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Random walks on the Sierpinski Gasket

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Résumé. — Nous calculons les fonctions de Green des marches aléatoires sur un réseau de Sierpinski. Dans le cas des marches fermées, nous étudions leur variation en fonction de la position et du paramètre de transition nu. Ces fonctions sont continues mais non différentiables sur un réseau infini.

Abstract. — The generating functions for random walks on the Sierpinski gasket are computed. For closed walks, we investigate the dependence of these functions on location and the bare hopping parameter. They are continuous on the infinite gasket but not differentiable.

1. Preliminaries and review of known results.

Consider a general lattice of points (vertices) connected by edges. A random walker steps from point to point along edges. The generating function for random walks is

\[ G(P_1, P_2, z) = \sum_n \#(n) z^n \]  

where \( P_1 \) and \( P_2 \) are two points on the lattice, and \( \#(n) \) is the number of walks of length \( n \) which go from \( P_1 \) to \( P_2 \). The length is just the number of steps taken. As \( n \) becomes much larger than the « distance » between \( P_1 \) and \( P_2 \), one expects

\[ \#(n) \sim z_n^{-d} n^{-d/2}, \]  

which translates to

\[ G(z) \sim (1 - z/z_c)^{-1 + d/2} \]  

as \( z \to z_c \) if \( d < 2 \). \( d \) is called the spectral dimension of the lattice. Hereafter, we shall call \( G \) the Green function. Another Green function of interest is that of first passage random walks, corresponding to all walks from \( P_1 \) to \( P_2 \) which go \( P_1 \) to visit \( P_2 \) only at the last step. One then has

\[ \Gamma(P_1, P_2, z) = G(P_1, P_2, z) / G(P_2, P_1, z). \]  

It is possible to extract from these Green functions the average number of distinct sites visited and recurrence times [1]. All first passage Green functions will be denoted by Greek letters, and others by Roman letters.

Let us now apply this to the Sierpinski gasket in two dimensions. Unless specified otherwise, we consider the vertices of the triangles of a gasket which has a smallest triangle size, but is infinite in extent. This is really for clarity only, as the properties of the « coarse » gasket are essentially the same as those of an infinitely fine gasket if one considers only quantities living on the coarse levels.

Consider the first passage Green function of walks which always remain within a given triangle at level \( l \). Level 1 corresponds to the smallest triangles. Let us determine a recurrence relation in \( l \) for these Green functions. In figure 1, we have labelled the vertices of the triangles at level \( l \) and \( l + 1 \). Denote by \( \gamma(l, z) \) (\( \delta(l, z) \)) the Green function for the first passage walks inside the triangle at level \( l \) which go from \( x \) to \( y \) (\( x \) to \( x \)). We have used lower case letters for these Green functions to indicate that the corresponding walks are constrained to remain inside...
Fig. 1. — Labelling of the vertices of triangles of two different levels.

a particular triangle. Define \( k_0 \) to be the Green function for walks from \( x \) to \( x \) which never reach \( X \), \( Y \), or \( Z \), and \( k \) for similar walks which go from \( x \) to \( y \). One then has

\[
\begin{align*}
  k_0 &= f^{-1} + f^{-1} 2 \gamma k \\
  k &= f^{-1} \gamma (k_0 + k)
\end{align*}
\]

where

\[
f = 1 - 2 \delta
\]  

Denoting \( \gamma = \gamma(l, z) \), \( \tilde{\gamma} = \gamma(l + 1, z) \), etc., we have the recursions:

\[
\begin{align*}
  \tilde{\delta} &= \delta + 2 \gamma (k_0 + k) \\
  \tilde{\gamma} &= \gamma k_0 + 3 \gamma k \gamma
\end{align*}
\]

