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1. Introduction.

It is well known that long polymers in a good solvent are well described by the two-parameter continuous model, introduced originally by Edwards [1]. This model has been first studied indirectly by field theoretic methods [2] (as well as the corresponding three-parameter model for long polymers in a θ solvent [3]). Since then, more direct renormalization methods have been devised for the two-parameter continuous model [4-6]. In [4], the essential role is held by a dimensionless second virial coefficient « g », which replaces the Zimm et al.-Yamakawa...
parameter $z$ ($^1$) in the renormalization process. In [5], the dimensional renormalization used is actually exactly that of the associated $(\phi^2)^2$ field theory [6, 7]. In [6], we recently showed that a direct dimensional renormalization can be performed in terms of a minimally renormalized parameter $z_R$, which is substituted for $z$. These renormalization methods work to all orders, and are equivalent [6, 7]. (For the three parameter model, a similar direct renormalization method also exists [8]). With all these renormalization methods, applied to Edwards’ continuous model of polymer chains in a good solvent, much attention has been paid until now to the direct evaluation of critical exponents, and to the perturbative calculation of universal relations between (« renormalized ») physical variables. These exponents or relations are universal in the asymptotic limit of very long polymer chains: whilst being obtained in the framework of the continuous polymer model, they could in principle be calculated as well in any other model. Here we want to study other quantities, which have not been determined before, i.e. the critical amplitudes which appear in the scaling laws describing the properties of very long polymer chains in a good solvent. We shall calculate these critical amplitudes in the framework of the two-parameter continuum model. One should realize that they are associated with this model, and are not universal. However some combinations of these amplitudes are universal. But they are quite interesting as such, for the knowledge of Edwards’ model. The latter is actually the simplest model for interacting polymer chains and has fascinating properties on its own account, which also captured recently the interest of mathematicians [9]. A good example of the theoretical interest of absolute amplitudes is given by reference [10], where the (non-universal) end-to-end swelling curve $I_0^\beta(z)$ was numerically calculated for the same continuum model, in three dimensions and for all values of $z$. This was done with the help of the $O(z^2)$ expansion of $I_0^\beta$ given by reference [11]. Here, by a quite different method, we shall calculate analytically the asymptotic form $I_0^\beta(z, \epsilon) = A_0(\epsilon) z^{(2D-1)/2\epsilon}$ of this swelling for $z \to \infty$, and compare it to the numerical results. Some critical amplitudes have been considered in [11], but they were not explicitly calculated.

Here we shall consider in all generality dimensionless quantities, associated with polymer chains, which are of the form $I(z, \epsilon)$, i.e. functions of Yamakawa’s parameter $z$ and of dimension $d = 4 - \epsilon$. We assume that they are given in the asymptotic limit of very long chains by a power law behaviour

$$I(z, \epsilon) = A(\epsilon) z^\sigma(\epsilon) \quad (1.1)$$

where $\sigma(\epsilon)$ is a critical exponent given by usual Wilson-Fisher $\epsilon$-expansion. Examples of such functions are given by the end-to-end swelling $I_0$, the gyration swelling $I_G$, or the dimensionless part $I_N$ of the general connected partition function $I_N$ of $N$ chains. We show how to calculate directly the « $\epsilon$-expansion » of $A(\epsilon)$, starting from the double $\epsilon$ Laurent expansion, and $z$ series expansion of $I(z, \epsilon)$. We give explicitly the formal expansion of $A(\epsilon)$ to order $\epsilon^2$. (In the Appendix we calculate it formally to $O(\epsilon^3)$). We use for this our $z$-dimensional renormalization method, introduced in reference [6], which is for our present purpose the most efficient. Actually it is the « minimal » method for this problem ($^2$). The amplitudes $A(\epsilon)$ are in general not universal. However, given a set of quantities $\{I_i(z)\}$, scaling with critical exponents $\{\sigma_i\}$, any set of coefficients $\{a_i\}$ such that $\sum \sigma_i \sigma_i = 0$, gives rise to a universal combination

$$\lim_{z \to \infty} \prod_i I_i^a(z) = \prod_i A_i^a(\epsilon). \quad (1.2)$$

Therefore, whilst critical amplitudes are not universal, but associated with Edwards’ model, some combinations of them of the type (1.2) are universal. In critical phenomena, this is well known and has been extensively studied [12]. For polymers, a trivial example is the gyration end-to-end swellings ratio $N = \overline{I_0}/I_0$, and other examples will appear below.

Let us finally note that the method we shall give here for calculating the critical amplitudes of the two parameter model, could be applied equally well to tricritical amplitudes associated with the three-parameter model, describing polymer chains near the $\theta$-point, below three dimensions [8].

This article is organized as follows. In section 2, we recall the expressions of polymer physical quantities of interest in the framework of the continuous model, and their expected scaling behaviour. We also briefly describe the $z$-dimensional renormalization method of [6]. In section 3, we show how to calculate directly the critical amplitude associated with any scaling quantity $I(z, \epsilon)$. We give its formal expression to order $O(\epsilon^2)$. The last section is devoted to the calculation of the critical amplitudes associated with the swellings of a single chain, and with the partition function of $N$ chains. We compare our results to numerical evaluations. Also, as a consequence, the universal virial expansion of the osmotic pressure is given, up to fourth order. The entropy of a single continuous chain is also calculated. In the Appendix, we give the formal expression of any critical amplitude to order $O(\epsilon^3)$.


($^2$) Using the methods of [4] or [5] would lead to significantly more complicated algebra.
2. The continuous model and its critical amplitudes.

The weight describing $N$ identical continuous chains is

$$P \{ r \} = \exp \left\{ -\frac{1}{2} \sum_{a=1}^{N} \int_{0}^{\infty} \left( \frac{dr_a}{ds} \right)^2 ds - \frac{1}{2} b \sum_{a \neq a'}^{N} \int_{0}^{\infty} ds \int_{0}^{\infty} ds' \delta^d(r_a(s) - r_{a'}(s')) \right\}$$

(2.1)

where $r_a(s)$ is the $d$-dimensional configuration of the $a$ chain, $a = 1, \ldots, N$. The common Brownian area $S$ of the chains is such that their Brownian square end-to-end distance reads $R^2 = dS$. The dimensionless Zimm-Yamakawa parameter $z$ reads then

$$z = (2\pi)^{-d/2} bS^{-d/2}$$

(2.2)

and defines the strength of the interaction. We use here dimensional regularization, and we set $d = 4 - \epsilon, \epsilon > 0$.

The connected generalized partition functions associated with the continuous model are, in Fourier space

$$(2\pi)^d \delta^d(k_1 + \cdots + k_{2N}) \overline{\mathcal{Z}}_N(k_1, \ldots, k_{2N}, S, b, \epsilon) =$$

$$\left\{ \begin{array}{ll}
\int d \{ r \} P \{ r \} \exp \left[ i \sum_{a=1}^{N} \left[ k_{2a-1} \cdot r_a(S) + k_{2a} \cdot r_a(0) \right] \right] & \\
\int d \{ r \} P_0 \{ r \} \delta^d(r_1(0)) - \delta^d(r_N(0)) & \\
\end{array} \right\}$$

(2.3)

where $P_0$ is the weight of free Brownian chains ($b = 0$). Here the momenta are injected at the extremeties of the chains, but similar partition functions can be defined with insertions of wave-vectors along the chains. At zero momenta, one gets the connected partition functions $\mathcal{Z}_N$ themselves, for $N$ chains

$$\overline{\mathcal{Z}}_N(\{ k = 0 \}, S, b, \epsilon) = \mathcal{Z}_N(S, b, \epsilon).$$

(2.4)

Let us note that these partition functions (2.3) are first normalized by the (infinite) Brownian partition functions. In place of dimensional regularization, one could introduce a minimal cut-off area $s_0$ between two interaction points along a same chain. Then the corresponding cut-off regularized partition function

$$^{+}\mathcal{Z}_N(\ldots, s_0)$$

reads in terms of the dimensionally regularized one $\mathcal{Z}_N$, in the limit $s_0 \to 0$

$$^{+}\mathcal{Z}_N(\ldots, s_0, \epsilon) = \exp \left[ NC \left( z_0 \right) S/s_0 \right] \times \mathcal{Z}_N(\ldots, \epsilon) \left|_{\text{dim. reg.}} \right.$$  

(2.5)

where $z_0 = (2\pi)^{-d} bS_0^{-d/2}$ is the small Zimm-Yamakawa parameter associated with the cut-off $s_0$, and where $C \left( z_0 \right)$ is a Taylor series in $z_0$. Thus one sees that in dimensional regularization, the power $\mu^{NS/s_0}$ of the « connectivity constant » $\mu = e^C \left( z_0 \right)$ has been already factorized out from the partition functions. This is important for the later definition of absolute critical amplitudes for dimensionally regularized polymer partition functions $\mathcal{Z}_N$.

