 6$, cold lakes join the external cold ocean and disappear, leaving only one hot island with a fractal coast.

3. Triangular lattice.

The renormalization method developed here is directly derived from the substitution method proposed for the Kagomé lattice and the procedure is detailed in figure 6 (let us define $x = e^{\beta}$).

i) First, each bond is replaced by two equal parallel bonds such that $x_0^2 = x$. 

Fig. 3. — Complex planes of Julia and Fatou sets obtained when the R.S. is applied to the Kagomé lattice, with $q = 2$ (Ising model). a) Fatou set in view to precise « hot » and « cold » regions: the points of the dark region flow towards $z_{mc}$ (« cold » regions), the points of the bright regions towards $z_h$ (« hot » regions). b) Julia set when the transformation is used without any precaution. c) « Clean » Julia set when the unphysical solutions are avoided.
ii) Then the substitution approximation (closed triangle → open triangle) is applied to obtain \( x' \):

\[
\frac{x^4}{[x^4 + (q-1)x^3 + 2(q-1)x + (q-1)(q-2)]} = \frac{x'^2}{[x'^2 + 2(q-1)x' + (q-1)^2]}
\]

(3.1a)

or, with \( u = 1/x \):

\[
(q-2)u_0^4 + 2u_0^3 + u_0^2 = \tilde{u} (2 + (q-1)\tilde{u}).
\]

(3.1b)
As for the Kagomé lattice, this transformation is not rational and we are confronted with similar difficulties.

iii) Double bonds between neighbouring sites are replaced by a single bond. The corresponding equation is: \( \tilde{x}^2 = x \).

iv) \( x' \) is deduced from \( x \) with the same relation as for the Kagomé lattice:

\[
x' = (\tilde{x}^2 + q - 1) / (2\tilde{x} + q - 2)
\]

(3.2)

The calculations are even more complicated because sixteen solutions are obtained, corresponding to a given initial \( x' \). The expression \( X'(q) \), analogous to \( Z'(q) \) obtained with the Kagomé lattice, could be plotted. But instead of two curves as in the Kagomé case, \( X'(q) \) is represented by four curves (corresponding to the four paths in the F.R. procedure): one equation of the second degree to obtain \( \tilde{u} \) (or \( \tilde{x} \)):

\[
\tilde{u} = -1 \pm \sqrt{1 + (q - 1) \left[ (q - 2) \mu_0^4 + 2\mu_0^2 + \mu_0^2 \right] / (q - 1)}
\]

(3.3)

and then:

\[
\tilde{x} = \pm \sqrt{\tilde{x}}.
\]

(3.4)

Happily for \( q \geq 2 \) and starting from a positive real (physical) value of \( x \), only one physical solution is obtained by selecting the plus signs in (3.3) and (3.4). As precedingly with the Kagomé lattice this rule will be adopted and generalized to the complex plane. Only the Ising model \( q = 2 \) is considered; the selected roots flow by F.R. towards physical stable fixed points: \( X_\infty, X_1 \). In fact, two « hot » fixed points, \( X_{-1} = -1 \) and \( X_{+1} = 1 \) are found and it is impossible to distinguish between them. The Fatou set of this transformation is presented in figure 7.

The critical parameters and exponents are given in the table for the two lattices: the substitution renormalization gives very good approximations for the positions of the unstable fixed points, especially for small \( q \) (better than 0.1% for \( q = 1 \), 1% for the Ising model). This agreement is better than for the M.K. model on the triangular lattice.

4. Conclusion.

The method of renormalization by substitution brings noticeable improvements in the determination of unstable fixed (critical) points. The price to pay is a heaviness of the M.K. method and, overall, the introduction of a non rational transformation. However, one must note that, if this last problem was tedious for drawing the Julia or Fatou sets, it does not appear for realistic physical problems: as a matter of fact every point of the positive real axis flows towards a physical stable fixed point. The introduction of the Julia-Fatou sets shows clearly that the substitution transformation (with its criterion of choice of the « physical » solution), as the M.K. procedure, is equivalent to a transformation of the initial Euclidean lattice into a self-similar one.

Finally, our method presents a quantitative improvement over the M.K. one without introducing too much complication and it would be interesting to test it on more sophisticated models (for example a spin 1 model such as the BEG model [17, 18], corresponding to the Potts model for \( q = 3 \)).

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