Eliminating \( k_0 \) and \( k \) leads to

\[
\gamma = \frac{\gamma^2 (f + 2 \gamma)}{(f + \gamma) (f - 2 \gamma)}
\]

\[
\tilde{\gamma} = \frac{\gamma^2 (f + 2 \gamma)}{(f + \gamma) (f - 2 \gamma)}
\]

\[
f^{-1} \text{ can be thought of as a staying amplitude and } \gamma \text{ as a hopping amplitude. These recursions are non-linear, and homogeneous of first degree. Thus the } l \text{th iterate of } (\lambda \gamma(1), \lambda f(1)) \text{ is simply } \lambda \text{ times the } l \text{th iterate of } (\gamma(1), f(1)). \text{ It is then sufficient to study the iterates of the ratio}
\]

\[
r(l, z) = \frac{f(l, z)}{\gamma(l, z)}
\]

One finds \( \tilde{r} = r(r - 3) \) which is remarkably simple. This recursion has two fixed points: \( r = 0 \) and \( r = 4 \), both of which are unstable. On the smallest triangles, one has \( f(1) = 1, \gamma(1) = z \). The region of physical interest is \( r > 4 \), i.e., \( z < z_c = 1/4 \). If \( \gamma(1) = z_c \), one has a massless Green function, i.e., scale invariance. As one iterates for \( \gamma(1) < z_c \), \( r \) increases indefinitely, corresponding to \( \gamma \to 0 \), and \( f \) has a finite limit as expected. The asymptotics of the random walks on the gasket can be extracted by a scaling argument. Near the fixed point, the recursions reduce to

\[
\begin{align*}
  \gamma(n, \frac{\gamma(1)}{f(1)}) &= \gamma(1) (3/5)^{n-1} \\
  f(n, \frac{\gamma(1)}{f(1)}) &= f(1) (3/5)^{n-1}; \\
  s(n) &= s(1) 5^{n-1}
\end{align*}
\]

with \( s = r - 4 \). Apply equation (1.2) to the Green function \( G \) for closed walks at the vertex of a very large triangle:

\[
G_\infty \left( \frac{\gamma(1)}{f(1)} \right) = \lim_{n \to \infty} f(n) \sim (1 - 4z)^{1-2/2} s(1)^{1-2/2}.
\]

Using

\[
G_\infty \left( \frac{\gamma(1)}{f(l)} \right) = \lim_{n \to \infty} f(n) \sim \left( \frac{3}{5} \right)^{-1} G_\infty \left( \frac{\gamma(1)}{f(1)} \right)
\]

one has

\[
\left( \frac{3}{5} \right)^{-1} = (5^{-1})^{-1} + 2/2, \text{ i.e. } \tilde{d} = 2 \ln 3/\ln 5.
\]

This is the spectral dimension (as first introduced in [2] of the two dimensional gasket. In arbitrary dimension, one has \( \tilde{d} = 2 \ln (d+1)/\ln (d+3) \). It is useful to visualize the evolution of \( \gamma \) and \( f \) under the recursion. In figure 2, we show how for \( r > r_c \), \( f \) and \( \gamma \) converge to their limits. This completes the review of known results [3-6] for random walks on the gasket.
2. Location and bare lattice dependence of closed walk Green functions.

The previous derivation was for the closed walk Green function from the vertex of a triangle of ever increasing size. We wish here to find the corresponding quantity for a fixed lattice point. It is convenient to label the triangles of the gasket by their ternary representation as in [7]. This is illustrated in figure 3. The three triangles in the largest triangle are labelled 0, 1 and 2. Inside each of these, this labelling is repeated after an appropriate rotation. This iteratively labels all the triangles of the lattice. Any triangle is in one-to-one correspondence with a sequence of ternary digits. This has the advantage of mapping the gasket onto the interval [0, 1] while keeping nearby triangles typically nearby on the interval. Now let us find the dependence of the closed walk Green function on the triangle location. To do this, we relate the Green functions at one level to those at the next lower level. Using the labelling of figure 1, we can find e.g., the closed walk Green function $G(z, z)$ in terms of $G(X, X)$, $G(X, Y)$ etc. A walk from $z$ to $z$ will either never reach $X$, $Y$ or $Z$, or there will be a first time it does, and then there will be a last time also.