The end-to-end mean square distance of a single chain is then given by

$$R^2 = dX^2 = -\frac{\partial^2}{\partial k^2} \overline{\mathcal{Z}}_1(k, -k, S, b, \epsilon) \bigg|_{k = 0}$$

where $X$ is the size of the chain. For dimensional reasons one has

$$X^2 = \mathcal{Z}_0(z, \epsilon) S.$$  

(2.6)

This end-to-end swelling $\mathcal{Z}_0$, with respect to the Brownian size of a chain, is given by a double Taylor expansion in $z$, and Laurent expansion in $\epsilon$, which reads to first orders

$$\mathcal{Z}_0(z, \epsilon) = 1 + z \left( \frac{2}{\epsilon} - 1 + \mathcal{O}(\epsilon) \right) + z^2 \left( -\frac{6}{\epsilon^2} + \frac{11}{2\epsilon} + \mathcal{O}(1) \right) + \cdots.$$  

(2.7)
In the asymptotic limit of Kuhnian chains, i.e. the limit of very long chains with excluded volume, the swelling $\mathcal{X}_0$ scales like

$$\mathcal{X}_0(z, \varepsilon) = \mathcal{X}_0(\varepsilon) \frac{z^{2\nu - 1}}{2\nu},$$

(2.8)

where $\nu$ is the usual critical index of the $n$-component field theory $(\varphi^2)^n$ in the limit $n = 0$ [14], given by the Wilson $\varepsilon$-expansion

$$2\nu - 1 = \frac{\varepsilon}{8} + \frac{15}{4} \left(\frac{\varepsilon}{8}\right)^2 + \cdots.$$

(2.9)

Here we shall calculate $\mathcal{X}_0(\varepsilon)$ starting simply from equation (2.7) and using renormalization theory.

In the same way, with the gyration square radius

$$R_G^2 = \frac{1}{2S^2} \int_0^L ds \int_0^L ds' \langle \| \mathbf{r}(s) - \mathbf{r}(s') \|^2 \rangle$$

(2.10)

is associated a gyration swelling

$$\mathcal{X}_G = 6 \frac{R_G^2}{ds},$$

(2.11)

defined with respect to the Brownian value $R_G^2 = ds/6$. We have calculated its double Laurent expansion and find

$$\mathcal{X}_G(z, \varepsilon) = 1 + z \left(\frac{2}{\varepsilon} - \frac{13}{12} + \mathcal{O}(\varepsilon) \right) + z^2 \left(-\frac{6}{\varepsilon^2} + \frac{6}{\varepsilon} + \mathcal{O}(1) \right) + \cdots.$$

(2.12)

Again, the gyration swelling of Kuhnian chains, i.e. in the limit $z \to \infty$, scales like

$$\mathcal{X}_G(z, \varepsilon) = \mathcal{X}_G(\varepsilon) \frac{z^{2\nu - 1}}{2\nu},$$

(2.13)

with the same index $\nu$ [13, 4, 6].

Let us finally consider the connected partition function $\mathcal{Z}_N$ of $N$ continuous polymer chains. In the tree-approximation, it reads explicitly

$$\mathcal{Z}_N^{(\text{tree})} = C_N (-bS^2)^{N-1},$$

(2.14)

where $C_N$ is the number of different tree-diagrams made of $N$ labelled chains, interacting with two-body interactions (Fig. 1). One has, in first orders

$$C_1 = 1, C_2 = 1, C_3 = 3, C_4 = 16,$$

(2.15)

and more generally, these tree-numbers can be calculated by means of a Legendre transform ($^3$)

($^3$) If one defines the generating function

$$A(x) = \sum_{N=1}^\infty C_N \frac{x^N}{N!},$$

its Legendre transform $B(y)$ defined by

$$A(x) + B(y) = xy, y = \frac{dA(x)}{dx}, x = \frac{dB(y)}{dy},$$

reads simply $B(y) = \frac{1}{2} (\ln y)^2$. Therefore one has

$$A(x) = \ln y - \frac{1}{2} (\ln y)^2, \quad xy = \ln y,$$

by series expansion in $x$ yields the set \( \{ C_N \} \), step by step.

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Fig. 1. — Tree-diagrams contributing to $\mathcal{Z}_4$ with their weights.
Now, the total connected partition function $Z_N$ can certainly be written as

$$Z_N = C_N (-bS^2)^{N-1} X_N(z, \epsilon)$$  \hspace{1cm} (2.16)

where the dimensionless function $X_N$ has a series expansion in powers of $z$ and is such that $X_N(z = 0, \epsilon) = 1$. We have recently studied these « renormalization » factors $X_N$ for general $N$ [15]. Their double Laurent expansion is

$$X_N(z, \epsilon) = 1 + \left( \frac{A_N}{\epsilon} + A_N' \epsilon + \mathcal{O}(\epsilon^0) \right) z + \left( \frac{B_N}{\epsilon^2} + \frac{B_N'}{\epsilon} + \mathcal{O}(1) \right) z^2 + \cdots.$$ \hspace{1cm} (2.17)

On the other hand, their Kuhnian scaling asymptotic form is

$$X_N(z, \epsilon) = A_N(\epsilon) z^{\nu_N} \hspace{1cm} (z \to \infty)$$ \hspace{1cm} (2.18)

where the critical exponent $\nu_N$ is [6]

$$\nu_N = \frac{2}{\epsilon} \left[ (\nu d - 2) (N - 1) + (\nu - 1) N \right],$$ \hspace{1cm} (2.19)

where $\gamma$ is the usual critical exponent of the partition function of a single chain. Indeed, owing to (2.16) (2.18) (2.19) one has identically

$$X_1(S, b, \epsilon) = X_1(z, \epsilon) = A_1(\epsilon) z^{(\gamma - 1) \frac{1}{2}} \hspace{1cm} (z \to \infty).$$ \hspace{1cm} (2.20)

The Wilson-Fisher $\epsilon$-expansion of $\gamma$-1 is

$$\gamma - 1 = \frac{e}{8} + \frac{13}{4} \left( \frac{e}{8} \right)^2 + \cdots.$$ \hspace{1cm} (2.21)

The result (2.19) is essential (for a proof, see [6] Appendix C). Using the latter, the $\epsilon$-expansions (2.9) (2.21), and our renormalization scheme [6], we have found [15] that the Laurent expansion (2.17) of the « renormalization » factors $X_N$, necessarily satisfies the equalities for $N \geq 1$

$$A_N = 4 - 2N, \hspace{1cm} B_N = 2(N - 2)(N + 2)$$ \hspace{1cm} (2.22)

and the relation

$$B_N' = 11N - 12 - (4 + 2N) A_N'.$$ \hspace{1cm} (2.23)

Thus we see that in this order, the expression (2.19) of the critical index $\nu_N$ determines all coefficients, except one, for instance $A_N'$. The latter must be calculated directly. For $N = 1, 2$ $X_1, X_2$ read [4]

$$X_1 = 1 + \left( \frac{2}{\epsilon} + 1 + \cdots \right) z + \left( \frac{6}{\epsilon^2} - \frac{7}{\epsilon} + \cdots \right) z^2 + \cdots$$ \hspace{1cm} (2.24)

$$X_2 = 1 + (1 + 4 \ln 2 + \cdots) z + \left( \frac{1}{\epsilon} (2 - 32 \ln 2) + \cdots \right) z^2 + \cdots$$ \hspace{1cm} (2.25)

in agreement with (2.22) (2.23). For $N = 3, 4$ we have calculated the missing coefficient $A_3' [15]. We found

$$A_3' = \frac{4}{3} - 8 \ln 2 + 9 \ln 3$$ \hspace{1cm} (2.26)

information which we shall use later.

$$A_4' = \frac{28}{15} + \frac{100}{3} \ln 2 - \frac{81}{5} \ln 3,$$ \hspace{1cm} (2.27)
3. Direct calculation of a critical amplitude.

Let us consider a general scaling function \( \mathcal{F}(z, \varepsilon) \) given by a double Laurent expansion

\[
\mathcal{F}(z, \varepsilon) = 1 + \left( \frac{A}{\varepsilon} + A' + \mathcal{O}(\varepsilon) \right) z + \left( \frac{B}{\varepsilon^2} + \frac{B'}{\varepsilon} + \mathcal{O}(1) \right) z^2 + \mathcal{O}(z^3/\varepsilon^3) \tag{3.1}
\]

which embraces the cases (2.7), (2.12), (2.17). Let us note that the case of a general function \( \mathcal{F}(z, \varepsilon) \) can always be reduced to form (3.1) by factorizing out its power law behaviour near the origin \( z \to 0 \). We assume it to have the asymptotic behaviour for \( z \) large

\[
\mathcal{F}(z, \varepsilon) \to \mathcal{A}(\varepsilon) z^\sigma(\varepsilon) + \ldots \quad \text{as} \quad z \to \infty \tag{3.2}
\]

First, the critical exponent \( \sigma \) can be determined by considering the logarithmic derivative

\[
\sigma(z, \varepsilon) = z \frac{\partial}{\partial z} \ln \mathcal{F}(z, \varepsilon) \bigg|_\varepsilon. \tag{3.3}
\]

One should notice that, owing to definition (2.2), \( \mathcal{F} \) scales in terms of the area \( S \), like \( S^{\sigma}(\varepsilon) \varepsilon^{\sigma(\varepsilon)/2} \). Owing to (3.1), \( \sigma(3.3) \) reads to the same order.

\[
\sigma(z, \varepsilon) = \sigma[z_\text{R}, \varepsilon] = \frac{1}{\varepsilon} \left( (A + A') \varepsilon \right) z_\text{R} + \left[ \left( 8 A + 2 B - A^2 \right) \frac{1}{\varepsilon} + 2 \left( 4 A' + B' - A A' \right) \right] z_\text{R}^2 + \ldots. \tag{3.7}
\]

Thus we find the identity

\[
8 A + 2 B - A^2 = 0, \tag{3.8}
\]

which is implied by the absence of poles in \( 1/\varepsilon \) in \( \varepsilon \sigma(z, \varepsilon) \). Equation (3.8) can be checked in all examples (2.7), (2.12), (2.24), (2.25).

