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Negative viscosity effect in three-dimensional flows

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Résumé. — On étudie la stabilité d’un écoulement tridimensionnel, unidirectionnel, périodique. On montre qu’à nombre de Reynolds suffisamment grand, un écoulement de ce type est instable vis-à-vis de perturbations de grande longueur d’onde. Cette instabilité peut être interprétée en termes de viscosité effective négative pour l’écoulement à grande échelle correspondant.

Abstract. — The paper treats the stability of a periodic unidirectional three-dimensional flow. It is shown that at sufficiently high Reynolds numbers a flow of this type is unstable to long-wave disturbances. This instability can be interpreted in terms of negative effective viscosity of the corresponding large-scale flow.

1. Introduction.

One of the most intriguing problems in the theory of hydrodynamic turbulence is the formation of large-scale structures in a liquid performing random (turbulent) motion at small scales. It is natural to explain the spontaneous formation of large-scale structures as a manifestation of long-wave instability of the corresponding small-scale flow. In a theoretical approach to the description of this phenomenon, it is convenient to assume that the small-scale flow is driven by an external force field which varies quasi-periodically (or randomly) with respect to the space coordinates. The simplest situation is that of a strictly periodic field. Previous papers [1, 2] treated two-dimensional problems of the stability of a periodic eddy system driven by an appropriate applied force. It was shown that large-scale flow will occur only if the small-scale flow is sufficiently anisotropic. An example of such a flow is unidirectional flow with velocity a periodic function of one of the space coordinates. If the small-scale flow is sufficiently isotropic, there is no long-wave instability and consequently no formation of large-scale structures.

In the long-wave approximation, the rate of instability is a quadratic function of the wavenumber of the disturbances. For this reason, the evolution of long-wave disturbances may be interpreted in terms of effective viscosity of the large-scale flow. In the unstable case, this viscosity will be negative in certain directions.

The possibility of large-scale eddies forming from small-scale ones in two-dimensional systems is usually connected with the conservation of enstrophy (the so-called inverse cascade theory). In three-dimensional systems there is no conservation of enstrophy. It is therefore of considerable interest to determine whether large-scale structures may also be generated in the three-dimensional case.

To this end, we propose in this paper to investigate the long-wave stability of unidirectional three-dimensional flow of a viscous liquid driven by an applied force periodic in two space coordinates:

$$f = (0, 0, R^{-1} \sin \alpha x \sin \beta y)$$

where $\alpha^2 + \beta^2 = 1$; $R$ is the Reynolds number; $\alpha$, $\beta$ are suitably scaled space frequencies. This flow is obviously just a three-dimensional version of the so-called Kolmogorov flow, discussed in detail in our previous papers [1, 2]. As in the two-dimensional case, the problem proves amenable to exact analytical solution. As we shall show below, if $\alpha \neq \beta$ and the Reynolds number $R$ is sufficiently large, the principal flow induced by force (1) is unstable to long-wave disturbances. Thus, a small-scale three-dimensional periodic flow may generate large-scale structures, just as in two-dimensional systems.
2. Stability analysis.

In terms of suitably selected nondimensional variables, the Navier-Stokes equations for the flow of a liquid in a field \( \mathbf{f} \) may be written as follows:

\[
\nabla \cdot \mathbf{v} = D.
\]

(2)

If the field is that given by (1), equation (2) has the following solution:

\[
\mathbf{v} = \mathbf{v}^0 = (0, 0, \sin \alpha x \sin \beta y).
\]

(3)

\[ p = p^0 = \text{Const.} \]

For a small perturbation

\[
\mathbf{V} = \mathbf{v} - \mathbf{v}_0, \quad P = p - p_0
\]