This decomposition leads to

$\begin{align*}
G(z, z) &= k(z, z) + \\
&+ \sum_{A, B} k(z, A) G(A, B) k(B, z)
\end{align*}$  \hfill (2.1)

with $A, B \in \{X, Y, Z\}$ and $k(z, A)$ is the Green function for walks from $z$ to $A$ which never encounter the corners $X$, $Y$ or $Z$ except perhaps at the last step. Then, using the definitions in the previous section, we have

$\begin{align*}
k(z, z) &= k_0 \\
k(z, y) &= k \\
k(z, X) &= (k_0 + k) \gamma = h \\
k(z, Z) &= 2k \gamma = h'
\end{align*}$  \hfill (2.2)

etc... The recursions for $k_0$ and $k$ were given previously. Equation (2.1) and its generalisations (e.g. to $G(z, y)$) give a linear recursion relation for the six Green functions of a smaller triangle in terms of those of the larger one. We find, using the definitions in (2.2),

\begin{equation}
F_n = L_n + M_n R_{n+1}^T F_{n+1}
\end{equation}

\begin{equation}
F_n = \begin{pmatrix} G(x, x) \\ G(y, y) \\ G(z, z) \\ G(y, z) \\ G(z, x) \\ G(x, y) \end{pmatrix}, \\
L_n = \begin{pmatrix} 0 \\ k_0 \\ k_0 \\ k \\ 0 \\ 0 \end{pmatrix}, \\
R_n = \begin{pmatrix} R_n & 0 \\ 0 & R_n \end{pmatrix}
\end{equation}
$R_{n+1}$ is the rotation necessary to make the smaller triangle have the label 1 at level $n+1$: $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for triangle 1, $R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ for triangle 0, and $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for 2. Starting on an infinitely large triangle, one has $k(z, z) = G(X, X) = G_\infty$, and $G(X, Y) = 0$ for $X \neq Y$. Upon iteration, one obtains the value of $G(z, z)$ for any point of interest. We have compared the value of $G(X, X)$ for different points on the gasket. A point is called periodic if its ternary representation is periodic. (We are now considering a gasket which has an infinitely fine mesh.) We have looked at $G(X, X)$ for all periodic points up to period 8. $G(X, X)$ is a minimum at the corners of large triangles, $G_\infty$ being the absolute minimum. The largest value for $G(X, X)$ is at the « spiral » points 0.000... and 0.222... $G(X, Y)$ is continuous in $X$ and $Y$ as discussed in the next section so for numerical purposes, one need not be precisely at a periodic point.

Figure 4 shows the value of $G(X, X)/G_\infty$ as a function of the ternary representation of $X$. In practice it is enough to determine $G$ iteratively till one reaches the scaling region, $s \sim 0$. Then one can average inside the triangle at that level analytically by supposing that linear scaling is exact. The ternary representation introduces discontinuities in figure 4 because two points can be far away from each other and yet have close ternary representations. (Example : 0.100000... and 0.02222... Similarly, there are points physically close to each other which have very different ternary representations, such as 0.221111... and 0.001111...) In figure 5, we show the relative number of occurrences of a value of $G$ versus its value normalized to its minimum value, $G_\infty$. The distribution is highly asymmetrical, with a peak around 1.11.