Now, the asymptotic finite value of the index \( \sigma \) is found from the fixed point value of \( z_\text{R} \), which we have determined in [6]

\[
z_\text{R}^* = \frac{\varepsilon}{8} + \frac{17}{4} \left( \frac{\varepsilon}{8} \right)^2 + \ldots. \tag{3.9}
\]

Thus we have in all generality

\[
\sigma(\varepsilon) = \sigma[z_\text{R}^*, \varepsilon] = \frac{A}{8} + \frac{\varepsilon}{8} \left( \frac{17}{8} A + 8 A' + B' - A A' \right) + \ldots. \tag{3.10}
\]

\(^{(4)}\) It is \( \varepsilon \sigma[z_\text{R}, \varepsilon] \), i.e. the critical exponent associated with \( S \), which is regular in \( \varepsilon, z_\text{R} \), and not \( \sigma \), associated with \( z \) [6].
This relation (3.10) is important. So we see that in general the first order term of the $e$-expansion of $\sigma (e)$ is a constant, when $A \neq 0$. A particular situation occurs when $A = 0$, and $\sigma (e)$ is then of order $\mathcal{O} (e)$. This is precisely the case of the two-chain partition function $\mathcal{F}_2$, for which $A_2 = 0$ (see (2.22), (2.25)).

Knowing $\sigma (e)$, let us now turn to the determination of the critical amplitude $\mathcal{A} (e)$ appearing in (3.2). As we shall see below, a key point is then to define a function $h(z, e)$ such that

$$h(z, e) = \left[ \frac{z}{h(z, e)} \right]^{-\sigma (e)}$$

where $\sigma (e)$ is here the fixed point value (3.10) of the critical exponent $\sigma$, and not the effective exponent $\sigma (z, e)$. Thus $\sigma (e)$ is a pure number.

Owing to the definition (3.2) of the critical amplitude, we shall have

$$\mathcal{A} (e) = \lim_{z \to \infty} [h(z, e)]^{-\sigma (e)}$$

which shows that the limit $\lim_{z \to \infty} h(z, e)$ is finite. On the other hand, we have trivially

$$h(z, e) = z [\mathcal{F}(z, e)]^{1/\sigma}.$$  

Since $\mathcal{F}(z, e)$ has a Laurent series expansion (3.1) in $z$, and $\sigma$ is a pure number, $h(z, e)$ will also have a Laurent series expansion in $z$ of the form (3.1). Moreover, since it has a finite limit for $z \to \infty$, it will then be renormalizable [6] by the same minimal substitution $z \to z_R$ (3.6), i.e. all poles in $1/e$ will disappear. Actually this will hold true only if $\sigma (e) = \mathcal{O} (1)$ has a non vanishing limit for $e = 0$, which corresponds to a coefficient $A, A \neq 0$, in (3.1). In the opposite case $A = 0$, $\sigma (e) = \mathcal{O} (e^2)$, $h(z, e)$ does not have anymore the Laurent form (3.1), and one has to use a different method given below.

Let us calculate formally the $z$-series expansion of $h(z, e)$ from (3.1):

$$h(z, e) = z \left[ 1 - \frac{1}{\sigma} (A/e + A') + \left[ \frac{1}{\sigma} \left( \frac{1}{\sigma} + 1 \right) \frac{1}{2} \left( \frac{A^2}{e^2} + 2 \frac{AA'}{e} \right) - \frac{1}{\sigma} \left( \frac{B}{e^2} + \frac{B'}{e} \right) \right] z^2 + ... \right].$$  

Substituting now the renormalized variable $z_R$ (3.6) to $z$, we find

$$h[z_R, e] = z_R \left[ 1 + \left[ \frac{A}{e} - \frac{1}{\sigma} (A/e + A') \right] z_R + \frac{[64 - 17 - 16 \frac{1}{\sigma} (A/e + A')] z_R^2 + ... \right].$$

Now, we have to expand $\sigma (e)$ given by (3.10) in powers of $e$. Then two cases occur. If $A \neq 0$, then $\sigma (e) = A \mathcal{O} (e)$ (see (3.10)), and one immediately sees that the linear term $\mathcal{O} (z_R)$ in $h/z_R$ is pole free, as expected. On the contrary, if $A = 0$, then (3.10) gives $\sigma (e) = \mathcal{O} (e^2)$, and equation (3.15) is not renormalized. For instance the same term $\mathcal{O} (z_R)$ in $h/z_R$ has a pole in $1/e$, and the method breaks down.

So we consider first the general case :

a) $A \neq 0$

We expand $\sigma (e)$ (3.10) inside (3.15) and obtain after some algebra the double $(z_R, e)$ series expansion of $h$

$$h[z_R, e] = z_R \left[ 1 + z_R \left( \frac{17}{4} + 8 \frac{A'}{A} + \frac{2 B'}{A} - 2 A' \right) + \frac{2}{e^2} \frac{1}{e} + \mathcal{O} (1) \right] z_R^2 + ... \right].$$

Indeed a systematic calculation using (3.8) (3.10) shows that the full coefficient of $z_R^2$ in $h/z_R$ (Eq. (3.16)) has vanishing contributions to orders $\mathcal{O} \left( \frac{1}{e^2} \right)$ and $\mathcal{O} \left( \frac{1}{e} \right)$, and is finite when $e \to 0$. This was
expected from the very principles of direct dimensional renormalization [6].

Finally, the fixed point value of the variable $h$ is found from (3.9) (3.16) and reads

$$h(\varepsilon) = h(z \to \infty, \varepsilon) = h\left[ z_R, \varepsilon \right] =$$

$$= \frac{\varepsilon}{8} + \left( \frac{\varepsilon}{8} \right)^2 \mathcal{A} + \mathcal{O}(\varepsilon^3) \quad (3.17)$$

with

$$\mathcal{A} = \frac{17}{2} + \frac{8A'}{A} + \frac{2B'}{A} - 2A' \cdot \quad (3.18)$$

This expression is one of the main results of this article. Of course, it is valid only for $A \neq 0$.

Owing to equation (3.11), the function $\mathcal{X}(z)$ reads then asymptotically

$$\mathcal{X}(z, \varepsilon) = \left( \frac{z}{h(\varepsilon)} \right)^{\sigma(\varepsilon)} \quad (3.19)$$

and the critical amplitude is therefore

$$\mathcal{A}(\varepsilon) = [h(\varepsilon)]^{-\sigma(\varepsilon)} \quad (3.20)$$

where $\sigma(\varepsilon)$ is the critical exponent (3.10), whilst $h(\varepsilon)$ has the regular (asymptotic) $\varepsilon$-series expansion (3.17) (3.18). It is worth noting at this stage that the $z$-minimal subtraction scheme [6] (3.6), used here, leads to remarkably simple calculations. The renormalization schemes of [4], [5] do not renormalize $z$ minimally as in equation (3.6) and they would lead to lengthier calculations.

Clearly, the results yet obtained become singular when $A = 0$. For $A = 0$ the method must be amended in the following way.

b) $A = 0$

In this case, according to equation (3.8) one has also $B = 0$. Thus the function $\mathcal{X}(z)$ (3.1) has a double Laurent expansion of the form

$$\mathcal{X}(z, \varepsilon) = 1 + [A' + \mathcal{O}(\varepsilon)] z +$$

$$+ \left[ \frac{B'}{\varepsilon} + \mathcal{O}(1) \right] z^2 + \cdots. \quad (3.21)$$

Accordingly, equation (3.10) shows that the associated critical index $\sigma$ reads

$$\sigma(\varepsilon) = \frac{\varepsilon}{8} \left( 8A' + B' \right) + \mathcal{O}(\varepsilon^2). \quad (3.22)$$

Therefore, we can write $\sigma$ under the form

$$\sigma(\varepsilon) = \varepsilon \sigma'(\varepsilon)$$

$$\sigma'(\varepsilon) = \frac{1}{32} \left( 8A' + B' \right) + \mathcal{O}(\varepsilon). \quad (3.23)$$

Thus

$$\sigma'(\varepsilon) = \mathcal{O}(1) \quad (3.24a)$$

provided that

$$8A' + B' \neq 0. \quad (3.24b)$$

Then, in this case, the solution to the determination of the critical amplitude is found by setting, in place of (3.11)

$$\mathcal{X}(z, \varepsilon) = \left[ \frac{z}{h'(z, \varepsilon)} \right]^{\sigma'(\varepsilon)}, \quad (3.25)$$

which defines a new function $h'(z, \varepsilon)$. The critical amplitude of $\mathcal{X}$ will be the finite limit

$$\mathcal{A}(\varepsilon) = \lim_{z \to \infty} [h'(z, \varepsilon)]^{-\sigma'}. \quad (3.26)$$

Moreover the choice of $h'$ is such that the latter has a renormalizable double Laurent expansion in $z$ and $\varepsilon$. One has indeed

$$h'(z, \varepsilon) = z^{\varepsilon} [\mathcal{X}(z, \varepsilon)]^{-1/\sigma'} \quad (3.27)$$

and since $\sigma'$ is a pure number of order $\mathcal{O}(1)$, one can expand $h'$ in a Laurent expansion of the form (3.21). We have, owing to (3.21)

$$h'(z, \varepsilon) = z^\varepsilon \left\{ 1 - \frac{A'}{\sigma'} z - \left[ \frac{B'}{\sigma'} \frac{1}{\varepsilon} + \mathcal{O}(1) \right] z^2 + \cdots \right\}. \quad (3.28)$$

We substitute now the fundamental minimal relation (3.6) giving $z$ in terms of $z_R$, and we find

$$h'[z_R, \varepsilon] = z_R^\varepsilon \left\{ 1 + \left( 8 - \frac{A'}{\sigma'} \right) z_R + \left[ \frac{32 - 8A' + B'}{\sigma'} \frac{1}{\varepsilon} + \mathcal{O}(1) \right] z_R^2 + \cdots \right\}. \quad (3.29)$$