(4)

we obtain from (2) and (3) a system of linear equations:

\[
\begin{align*}
\epsilon^2 \frac{\partial V_x}{\partial t} - \epsilon^2 \frac{1}{R} \frac{\partial^2 V_x}{\partial z^2} + \epsilon \sin \alpha x \sin \beta y \frac{\partial V_x}{\partial z} &= - \frac{\partial P}{\partial x} + \frac{1}{R} \Delta_{||} V_x, \\
\epsilon^2 \frac{\partial V_y}{\partial t} - \epsilon^2 \frac{1}{R} \frac{\partial^2 V_y}{\partial z^2} + \epsilon \sin \alpha x \sin \beta y \frac{\partial V_y}{\partial z} &= - \frac{\partial P}{\partial y} + \frac{1}{R} \Delta_{||} V_y, \\
\epsilon^2 \frac{\partial V_z}{\partial t} - \epsilon^2 \frac{1}{R} \frac{\partial^2 V_z}{\partial z^2} + \epsilon \sin \alpha x \sin \beta y \frac{\partial V_z}{\partial z} + \alpha V_x \cos \alpha x \sin \beta y + \beta V_y \sin \alpha x \sin \beta y &= \frac{1}{R} \Delta_{||} V_z,
\end{align*}
\]

(7)

Elimination of \( V_z \) transforms the equation system (6) to a more convenient form for further analysis:

\[
\begin{align*}
\epsilon^2 \frac{\partial V_x}{\partial t} - \epsilon^2 \frac{1}{R} \frac{\partial^2 V_x}{\partial z^2} + \epsilon \sin \alpha x \sin \beta y \frac{\partial V_x}{\partial z} &= - \frac{\partial P}{\partial x} + \frac{1}{R} \Delta_{||} V_x, \\
\epsilon^2 \frac{\partial V_y}{\partial t} - \epsilon^2 \frac{1}{R} \frac{\partial^2 V_y}{\partial z^2} + \epsilon \sin \alpha x \sin \beta y \frac{\partial V_y}{\partial z} &= - \frac{\partial P}{\partial y} + \frac{1}{R} \Delta_{||} V_y, \\
\epsilon^2 \frac{\partial^2 P}{\partial z^2} + 2 \epsilon \cos \alpha x \sin \beta y \frac{\partial V_x}{\partial z} + 2 \epsilon \beta \sin \alpha x \sin \beta y \frac{\partial V_y}{\partial z} &= - \Delta_{||} P.
\end{align*}
\]

(8)

To solve system (8), we expand the solution as an asymptotic series in \( \epsilon \):

\[
V_x = V_x^{(0)} + \epsilon V_x^{(1)} + \epsilon^2 V_x^{(2)} + \cdots
\]

\[
V_y = V_y^{(0)} + \epsilon V_y^{(1)} + \epsilon^2 V_y^{(2)} + \cdots
\]

\[
P = P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \cdots
\]

(9)

In the zeroth approximation, we obtain from (9)

\[
\frac{1}{R} \Delta_{||} V_x^{(0)} = \frac{\partial P^{(0)}}{\partial x},
\]

\[
\frac{1}{R} \Delta_{||} V_y^{(0)} = \frac{\partial P^{(0)}}{\partial y},
\]

(10)

\[
\Delta_{||} P^{(0)} = 0.
\]

The solution of this system bounded with respect to \( x, y \) is

\[
V_x = V_x^{(0)}(\tau, \xi)
\]

\[
V_y = V_y^{(0)}(\tau, \xi)
\]

\[
P = P^{(0)}(\tau, \xi)
\]

(11)

These functions are as yet unknown, and our aim is to derive a suitable closed system of equations. To this end we proceed to higher-order approximations.
For the first approximation we have from (9)

\[ \sin \alpha x \sin \beta y \frac{\partial V_x^{(0)}}{\partial x} = -\frac{\partial P^{(1)}}{\partial x} \frac{\partial V_x^{(1)}}{\partial x} + 2 \alpha \cos \alpha x \sin \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} + 2 \beta \sin \alpha x \cos \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} = -\Delta_n P^{(1)}. \]

Hence, the solution bounded in \( x, y \) is

\[ V_x^{(1)} = R \left[ (x^2 - \beta^2) \sin \alpha x \sin \beta y \frac{\partial V_x^{(0)}}{\partial \zeta} - 2 \alpha \beta \cos \alpha x \cos \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} \right] + V_x(\tau, \zeta), \]

\[ V_y^{(1)} = R \left[ (\beta^2 - \alpha^2) \sin \alpha x \sin \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} - 2 \alpha \beta \cos \alpha x \cos \beta y \frac{\partial V_x^{(0)}}{\partial \zeta} \right] + V_y(\tau, \zeta), \]

\[ P^{(1)} = 2 \alpha \cos \alpha x \sin \beta y \frac{\partial V_x^{(1)}}{\partial \zeta} + 2 \beta \sin \alpha x \cos \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} + P(\tau, \zeta). \]