Equation (2.1) can also be used to find the average of $G(x, x)$ or $G(x, y)$ over all points of the lattice which are the vertices of triangles of a given size. Averaging over points $X, Y, Z, x, y, z$, one obtains

$$
\begin{pmatrix}
\hat{\theta}_n \\
\hat{\theta}_n^{+1}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{3} (\theta_n + 2k) \\
\frac{1}{3} (\theta_n^{+1} + 2k)
\end{pmatrix} + \begin{pmatrix}
\frac{2}{3} h^2 + h' \\
\frac{2}{3} (h + h')^2
\end{pmatrix} \begin{pmatrix}
\hat{\theta}_n \\
\hat{\theta}_n^{+1}
\end{pmatrix} (2.6)
$$

where $\hat{\theta}$ and $\theta$ refer to averages of $G(X, X)$ and $G(X, Y)$ over vertices and edges of triangles of the same size. For the largest size triangles, we have $\theta = G_\infty$, $\hat{\theta} = 0$. These increase thereafter approaching the value for the infinitesimal triangles $\theta = \theta = 1.1021 G_\infty$. The averaging over points $X, Y, Z, x, y, z$ is simple because equation (2.1) is linear in the $G$'s, and the coefficients do not depend on the triangle location, only on its size. It is possible to generalize equation (2.6) to higher moments of $G$ with this same property. For instance, the symmetric averages of the bilinears in $G$ and $G$ obey a recursion given by a 6 x 6 matrix.

Let us now discuss the dependence of the above quantities on the value of $s$ at the bare level. First take the case of $G_\infty$. As one approaches the scaling region, one has

$$
G_\infty = \phi(s_1) s_1^{-1+2/3} (2.7)
$$

Since for small $s$, one has $s_{n+1} = 5 s_n$, it is convenient to parametrise $s_1$ by $s_1 = (1/5)^q$ with $q$ a real number. In the scaling region, $\phi$ becomes a periodic function of $q$ with period 1. On Euclidean lattices, $\phi$ becomes $q$ independent. It is clear that there should be some dependence in the case of the gasket: for a given $s_1$, the Green functions fall off with a characteristic scale which lies between two consecutive triangle sizes. Changing the correlation length affects the Green functions differently depending on their origin. We have found the dependence of $\phi$ on $q$ by numerical investigation. The Fourier decomposition gives the result

$$
\phi(q) = 1.202 + 1.465 \times 10^{-5} \sin 2 \pi q \\
+ 1.617 \times 10^{-7} \cos 2 \pi q - 2.7 \times 10^{-12} \sin 4 \pi q \\
- 5.38 \times 10^{-11} \cos 4 \pi q \ldots (2.8)
$$

The various Fourier components are suppressed by approximately $10^{-5}$ at each order. This oscillatory behaviour occurs for the spiral points and for the average over the lattice of $\theta$ also and again one has a factor
$10^{-5}$ suppression. We do not really understand why the $q$ dependence is so weak. However, the dependence is much greater for most other periodic points. For a point of period $p$, the periodicity of $\phi$ is $p$ at most, and we have found that the first Fourier coefficients are typically between $10^{-3}$ and $10^{-2}$. This $s$ dependence reflects itself in the distribution of $G(X, X)$. Figure 5 showed the distribution for $q$ integer. We have seen changes in this distribution of order 1 percent as $q$ is varied. It is interesting to note that a factor $10^{-5}$ was also found in the oscillation of the free energy for an Ising model on a diamond hierarchical lattice [8].

3. General Green functions.

The previous section gave an algorithm for finding the Green function between two vertices of a triangle. This enables us now to give $G(M, N)$ between any two points $M$ and $N$ on the gasket. Given $M$ and $N$, there is a smallest triangle which contains both of them, as depicted in figure 1. Walks between $M$ and $N$ can be labelled by their first and last landing on the vertices of the triangles containing $M$ and $N$. This gives

$$G(M, N) = \sum_{P_1, P_2} \eta(M, P_1) G(P_1, P_2) \eta(P_2, N)$$

(3.1)

where $P_1(P_2)$ is a vertex of the triangle $M(N)$, and $\eta$ is the Green function for first passage walks restricted to the inside of the triangle $M(N)$. All of the $G$'s in the above equation have been obtained in the previous section except for $G(x, X)$ and $G(Y, y)$. For these one has

$$G(x, X) = 2kG(x, X) + (k_0 + k) \gamma(G(Y, X) + G(Z, X))$$

$$G(Y, y) = 2kG(Y, Y) + (k_0 + k) \gamma(G(Z, Y) + G(X, Y))$$

(3.2)