We have now to expand $\sigma'$ in series of $\varepsilon$, using (3.23), and we find

$$h'[z_R, \varepsilon] = z_R^\varepsilon \left\{ 1 + 8 z_R \left( 1 - \frac{4A'}{8A' + B'} + \mathcal{O}(\varepsilon) \right) + \left[ \frac{0}{\varepsilon} + \mathcal{O}(1) \right] z_R^2 + \cdots \right\}. \quad (3.30)$$
Thus, as expected, the pole in \(1/\epsilon\) present in the double Laurent expansion in \(z, \epsilon\), disappears from the double (Taylor) expansion in \(z^R, \epsilon\). Now, the fixed point value of \(h'\) is obtained from (3.9) and the critical amplitude is given by

\[
A'(\epsilon) = [h'(\epsilon)]^{-\sigma'(\epsilon)}
\]

where we recall that \(\sigma'(\epsilon) = \frac{\sigma(\epsilon)}{\epsilon} = \frac{8A' + B'}{32} + O(\epsilon)\). This gives the \(\epsilon\)-expansion of the critical amplitude in the particular case \(A = 0, 8A' + B' \neq 0\). One notices that here a power \(\epsilon^{\sigma(1)}\) appears in \(A\), in contrast to the general result (3.17) (3.20) where a power \(\epsilon^{\sigma(1)}\) appeared.

Let us stress that the particular result (3.31) (3.32) holds only for \(A = 0, 8A' + B' \neq 0\). If, furthermore \(8A' + B' = 0\), then one has to perform another calculation, since then the critical index \(\sigma'\) (3.23) becomes of order \(\epsilon\) at least \(\sigma'(\epsilon) = O(\epsilon)\). More generally, consider a diverging function \(X(z, \epsilon)\), whose critical exponent \(\sigma\) has a Wilson-Fisher \(\epsilon\)-expansion

\[
\sigma(\epsilon) = \epsilon^n \sigma(n) (\epsilon),
\]

\[
\sigma(n) (\epsilon) = O(1)
\]

This occurs for instance if the double \((\epsilon, z)\) expansion of \(X(z, \epsilon)\) starts like

\[
X(z, \epsilon) = 1 + z[A^{[n]} \epsilon^{n-1} + O(\epsilon^n)] + O(z^2)
\]

\[
(n \neq 0, A^{[n]} \neq 0)
\]

(3.34)

(note that poles in \(1/\epsilon\) appear at order \(z^{n+1}\) only). This form (3.34) is not the only possibility: for instance for \(A = 0, 8A' + B' = 0\), one is in the case \(n = 1\), even if \(A' \neq 0, B' \neq 0\).

Then we shall set

\[
X(z, \epsilon) = \left[ z^{\sigma}/h^{[n]}(z, \epsilon) \right]^{\sigma(n)}
\]

and \(h^{[n]}(z, \epsilon) z^{-\sigma}\) will have a renormalizable \((z, \epsilon)\) Laurent expansion, whose fixed point value will be calculated with \(z_{\epsilon}^{[n]}\) as above. One will find necessarily at the fixed point

\[
h^{[n]}(\epsilon) = \left( \frac{\epsilon}{8} \right)^{\sigma(n)} \left[ 1 + \beta^{[n]}(\epsilon) \right]
\]

where \(\beta^{[n]}\) is a pure number.

Let us finally write these results under the general form (use (3.33) (3.35) (3.36))

\[
X(z, \epsilon) = \left( \frac{8}{\epsilon} \right)^{\sigma(\epsilon)} \times
\]

\[
1 - \beta^{[n]} + \frac{4A'}{8A' + B'}
\]

(3.37)

where \(\beta^{[n]}(0)\) is the first non vanishing coefficient of the \(\epsilon\)-expansion of \(\sigma(\epsilon)\)

\[
\sigma(\epsilon) = \sigma(n) (\epsilon) \epsilon^n + O(\epsilon^{n+1}),
\]

\[
\sigma(\epsilon) = \sigma(n) (\epsilon) \epsilon^n + O(\epsilon^{n+1}),
\]

\[
\sigma(n) (\epsilon) = O(1)
\]

(3.38)

In particular, our results (3.10) (3.17) (3.18) and (3.31) correspond to the cases \(n = 0, n = 1\) respectively, and for the latter

\[
\begin{align*}
\sigma(\epsilon) &= A + \epsilon \left( \frac{17}{8} A + 8A' + B' - AA' \right) + \cdots \\
\sigma(n) (0) &= A + \epsilon \left( \frac{17}{8} A + 8A' + B' - AA' \right) + \cdots \\
\beta^{[n]}(0) &= \frac{17}{2} A + \frac{2B'}{A} - 2A' \\
\beta^{[n]}(0) &= \frac{17}{2} A + \frac{2B'}{A} - 2A'
\end{align*}
\]

(3.39)

(3.40)

In the Appendix, we give the formal general next order result for a critical amplitude, for \(A = 0, \beta(\epsilon) = [h(\epsilon)]^{-\sigma}, \) with \(h(\epsilon)\) calculated to third order \(O(\epsilon^3), \) or for \(A = 0, \beta(\epsilon) = [h(\epsilon)]^{-\sigma}, \) with \(h(\epsilon)\) calculated to second order \(O(\epsilon^2), \)

We note in (3.37) that \(z\) enters in a critical asymptotic function only in the combination \(\frac{8z}{\epsilon}, \) for \(\epsilon > 0.\) This strongly suggests that, in exactly four dimensions, \(\epsilon = 0, \) corresponding powers of logarithms will appear [15]. Indeed one has (see (2.2))

\[
\frac{2z}{\epsilon} \sim \frac{S_{\epsilon z^2}}{(\epsilon/2)}
\]

which can be (formally) replaced by

\[
\lim_{\epsilon \to 0} \left( \frac{2S_{\epsilon z^2}}{\epsilon - \epsilon} \right) = \ln S.
\]

Then, one expects, for ins-
tance in the case $n = 0$ (3.39), where $\sigma^{[0]}(\epsilon = 0) = A/8$, that the quantity $\mathcal{X}(S)$ diverges logarithmically in four dimensions like

$$\mathcal{X}(S) = (4 \ln S) \sigma^{[0]}(\epsilon) \times \sigma^{[0]}(0) = A/8$$

(3.41)

and this is indeed true, as we have shown in detail in [15].

Let us also mention that the method we have proposed here for polymers, works also for critical amplitudes in general phase transitions (described by a $\varphi^4$ field theory). Indeed, as we already stated in [6], we define then an analog of parameter $z : z' = gm^{-\epsilon}$, where $g$ is the coupling constant of $\varphi^4$, and $m$ the physical mass of the theory. This $z'$ can be renormalized minimally to a $z'_R$ (see appendix A in [6]). Hence, the critical amplitude of a dimensionless function $\mathcal{X}(z', \epsilon)$ could be calculated exactly in the same way as above. However, this method, as the minimal renormalization of $z' = gm^{-\epsilon}$, have not appeared elsewhere [6].

4. Applications

4.1 END-TO-END AND GYRATION SWELLINGS. —

We apply now our results (3.17) (3.19) or (3.31) (3.32) to the various quantities $\mathcal{X}_0, \mathcal{X}_G, \mathcal{X}_N$, characterizing polymer chains and described in section 2. The double Laurent expansion of the end-to-end swelling $\mathcal{X}_0(z, \epsilon)$ is given by (2.7), and thus

$$A_0 = 2, A'_0 = -1, B_0 = -6, B'_0 = 11/2.$$ 

Therefore, owing to (3.17) (3.19) $\mathcal{X}_0$ asymptotically reads

$$\mathcal{X}_0(z, \epsilon) = \left( \frac{z}{h_0(\epsilon)} \right)^{\sigma_0}$$

(4.1)

with

$$h_0(\epsilon) = \frac{\epsilon}{8} \left( 1 + 13 \frac{\epsilon}{8} + \cdots \right)$$

(4.2)

$$\sigma_0(\epsilon) = \frac{2}{\epsilon} \left( 2 \nu - 1 \right) = \frac{1}{4} \left( 1 + 15 \frac{\epsilon}{4} + \cdots \right).$$

(4.3)

The critical amplitude itself can be written as

$$\mathcal{A}_0(\epsilon) = \left[ h_0(\epsilon) \right]^{-\sigma_0} = \left( \frac{\epsilon}{8} \right)^{-\sigma_0} \left( 1 - 3 \frac{\epsilon}{8} + \cdots \right).$$

(4.4)

and therefore, for $\epsilon$ small, it starts like

$$\mathcal{A}_0(\epsilon) = \left( \frac{\epsilon}{8} \right)^{-114} + \cdots$$

(4.5)

In a similar way, the gyration swelling $\mathcal{X}_G$ is given by the double Laurent expansion (2.12), hence


Therefore $\mathcal{X}_G$ has the asymptotic form

$$\mathcal{X}_G(z, \epsilon) = \left( \frac{z}{h_G(\epsilon)} \right)^{\sigma_0}$$

(4.6)

with

$$h_G(\epsilon) = \frac{\epsilon}{8} \left( 1 + \frac{37 \epsilon}{3} + \cdots \right)$$

(4.7)

and where, of course, $\sigma_0$ is the same critical exponent as for $\mathcal{X}_0$, i.e. $\sigma_0 = \frac{2}{\epsilon} \left( 2 \nu - 1 \right)$. The critical
amplitude of the gyration swelling is therefore
\[ A_G(\epsilon) = \left[ h_G(\epsilon) \right]^{-\sigma_0} = \left( \frac{\epsilon}{\bar{\theta}} \right)^{-\sigma_0} \left( 1 - \frac{37 \epsilon}{128} + \cdots \right). \] (4.8)

Incidentally, we note that the universal ratio \[ \Xi = 6 \frac{R_G^2}{\bar{R}^2} = \frac{A_G}{A_0} \] takes asymptotically the value (use (4.4) (4.8))
\[ \Xi = A_G(\epsilon) / A_0(\epsilon) = 1 - \frac{\epsilon}{96} + \cdots, \] (4.10)
in agreement with a well-known result [13].