In the second approximation (9), yields

\[ \frac{\partial V_x^{(0)}}{\partial \tau} = -\frac{1}{R} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} + \sin \alpha x \sin \beta y \frac{\partial V_x^{(1)}}{\partial \zeta} = -\frac{\partial P^{(2)}}{\partial x} + \frac{1}{R} \Delta_n V_x^{(2)}, \]

\[ \frac{\partial V_y^{(0)}}{\partial \tau} = -\frac{1}{R} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} + \sin \alpha x \sin \beta y \frac{\partial V_y^{(1)}}{\partial \zeta} = -\frac{\partial P^{(2)}}{\partial y} + \frac{1}{R} \Delta_n V_y^{(2)}, \]

\[ 2 \alpha \cos \alpha x \sin \beta y \frac{\partial V_x^{(1)}}{\partial \zeta} + 2 \beta \sin \alpha x \cos \beta y \frac{\partial V_y^{(0)}}{\partial \zeta} = -\Delta_n P^{(2)}. \]

Hence

\[ P^{(2)} = \frac{R(x^2 - 3 \beta^2)}{8 \alpha} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} \sin 2 \alpha x + \frac{R(\beta^2 - 3 \alpha^2)}{8 \beta} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} \sin 2 \beta y - \frac{\alpha R}{8} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} \sin 2 \alpha x \cos 2 \beta y - \frac{\beta R}{8} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} \sin 2 \beta y \cos 2 \alpha x, \]

\[ \frac{1}{R} \Delta_n V_x^{(2)} = \frac{\partial V_x^{(0)}}{\partial \tau} + \frac{R(x^2 - \beta^2)}{4} - \frac{1}{R} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} - 2 \beta R \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} \cos 2 \alpha x + \frac{(\beta^2 - \alpha^2) R}{4} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} \cos 2 \beta y - \frac{\beta R}{4} \cos \alpha x \cos 2 \beta y \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} - \frac{\alpha R}{4} \sin 2 \alpha x \sin 2 \beta y \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2}, \]

\[ \frac{1}{R} \Delta_n V_y^{(2)} = \frac{\partial V_y^{(0)}}{\partial \tau} + \frac{R(\beta^2 - \alpha^2)}{4} - \frac{1}{R} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} - \frac{x^2 R}{2} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} \cos 2 \alpha x + \frac{(x^2 - \beta^2) R}{4} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} \cos 2 \beta y + \frac{\alpha R}{4} \sin 2 \alpha x \sin 2 \beta y \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} - \frac{\alpha R}{4} \cos \alpha x \cos 2 \beta y \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2}. \]

A bounded solution exists subject to following conditions:

\[ \frac{\partial V_x^{(0)}}{\partial \tau} + \frac{R(x^2 - \beta^2)}{4} - \frac{1}{R} \frac{\partial^2 V_x^{(0)}}{\partial \zeta^2} = 0, \quad (16) \]

\[ \frac{\partial V_y^{(0)}}{\partial \tau} + \frac{R(\beta^2 - \alpha^2)}{4} - \frac{1}{R} \frac{\partial^2 V_y^{(0)}}{\partial \zeta^2} = 0. \quad (17) \]

Thus, we have obtained equations for the velocity of the large-scale flow. The coefficients of \( (V_x^{(0)})_{t\zeta}, \) \( (V_y^{0})_{t\zeta} \) may be interpreted as effective viscosities in the \( z \)-direction.

If \( R > R_c = 2 \sqrt{\alpha \beta} \), one of the effective viscosities is negative, leading to the formation of large-scale secondary flow.

In conclusion, we would like to note that the possibility of constructing the exact asymptotics (16), (17) is due to the unidirectional nature of the basic flow (1). If the basic flow is not unidirectional
the corresponding mathematical problem may well be analytically untreatable. To get a closed set of the amplitude equations in this case, less accurate methods (e.g., Galerkin's) may be helpful [2].

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References