Thus all the $G$'s have known expressions. We proceed to find the recursion for $|\eta\rangle_n = (\eta(M, x), \eta(M, Y), \eta(M, Z))$. Using the notation of the previous section, one has

$$|\eta\rangle_{n+1} = R_{n+1} N_n |\eta\rangle_n$$

$$N_n = \begin{pmatrix} 1 & h_n & h_n \\ 0 & h_n & h_n \\ 0 & h_n & h_n \end{pmatrix}$$

(3.3)

At the smallest triangle level, one has the initial condition $|\eta\rangle_1 = (1, 0, 0)$. This establishes a method to compute $G(M, N)$. It may be of use to average certain Green functions either over triangles of the same size, or over all points inside a triangle. The linearity in $G$ of equation (3.1) permits an averaging over triangles, and similarly $|\eta\rangle$ can be averaged over the points inside the triangle. It is possible to calculate essentially any two-point Green function of interest.

Of particular interest are the differences of Green
functions. Consider for instance \( G(X, X) - G(Y, Y) \). How does this quantity behave as \( X \) and \( Y \) get close? (Hereafter, the gasket has an infinitely fine mesh.) In the scaling regime, one has the recursion:

\[
\Delta_n = \frac{1}{5} \begin{pmatrix} 6 \\ 6 \\ 2 \\ 0 \\ 0 \end{pmatrix} + V_n R_{n+1} \Delta_{n+1} \tag{3.4}
\]

with

\[
\Delta_n = \begin{pmatrix} G_{yy} - G_{xx} \\ G_{yx} - G_{xx} \\ G_{zx} - G_{xx} \\ G_{xy} - G_{xx} \end{pmatrix} \cdot f_n , \tag{3.5}
\]

\[
V_n = \begin{pmatrix} 4 & 1 & 4 & 4 & 8 \\ 1 & 4 & 4 & 8 & 4 \\ 2 & 2 & 5 & 6 & 6 \\ 0 & 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & 5 & 10 \end{pmatrix} .
\]

As the triangle size goes to zero, \( \Delta_n \) has a limit, so that

\[
G_{yy} - G_{xx} = (3/5)^l \tag{3.6}
\]

for triangles of size \( 2^{-l} \). This shows that \( G \) is continuous, but clearly it is not differentiable if one embeds the gasket in \( \mathbb{R}^2 \) since \( 3/5 > 1/2 \). This scaling behaviour in fact is true for all Green functions with a physical interpretation. To obtain an estimate of the strength of this non-differentiability, we looked at two quantities. The first, for a triangle of vertices \( x, y, z \),

\[
(G_{xx} - G_{yy})^2 + (G_{yy} - G_{zx})^2 + (G_{zx} - G_{xy})^2
\]

gives the average square of the gradient of \( G(x, x) \). The second,

\[
\langle G(p, p) \rangle_p - \frac{1}{3} \left( G_{xx} + G_{yy} + G_{zx} \right)
\]

with \( \langle \cdot \rangle_p \) meaning the average over all points inside the triangle \( x, y, z \), is an approximation to the Laplacian operator on the gasket. In table I, we give these quantities for periodic points of period \( \leq 3 \) and some others for illustration. The gradient square is minimized at the extrema of \( G \), e.g., at the spiral points. Also, it tends to be large along the edges of big triangles. Over all, the results agree with expectations.

### 4. Conclusions.

We have shown how to obtain the generating function for arbitrary random walks on the Sierpinski gasket. Of particular interest is the dependence on the origin of closed walk Green functions. We computed these Green functions, their gradient squared and Laplacian for a number of periodic points. This method of computation can be extended to other walks. The same analysis can be done e.g. for self avoiding walks. The critical exponents have been determined in reference [9] and we have found the amplitude modulations. This is presently being extended to an ensemble of repelling chains, i.e.; a polymer melt [10].

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