It is also interesting to compare our « exact » results (4.1) (4.2) for the critical amplitude of the swelling \( \mathcal{A}_0 \), to a numerical evaluation of the latter, which can be found in [10]. The result reads there, for large \( z \), and in three dimensions
\[ \ln \mathcal{A}_0(z) = 0.3538 \ln z + 0.4278 + \mathcal{O}(z^{-0.9468}). \] (4.11)
Here (4.1) reads
\[ \ln \mathcal{A}_0(z, \epsilon) = \frac{2}{\epsilon} (2 \nu - 1) \ln z - \frac{2}{\epsilon} (2 \nu - 1) \ln h_0(\epsilon) + \cdots, \] (4.12)
and for \( \epsilon = 1 \), (4.2) gives at this order
\[ h_0(\epsilon = 1) = 5/16. \] (4.13)

If we use the « exact » value [17] in three dimensions
\[ \nu = 0.588 \] (4.14)
which actually corresponds to the coefficient \( 2 (2 \nu - 1) = 0.3538 \) of equation (4.11), we find from (4.12) (4.13) the approximation
\[ \ln \mathcal{A}_0(z, \epsilon = 1) = 0.3538 \ln z + 0.4094, \] (4.15)
which compares quite well with (4.11). This agreement is interesting, since the asymptotic form (4.11) was numerically calculated in [10] by using the series expansion of \( \mathcal{A}_0(z) \) to \( \mathcal{O}(z^6) \) in exactly three dimensions, calculated by Muthukumar and Nickel [11]. So the information used there was quite different from the double Laurent expansion (2.7), which we started from in this article. This suggests, more generally, that the critical coefficients \( h(\epsilon) \), calculated to order \( \epsilon^2 \) here, give reasonable estimates of the critical amplitudes. The agreement is even more striking, if we consider the constant term in \( \ln \mathcal{A}_0(z) \) (4.12)
\[ C_0 = -\frac{2}{\epsilon} (2 \nu - 1) \ln h_0(\epsilon). \] (4.16)
If we use the second order approximation (4.3) for \( \sigma_0 = \frac{2}{\epsilon} (2 \nu - 1) \), we find in a crude way \( \sigma_0(\epsilon = 1) = 47/128 \), and our formula (4.13) (4.16) yields for \( d = 3 \)
\[ C_0 = 0.4271. \] (4.17)
Thus the constant term \( C_0 \) agrees with that of (4.11) within an accuracy of order \( 10^{-3} \) ! Therefore one could expect in general that critical amplitudes \( A(\epsilon) = [h(\epsilon)]^{-\sigma(\epsilon)} \), evaluated with the \( \epsilon \)-expansions of \( h(\epsilon) \) and \( \sigma(\epsilon) \) up to order \( \mathcal{O}(\epsilon^2) \), give good theoretical predictions, even for the physical three-dimensional space.

4.2 PARTITION FUNCTIONS. — Let us now turn to the general connected partition functions \( \mathcal{Z}_N \) of \( N \) chains \( (N > 1) \), considered in section 2, equation (2.16). They read
\[ \mathcal{Z}_N(b, S, \epsilon) = C_N \left(-bS^2\right)^{N-1} \mathcal{Z}_N(z, \epsilon) \] (4.18)
with asymptotically for \( z \) large

\[
\xi_N(z, \varepsilon) = \mathcal{A}_N(\varepsilon) z^{\sigma_N} \quad (z \to \infty)
\]

\[
\sigma_N = \frac{2}{\varepsilon} \left[ \left( \nu d - 2 \right) (N - 1) + (\gamma - 1) N \right].
\]

Owing to the \( \varepsilon \)-expansions (2.9) (2.21) of \( \nu \) and \( \gamma \), one has identically

\[
\sigma_N = \frac{1}{4} (2 - N) + \varepsilon \frac{1}{16} \left( 27 N - 14 \right) + \mathcal{O} \left( \varepsilon^3 \right).
\]

Thus for \( N \neq 2 \), we can apply the general result (3.17) (3.20) to get the critical amplitude:

\[
\mathcal{A}_N(\varepsilon) = \left[ h_N(\varepsilon) \right]^{-\sigma_N}
\]

with

\[
h_N(\varepsilon) = \frac{\varepsilon}{8} + \mathcal{B}_N \left( \frac{\varepsilon}{8} \right)^2 + \mathcal{O} \left( \varepsilon^3 \right)
\]

\[
\mathcal{B}_N = \frac{17}{2} + \frac{8 A_N}{A_N} + \frac{2 B_N}{A_N} - 2 A_N
\]

A simpler form of \( \mathcal{B}_N \) can be found by using the exact value of \( A_N \) (2.22), as well as the general relation (2.23). This gives after calculation the general result

\[
h_N(\varepsilon) = \frac{\varepsilon}{8} \left[ 1 + \left( \frac{5}{2} + \frac{4 A_N - 10}{N - 2} \right) \varepsilon + \mathcal{O} \left( \varepsilon^2 \right) \right]
\]

valid for any \( N, N \neq 2 \). Before applying these results to actual values of \( N \), let us consider the last pending case, where \( A_N = 0 \), i.e. the case \( N = 2 \).

Here, the critical exponent \( \sigma_2 \) (4.20) reads

\[
\sigma_2 = \frac{2}{\varepsilon} \left[ \left( \nu d - 2 \right) + 2 (\gamma - 1) \right].
\]

Its \( \varepsilon \)-expansion is, owing to (4.21)

\[
\sigma_2 = \frac{5}{16} \varepsilon + \mathcal{O} \left( \varepsilon^2 \right).
\]

The general relation (2.23) yields

\[
B_2 = 10 - 8 A_2.
\]

Hence, applying now (3.31), (3.32), which gives the critical amplitude in the case \( A = 0 \), we get

\[
\mathcal{A}_2(\varepsilon) = \left[ h_2(\varepsilon) \right]^{-\sigma_2}
\]

with

\[
h_2(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^4 \left[ 1 + \varepsilon \left( 1 - \frac{2}{5} A_2 \right) + \mathcal{O} \left( \varepsilon^2 \right) \right]
\]

valid for \( N, N \neq 2 \). Before applying these results to actual values of \( N \), let us consider the last pending case, where \( A_N = 0 \), i.e. the case \( N = 2 \).
or, owing to (4.21)

\[
\mathcal{A}_N(\varepsilon) \bigg|_{N=1} = \left( \frac{\varepsilon}{8} \right)^{-\sigma_N} \left\{ 1 + \left[ -\frac{5}{8} (N-2) + A'_N - \frac{10}{4} \right] \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right\}
\]  

(4.31)

\[
\sigma_N = \frac{2}{\varepsilon} \left[ (\nu d - 2) (N-1) + (\gamma - 1) N \right]
\]

\[
= \frac{1}{4} (2 - N) + \frac{\varepsilon}{8} \left[ \frac{1}{16} (27 N - 14) + \mathcal{O}(\varepsilon^2) \right].
\]

This expression is now regular when \(N = 2\), and it also gives the correct answer in this case, in agreement with equation (3.37). Indeed, equations (4.29) (4.30), transformed with the help of (4.27), coincide with (4.31) for \(N = 2\).

Let us stress the simplicity of these results, which are complete two-loop calculations. One has only to calculate explicitly the coefficient \(A'_N\), i.e. a one-loop term (see (2.17)). This is because we used the non trivial information [15] contained in the \(\varepsilon\)-expansion of indices \(\sigma_N\) to two-loop order, which in turn gave us relations (2.22), (2.23).

Explicit results for \(N = 1, 2, 3, 4\) are the following. For \(N = 1\), \(A'_1 = 1\) (see (2.24)), and (4.20), (4.22), (4.25) give

\[
\mathcal{A}_1(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{-\sigma_1} \left[ 1 + \frac{7}{8} \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right]
\]

(4.32)

\[
h_1(\varepsilon) = \frac{\varepsilon}{8} \left[ 1 + \frac{7}{2} \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right]
\]

(4.33)

\[
\sigma_1 = \frac{2}{\varepsilon} (\gamma - 1) = \frac{1}{4} \left[ 1 + \frac{13}{4} \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right].
\]

(4.34)

Using (4.31) gives the expanded result

\[
\mathcal{A}_1(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{-\sigma_1} \left[ 1 + \frac{7}{8} \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right]
\]

(4.35)

For \(N = 2\), we have \(A'_2 = 1 + 4 \ln 2\) (see (2.25)). Therefore, using equation (4.29) (4.30), we obtain

\[
\mathcal{A}_2(\varepsilon) = \left[ h_2(\varepsilon) \right]^{-\sigma_2/\varepsilon}
\]

(4.36)

\[
h'_2(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{\varepsilon} \left[ 1 + \varepsilon \left( \frac{3}{5} - \frac{8}{5} \ln 2 \right) + \mathcal{O}(\varepsilon^2) \right]
\]

(4.37)

\[
\sigma_2 = \frac{2}{\varepsilon} \left[ (\nu d - 2) + 2 (\gamma - 1) \right] = \frac{5}{16} \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2).
\]

(4.38)

The totally expanded result (4.31) gives for \(N = 2\)

\[
\mathcal{A}_2(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{-\sigma_2} \left[ 1 + \left( 4 \ln 2 - \frac{3}{2} \right) \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right]
\]

(4.39)

Finally, we know [15] the next order values of \(A'_N\) for \(N = 3, 4\), given in (2.26) (2.27). We therefore have for \(N = 3\)

\[
\mathcal{A}_3(\varepsilon) = \left[ h_3(\varepsilon) \right]^{-\sigma_3}
\]

\[
h_3(\varepsilon) = \frac{\varepsilon}{8} \left[ 1 + \left( \frac{-43}{6} - 32 \ln 2 + 36 \ln 3 \right) \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right] \approx \frac{\varepsilon}{8} (1 + 1.275 \varepsilon + \cdots)
\]

\[
\sigma_3 = \frac{2}{\varepsilon} \left[ 2 (\nu d - 2) + 3 (\gamma - 1) \right] = \frac{1}{4} \left( 1 - \frac{67}{4} \frac{\varepsilon}{8} + \cdots \right)
\]
or using (4.31), the expanded result
\[ \mathcal{A}_2(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{-\sigma_4} \left[ 1 + \frac{1}{4} \left( -\frac{43}{6} - 32 \ln 2 + 36 \ln 3 \right) \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right] \approx \left( \frac{\varepsilon}{8} \right)^{-\sigma_4} (1 + 0.336 \varepsilon + \ldots) . \] (4.40)

For \( N = 4 \), we find
\[ \mathcal{A}_4(\varepsilon) = \left[ h_4(\varepsilon) \right]^{-\sigma_4} \]
\[ h_4(\varepsilon) = \frac{\varepsilon}{8} \left[ 1 + \left( -\frac{113}{30} + \frac{200}{3} \ln 2 - \frac{162}{5} \ln 3 \right) \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right] \]
\[ \sigma_4(\varepsilon) = \frac{2}{\varepsilon} \left[ 3(\nu d - 2) + 4(\gamma - 1) \right] \]
\[ = -\frac{1}{2} \left( 1 - \frac{47}{4} \frac{\varepsilon}{8} + \ldots \right) . \]

Finally, expanding \( \sigma_4 \) yields
\[ \mathcal{A}_4(\varepsilon) = \left( \frac{\varepsilon}{8} \right)^{-\sigma_4} \left[ 1 + \frac{1}{2} \left( -\frac{113}{30} + \frac{200}{3} \ln 2 - \frac{162}{5} \ln 3 \right) \frac{\varepsilon}{8} + \mathcal{O}(\varepsilon^2) \right] \]
\[ \approx \left( \frac{\varepsilon}{8} \right)^{-\sigma_4} (1 + 0.428 \varepsilon + \ldots) . \] (4.41)

### 4.3 Virial Expansion of the Osmotic Pressure.

The osmotic pressure \( \Pi \) of a polymer solution is given, in grand canonical formalism, by the parametric representation
\[ \Pi \beta = \sum_{N=1} f^N \mathcal{Z}_N / N ! . \] (4.42)
\[ \mathcal{C} = f \frac{\partial}{\partial f} (\Pi \beta) \]

where \( f \) is the fugacity, conjugated to the number \( C \) of polymer chains per unit volume. The virial expansion of the osmotic pressure can then be written from dimensional analysis, as
\[ \Pi \beta / C = 1 + \sum_{n>1} \frac{g_{n+1}}{n} \left[ \mathcal{C} \left( 2\pi X^2 \right)^{d/2} \right]^n , \] (4.43)

where the dimensionless virial coefficients \( g_n, n \geq 2 \), are calculable in terms of the partition functions. In particular the dimensionless second virial coefficient \( g_2 = g \) reads explicitly \[ g = - \left( 2\pi X^2 \right)^{-d/2} \frac{\mathcal{Z}_2}{\left( \mathcal{Z}_1 \right)^{1/2}} . \] (4.44)

Using definitions (2.2), (2.6), (2.16) we may write it as
\[ g = 2\mathcal{Z}_0^{-d/2} \mathcal{Z}_2 \mathcal{Z}_1^{-1/2} . \] (4.45)

Therefore its fixed point value \( g^* \) for \( z \to \infty \), is given by the asymptotic limits (2.8), (2.18), (2.19)
\[ g^*(\varepsilon) = \mathcal{A}_0^{-d/2}(\varepsilon) \mathcal{A}_2(\varepsilon) \mathcal{A}_1^{-2}(\varepsilon) \] (4.46)

and it is therefore a universal ratio of critical amplitudes. Using (4.4), (4.35), (4.39) we find the second order value
\[ g^*(\varepsilon) = \frac{\varepsilon}{8} \left[ 1 + \left( \frac{25}{4} + 4 \ln 2 \right) \frac{\varepsilon}{8} + \ldots \right] , \] (4.47)

in complete agreement with the value already calculated in [4, 6] by quite different methods. This is of course a consistency check of our expressions for the critical amplitudes.

Numerically one has in \( d = 3, \varepsilon = 1 \), from (4.47) \( g^* \approx 0.266 \) [4]. We can find a different approximation if we use, instead of the fully \( \varepsilon \)-expanded expressions of the critical amplitudes, the expressions (4.4), (4.32), (4.36), and get
\[ g^* = h_0^{\sigma d/2} \left( h_2^* \right)^{-\sigma_2/\varepsilon} h_1^{2\sigma_1} . \] (4.48)

For critical indices \( \sigma_0 (4.3), \sigma_1, \sigma_2 (4.20) \) we may use the best known values of \( \nu, \gamma \) in 3d [17], \( \nu = 0.588, \gamma = 1.160 \), whilst \( h_0, h_1, h_2^* \) are calculated from their \( \varepsilon \)-expansions (4.2), (4.33), (4.37). Then we find \( g^* = 0.290 \). This value is probably too high [4]. We may find another \( \mathcal{O}(\varepsilon^2) \) approximation by using in (4.48) the second order \( \varepsilon \)-expansions of \( h_0, h_1, h_2^* \), and (4.3), (4.34) of \( d\sigma_0, \sigma_1, \sigma_2 \). For \( \sigma_2 \) we
keep the best known value, since its $\varepsilon$-expansion (4.38) has only one significant term. Then we find
\[ g^* = 0.237 , \]
which compares well to the approximate value $g^* = 0.233$ obtained in [4] by a quite different interpolation method. Let us note that the experimental universal value seems to be [18] $g^*_{\text{exp}} = 0.206$.

Let us return to the virial expansion (4.43) of the osmotic pressure. Since we have evaluated above the asymptotic partition functions $\mathcal{Z}_N$, $N = 1, \ldots, 4$, as well as $X^2$, we can obtain the virial coefficients $g^* = g_2, g_3, g_4$. The result, which we have calculated in [15], reads
\[ g_2 = \frac{\varepsilon}{8} + \left( \frac{25}{4} + 4 \ln 2 \right) \left( \frac{e}{8} \right)^2 = 0.266 \ (d = 3) \]
\[ g_3 = (-1 + 48 \ln 2 - 27 \ln 3) \left( \frac{e}{8} \right)^3 \approx 0.0051 \ (d = 3) \]
\[ g_4 = \left( \frac{14}{15} + \frac{5}{3} \ln 2 - \frac{2}{5} 3^6 \ln 3 \right) \left( \frac{e}{8} \right)^4 \]

As already noted in [15], the universal fourth virial coefficient is negative at this order in $\varepsilon$, and this is probably true also for $d = 3$.

4.4 ENTROPY OF A SINGLE CHAIN. — Consider the entropy of a long polymer chain, within the framework of the continuous model. We have found for the dimensionally regularized partition function (2.20), (4.35) the asymptotic behaviour
\[ \mathcal{Z}_1 (z, \varepsilon) = \left( z/h_1 \right)^{(\gamma - 1) \varepsilon} \]
\[ z \to \infty \]
\[ h_1 = \frac{\varepsilon}{8} \left( 1 + \frac{7}{2} \varepsilon + \mathcal{O} \left( \varepsilon^2 \right) \right) . \]

Now the full partition function, regularized with a minimal interaction area $s_0$, reads, owing to (2.5)
\[ + \mathcal{Z}_1 (b, S, s_0, \varepsilon) = e^{C (s_0) S/s_0} \mathcal{Z}_1 (z, \varepsilon) \]

with $z_0 = (2 \pi)^{-d/2} b s_0^{-d/2}$, $z = (2 \pi)^{-d/2} b S^2 S^{-d/2}$. The entropy is calculated in a standard way [3, 4], for the continuous model:
\[ + \mathscr{S} = \left( S \frac{\partial}{\partial S} + s_0 \frac{\partial}{s_0} - 3 b \frac{\partial}{\partial b} + 1 \right) \ln + \mathcal{Z}_1 \]
\[ \text{(Note that the elastic energy is included here). Equation (4.51) gives} \]
\[ + \mathscr{S} = \left[ 1 + \left( \frac{e}{2} - 3 \right) z_0 \frac{\partial}{\partial z_0} \right] C (z_0) S/s_0 + \left[ 1 + \left( \frac{e}{2} - 3 \right) z \frac{\partial}{\partial z} \right] \ln \mathcal{Z}_1 (z, \varepsilon) . \]

The first term yields the extensive part of the entropy, proportional to Brownian size $S$ of the chain. The second one, associated with the dimensionally regularized part $\mathcal{Z}_1$, gives a non extensive correction $\mathscr{S}$, and is independent of $s_0$. According to (4.50), we find for this non extensive contribution
\[ \mathscr{S} = \left( \frac{e}{2} - 3 \right) \frac{2}{e} (\gamma - 1) + \frac{2}{\gamma} (\gamma - 1) \ln \left( z/h_1 \right) . \]

The first term in (4.53) is exactly the internal energy of the chain, whilst the second is the free energy term. Let us insert in (4.53) approximation (4.33) for $h_1$:
\[ \mathscr{S} = \frac{2}{4 - d} (\gamma - 1) \left[ - \left( 1 + \frac{d}{2} \right) + \ln \left( \frac{8 z}{e} \left( 1 + \frac{7}{2} \frac{e}{8} + \mathcal{O} (e^2) \right) \right) \right] . \]

If furthermore, we use the $\varepsilon$-expansion (4.34) of $\gamma$, we arrive at the fully expanded result
\[ \mathscr{S} = \frac{2}{4 - d} (\gamma - 1) \ln \left( z/(4 - d) \right) + \frac{3}{4} (\ln 2 - 1) + (45 \ln 2 - 43) \frac{e}{2} . \]
Incidentally, this refines a result of [4] where the constant term were not calculated. We also note that Elderfield [2] calculated an entropy of a single chain defined by (in our notations)

\[ + \mathcal{F} = \left( 1 - b \frac{\partial}{\partial b} \right) \ln \mathcal{F}_1, \]  

which discards the elastic energy. Using (4.51) we find

\[ + \mathcal{F} = \left[ 1 - z_0 \frac{\partial}{\partial z_0} \right] C(z_0) S/z_0 + \mathcal{F} \]  

\[ \mathcal{F} = \left( 1 - b \frac{\partial}{\partial b} \right) \ln \mathcal{F}_1, \]  

where \( \mathcal{F} \) is the dimensionally regularized entropy corresponding to \( + \mathcal{F} \). Using equation (4.50) we find immediately

\[ \mathcal{F} = \frac{2}{\varepsilon} (\gamma - 1) \ln \left( z/h_1 \right) - \frac{2}{\varepsilon} (\gamma - 1). \]  

We may rewrite \( \mathcal{F} \) in terms of the swelling factor \( \mathcal{X}_0 \) (4.1). We obtain, using (4.2) (4.33)

\[ \mathcal{F} = (\gamma - 1) \left[ \ln \mathcal{X}_0^{2^{-1}} - \frac{2}{\varepsilon} \left( 1 - \frac{17}{16} \varepsilon \right) \right]. \]

We note that the result of Elderfield [2] for his subtracted entropy \( \mathcal{S} \), which should correspond to our dimensionally regularized entropy \( \mathcal{F} \), differs slightly from the latter. We have not yet found the origin of the difference. However, we note that our calculation uses only the values (4.2), (4.33) of \( h_0(\varepsilon) \) and \( h_1(\varepsilon) \) for obtaining (4.60). Now, these values (4.2) (4.33) for \( h_0, h_1 \) have already found a confirmation when they were used for recovering independently the known correct value \( g^* \) of the second virial coefficient \( g \) in section (4.3). This strengthens result (4.60).

5. Conclusion.

We have presented here a simple method for calculating the \( \varepsilon \)-expansion of any critical amplitude associated with the continuum polymer model. A critical amplitude was defined here to be a pure number, depending only on the space dimension \( d \), i.e. on \( \varepsilon = 4 - d \), and it is a non universal feature of the continuum model. We have given its explicit expression up to two-loop order (and formally to three-loop order). The « \( \varepsilon \)-dimensional renormalization method », which we previously introduced, and which renormalizes minimally Yamakawa’s parameter \( \varepsilon \), appeared to be most convenient for our purpose. A set of new results was found in this way, concerning the various swellings of a continuous interacting polymer chain, its exact entropy, and the asymptotic multichain partition functions. The method presented here is quite general, and could be applied to any model of continuous chains, like the three-parameter \( \theta \)-point model below three dimensions, or the continuous model of branched polymers. It should also be applicable to critical amplitudes of phase transitions.

Appendix.

THIRD ORDER CALCULATION OF THE CRITICAL AMPLEITUDES. — So far, we have given the results for a critical amplitude \( \mathcal{A} = h^{-\sigma} \), under the form of the \( \varepsilon \)-expansion of \( h \) to second order. But we already considered a part of the third order calculation, when we checked that the terms of order \( \mathcal{O} \left( \frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^3} \right) \) disappeared in the renormalized expression of \( h \). However, we did not calculate the constant coefficient of \( z_R^3 \), or terms of order \( \varepsilon^3 \). We give here a short account of this calculation.

For determining the critical amplitude of a quantity \( \mathcal{X} \) to third order, it is actually sufficient to know the double expansion of \( \mathcal{X}(z, \varepsilon) \) to the order

\[ \mathcal{X}(z, \varepsilon) = 1 + z \left( \frac{A}{\varepsilon} + A' + A'' \varepsilon + \mathcal{O}(\varepsilon^2) \right) \]

\[ + z^2 \left( \frac{B}{\varepsilon^2} + \frac{B'}{\varepsilon} + B'' + \mathcal{O}(\varepsilon) \right) + \mathcal{O}(z^3). \]  

(A.1)
In principle, terms of order $z^3$ would be required, but we shall use additional information on the critical index $\sigma$ governing $\mathcal{F}$, up to order $e^3$, and expression (A.1) of $\mathcal{F}$ will be sufficient. As in section 3, we define the function $h(z, e)$ (if $A \neq 0$) as

$$h(z, e) = z[\mathcal{F}(z, e)]^{-1/\sigma}$$  \hspace{1cm} (A.2)

where $\sigma = O(1)$. The $e$-expansion of $\sigma^{-1}$ is supposed to be known up to second order

$$\sigma^{-1} = \frac{8}{A} \left(1 + ae + a'e^2\right)$$  \hspace{1cm} (A.3)

where according to equation (3.10), $a$ is given by

$$a = -\frac{1}{4} \left(\frac{17}{8} + \frac{8A'}{A} + \frac{B'}{A} - A'\right).$$  \hspace{1cm} (A.4)

Knowing the next order term $a'e^2$ in $\sigma^{-1}$ is important. It corresponds for usual critical indices $\nu$, $\gamma$ to $O(e^3)$ terms. (One could calculate it by expanding $\mathcal{F}$ (A.1) to third order in $z$). But, in general, $\sigma$ is directly related to the usual critical indices $\gamma$ and $\nu$, whose $e$-expansion is known up to order $e^4$ or $e^5$ [16], from field theoretic calculations for the $(\varphi^2)_n$ model, with $0(n)$ symmetry. So, for polymers, it is sufficient to set $n = 0$, in the results, $n$ being the number of field components.

Inserting (A.1) into (A.2), one finds the double Laurent expansion of $h(z, e)$ to the same order:

$$h(z, e) = z \left[1 - \frac{1}{\sigma} \left(\frac{A}{e} + A' + A'' e\right) + \frac{1}{\sigma} \left(\frac{1}{\sigma} + 1\right) \frac{1}{2} \left(\frac{A^2}{e^2} + \frac{2AA'}{e} + 2AA'' + A^2\right) - \frac{1}{\sigma} \left(\frac{B}{e^2} + \frac{B'}{e} + B''\right)\right] z^2$$  \hspace{1cm} (A.5)

Again we substitute $z_R$ to $z$ using

$$z = z_R + \frac{8}{e} z_R^2 + \left(\frac{64}{e^2} - \frac{17}{e}\right) z_R^3$$

and we use also explicitly (A.3) (A.4). After some algebra, we find the renormalized regular double $e$, $z_R$ series expansion of $h$

$$h[z_R, e] = z_R \left[1 - \left(\frac{A'}{A} + a + a\frac{A'}{A} + a'e + \frac{A''}{A} e\right) \frac{8}{z_R} + \left[-8a'^2 + 4a^2 + aa' - 4a' - a\frac{B'}{A} - 16A'' + \left(\frac{8}{A} + 1\right) \left(\frac{A''}{A} + \frac{A'^2}{2A} - \frac{B''}{A}\right) \frac{8}{z_R}\right] \frac{8}{z_R^2}\right]$$  \hspace{1cm} (A.6)

where we used also (3.8). Since $a$ is given by (A.4), we see that this formal critical amplitude to third order in $z_R$, is only expressed in terms of the coefficients $A, A', A'', B', B''$ of $\mathcal{F}$ (A.1) expanded to second order in $z$, and in terms of $a'$. The only difficult number to obtain in practice could be $B''$.

Coefficient $a'$ appearing in the critical exponent $\sigma^{-1}$ will be found in general from field theoretic results.

Let us now turn to the fixed point value of $h$. We need for that the fixed point value $z_R^*$ of $z_R$ to third order, which we write

$$z_R^* = \frac{e}{8} + \frac{17}{4} \left(\frac{e}{8}\right)^2 + \xi \left(\frac{e}{8}\right)^3.$$  \hspace{1cm} (A.7)

The number $\xi$ has not yet been calculated. This could be done, with more work, by extending the minimal subtraction equation (3.6) to next order, following the same method as in reference [6]. Let us give now the formal $e$-expansion of $h[z_R^*, e]$. Using (A.6) (A.7), we find

$$h(e) = h[z_R^*, e]$$

$$= \frac{e}{8} + \mathcal{B} \left(\frac{e}{8}\right)^2 + \mathcal{B}' \left(\frac{e}{8}\right)^3 + O(e^4)$$  \hspace{1cm} (A.8)

$$= \mathcal{B} \left(\frac{e}{8}\right)^2 + \mathcal{B}' \left(\frac{e}{8}\right)^3 + O(e^4)$$
where

\[
\mathcal{B} = \frac{17}{2} + \frac{8A'}{A} + \frac{2B'}{A} - 2A'
\]  

\[
\mathcal{B}' = 8 \left\{ -8a^2 + 4a^2 + aA' - \frac{aA'}{A} + 3a' - \frac{ab'}{A} - \frac{17A''}{A} \right\} + \left( \frac{8}{A} + 1 \right) \left( A'' + \frac{1A''}{A} \right) + \frac{B''}{A} + \frac{17}{8} \left( \frac{17}{8} + \frac{4A'}{A} + \frac{B'}{A} - A' \right) \right) + \xi. 
\]  

In summary, for calculating the critical amplitude \( \mathcal{A}(\epsilon) = [h(\epsilon)]^{-\sigma} \) of a given quantity \( \mathcal{X} \) scaling like \( z^\sigma \), to third order, it is sufficient to determine the values of the coefficients \( A, A', A'', B = \frac{A^2}{2} - 4A, B', B'' \) of the double Laurent expansion of \( \mathcal{X}(z, \epsilon) \) (A.1). Of these coefficients, only \( B'' \) is difficult to obtain. Furthermore, as a general information, one needs also the value of the coefficient \( \xi \) of the \( O(\epsilon^3) \) contribution to \( z^\sigma \) (A.7). Formula (A.10) would then be a good starting point for a practical evaluation of \( h(\epsilon) \).

Let us illustrate briefly the results in two cases, the swelling \( \mathcal{X}_0 \) and the single chain partition function \( \mathcal{X}_1 \). For the end-to-end swelling \( \mathcal{X}_0(z, \epsilon) \) (2.6)-(2.8), we have found

\[
\mathcal{X}_0(z, \epsilon) = 1 + z \left( 2 - 1 + \frac{\epsilon}{2} \right) + z^2 \left( -\frac{6}{\epsilon^2} + \frac{11}{2} + \frac{B_0''}{\epsilon} \right)
\]  

and \( B_0'' \) could be calculated. The critical exponent \( \sigma_0 \) reads

\[
\sigma_0 = \frac{2}{\epsilon} \left( 2\nu - 1 \right).
\]  

From [16], one has

\[
\nu \mid_{\nu = 0} = \frac{1}{2} \left\{ 1 + \frac{\epsilon}{8} + \frac{15}{4} \left( \frac{\epsilon}{8} \right)^2 + \left[ -33\zeta(3) + \frac{135}{8} \right] \left( \frac{\epsilon}{8} \right)^3 + \cdots \right\}
\]  

and from this result, one finds

\[
\sigma_0^{-1} = 4 \left[ 1 + a_0 \epsilon + a'_0 \epsilon^2 + \cdots \right]
\]  

with

\[
a_0 = -\frac{1}{8}, \quad a'_0 = \frac{15}{4},
\]  

\[
\frac{1}{8^2} \left[ 33 \zeta(3) - \frac{45}{16} \right].
\]  

In the same way, for the single chain partition function (2.24), we have found

\[
\mathcal{X}_1(b, S, \epsilon) = \mathcal{X}_1(z, \epsilon) = 1 + z \left( \frac{2}{\epsilon} + 1 + \frac{\epsilon}{2} \right) + z^2 \left( -\frac{6}{\epsilon^2} + \frac{7}{\epsilon} + B_1'' \right)
\]  

with \( B_1'' \) not calculated here. The associated exponent \( \sigma_1 \) is

\[
\sigma_1 = \frac{2}{\epsilon} \left( \nu - 1 \right)
\]  

which reads from [16]

\[
\gamma - 1 \mid_{\gamma = 0} = \frac{\epsilon}{8} + \frac{13}{4} \left( \frac{\epsilon}{8} \right)^2 + \left[ \frac{97}{8} - 33 \zeta(3) \right] \left( \frac{\epsilon}{8} \right)^3 + \cdots.
\]
Therefore, let us finally turn to the case $A = 0$, where the above results are not applicable. Let us recall that this case is that of $\mathcal{H}_2 (2.25)$. If $A = 0$, one has to resort to equation (3.25)

$$\mathcal{H} (z, \varepsilon) = \frac{z^\varepsilon}{h \left( z, \varepsilon \right)} \sigma \left( \varepsilon \right)^{1/\varepsilon},$$

and calculate $h'(z, \varepsilon)$ by the same method as in section 3. We start from the expansion (A.1), where $A = 0$ and $B = 0$,

$$\mathcal{H} (z, \varepsilon) = 1 + \left( A' + A'' + \mathcal{O} \left( \varepsilon^2 \right) \right) z + \left( \frac{B'}{\varepsilon} + B'' + \mathcal{O} \left( \varepsilon \right) \right) z^2 + \mathcal{O} \left( z^3 \right).$$

(A.18)

The critical exponent $\sigma$ reads here

$$\sigma \left( \varepsilon \right) = \varepsilon \sigma' \left( \varepsilon \right) = \frac{\varepsilon}{32} \left( 8 A' + B' \right) + \mathcal{O} \left( \varepsilon^2 \right)$$

(A.19)

and we set

$$\sigma^{-1} \left( \varepsilon \right) = \frac{32}{8 A' + B'} \left[ 1 + a' \varepsilon + \mathcal{O} \left( \varepsilon^2 \right) \right]$$

(A.20)

where, again, $a'$ can be calculated in general from known values of the critical exponents $\nu$ and $\gamma$. Following the same method, one finds, after some calculations, the renormalized expression of $h'(z, \varepsilon)$ in terms of $z_R, \varepsilon$ :

$$h'(z, \varepsilon) = h' \left[ z_R, \varepsilon \right] = z_R \left[ 1 + \left( 8 - \frac{A'}{\sigma' \left[ 1 \right]} \right) z_R - \frac{1}{\sigma' \left[ 1 \right]} \left( A'' + a' A' \right) \varepsilon z_R + \frac{1}{\sigma' \left[ 1 \right]} \left( \frac{1}{\sigma' \left[ 1 \right]} + 1 \right) \frac{1}{2} A'^2 \right] z_R^2$$

(A.21)

where

$$\sigma' \left[ 1 \right] = \frac{8 A' + B'}{32}.$$  

(A.22)

Using now the value (A.7) of $z_R^*$ gives

$$z_R^* = \left( \frac{\varepsilon}{8} \right)^{\frac{\varepsilon}{8}} \left[ 1 + 34 \left( \frac{\varepsilon}{8} \right)^2 + \mathcal{O} \left( \varepsilon^3 \right) \right],$$

(A.23)

and we note that here the precise value of term $\varepsilon \varepsilon^3$ in $z_R^*$ only matters in next order, in contrast to the case $A \neq 0$. Inserting (A.23) into (A.21) yields finally, after some algebra, the $\varepsilon$-expansion of the fixed point value $h'(\varepsilon)$ :

$$h'(\varepsilon) = h' \left[ z_R^*, \varepsilon \right] = \left( \frac{\varepsilon}{8} \right)^{\frac{\varepsilon}{8}} \left[ 1 + \frac{\varepsilon}{8} \left( 8 - \frac{32 A'}{8 A' + B'} \right) \right]$$

(A.24)

$$+ \left( \frac{\varepsilon}{8} \right)^2 \left[ 83 - 32 a' - \frac{32}{8 A' + B'} \left[ \frac{49 A'}{4} + 16 A'' + 8 a' A' + B'' - \left( \frac{32}{8 A' + B'} + 1 \right) \frac{A'^2}{2} \right] \right].$$
Here again, the only difficult quantity to calculate in practice would be $B''$.

Let us partially illustrate these results for the case of the two-chain partition function $\mathcal{Z}_2, \mathcal{X}_2$ (2.25). One has (see (2.16))

$$\mathcal{Z}_2 = (-bS^2) \mathcal{X}_2$$

with

$$\mathcal{X}_2 = 1 + \left[ (1 + 4 \ln 2) + A'' \varepsilon \right] z$$

$$+ \left[ \frac{1}{\varepsilon} (2 - 32 \ln 2) + B'' \right] z^2 + \cdots$$

and thus

$$A' = 1 + 4 \ln 2, \ B' = 2 - 32 \ln 2$$

and

$$\sigma^{[1]} = \frac{8 A' + B'}{32} = \frac{5}{16}.$$  

The $a'$ coefficient of $\varepsilon$-expansion (A.19) (A.20) of $\sigma_2$ is obtained from (2.19)

$$\sigma_2 = \frac{2}{\varepsilon} \left[ (\nu d - 2) + 2 (\gamma - 1) \right],$$  

(A.27)

and owing to (A.13) (A.16)

$$\sigma_2 (\varepsilon) = \varepsilon \sigma_2' (\varepsilon) = \frac{5}{16} \varepsilon + \left[ \frac{43}{4} - 33 \xi (3) \right] \left( \frac{\varepsilon}{8} \right)^2$$

(A.28)

$$\sigma_2' = \frac{16}{5} \left[ 1 + \frac{\varepsilon}{4} \left[ 33 \xi (3) - \frac{43}{4} \right] + O (\varepsilon^2) \right].$$

Thus $a'$ reads

$$a' = \frac{10}{20} \left( 33 \xi (3) - \frac{43}{4} \right),$$

(A.29)

and a practical calculation of $A''_2$ and $B''_2$ would yield the complete value of $h_2$ (A.24).

References