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Direct or dimensional renormalizations of the tricritical polymer theory

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Résumé. — Nous étudions une méthode de renormalisation directe pour les chaînes polymères près du point thêta tricritique. Nous avions auparavant introduit cette méthode lors de calculs explicites tricritiques. A l'aide de la renormalisation dimensionnelle de la théorie des champs, dans la limite \( n = 0 \), nous prouvons que cette dernière, ainsi que la méthode de renormalisation directe, fournissent une théorie des polymères renormalisée à tous les ordres des deux premiers coefficients du viriel, et à tous les ordres en \( \varepsilon = 3 - d \), où \( d \) est la dimension de l'espace (\( d \approx 3 \)). Nous donnons, en exemple, des résultats nouveaux du développement en \( \varepsilon \) pour des chaînes polymères au point thêta.

Abstract. — We study a direct renormalization method for polymer chains near the theta-tricritical point, which we have introduced previously for explicit tricritical calculations. With the help of the dimensional renormalization of field theory in the limit \( n = 0 \), we prove that this latter dimensional renormalization as well as the direct one yield a finite renormalized polymer theory to all orders in the first two virial coefficients and to all orders in the parameter \( \varepsilon = 3 - d \), where \( d \) is the space dimension (\( d \approx 3 \)). We give, as examples, new tricritical results of the \( \varepsilon \)-expansion for polymer chains at the theta-point.

1. Introduction.

P. G. de Gennes [1] has indicated in 1975 that the model of polymer chains near the theta point [2, 3] has the properties of a tricritical system. Since then, there have been a series of theoretical works on this subject [4-8]. Most of them were using the equivalence between a \( (\phi^2)^3 \) field theory with a vanishing number of components \( n = 0 \) and the polymer system [3, 9, 10]. As shown in a previous work [7] this method leads to precise predictions concerning the tricritical behaviour of polymer chains near the theta point, or semi-attractive chains. In references [8], use was made of the 't Hooft Veltman renormalization scheme directly for polymers, but without general proof of validity. But the applicability of field theoretic methods to polymer theory near the tricritical point was criticized in the same references [8]. However, as we shall see, these critics were erroneous. The equivalence between \( (n = 0) \) field theory near the tricritical point and polymer theory near the theta point works perfectly. We know it for two different reasons.

First, we have recently [11] recalculated the properties of the polymer chain solutions near the theta point by introducing a direct renormalization method for the tricritical polymer theory, analogous to that introduced by J. des Cloizeaux [12] for polymers in a good solvent, i.e. in the critical limit. We found that this direct method, which does not rely on field theory, yields exactly the same results as obtained in a previous work [7], where use was made only of field theory. This contradicts the results of references [8].

The second reason for the validity of the equivalence is that the minimal dimension subtraction scheme of the tricritical field theory can be entirely transferred to the polymer theory for renormalizing it near the theta-point, as shown in this article. One can furthermore relate the direct renormalization method [11], quoted above for « tricritical polymers » to the dimensional one. So all these methods are equivalent,
as it will appear in this article, and they naturally must yield exactly the same results.

The aim of the present article is to present this direct renormalization method for semi-attractive chains, and to prove that it ensures renormalization to all orders in an expansion with respect to the coupling constants and \( \varepsilon \), where here \( \varepsilon = 3 - d \), \( d \) being the space dimension, \( d \leq 3 \). For this, we shall rely on the \((\varphi^2)^3\) field theory and the dimensional renormalization scheme [13] associated with it. Let us note that a similar result has been made recently by M. Benhamou and G. Mahoux [14] and by the author [15]. According to a remark by M. Benhamou [14], the Laplace-de Gennes transform of polymer partition functions into Green functions of field theory « commutes » with the minimal dimensional renormalization factors of t’ Hooft and Veltman [13]. Then, using this scheme for the \((\varphi^2)^3\), \( n = 0 \) field theory one can show that the direct renormalization method of reference [12] is « finite » to all orders [14,15], and one can also introduce simpler renormalization methods in polymer theory which are both direct and dimensional, and work to all orders [15].

In section 2, we present the dimensional renormalization scheme for the \((\varphi^2)^3\) field theory near the tricritical point, and apply it to polymers near the corresponding theta-point, obtaining some results derived independently in reference [14]. In section 3, we define the renormalization functions for the tricritical polymer theory, which are the logarithmic derivatives of the partition functions with respect to the Brownian area of the chains. We show that these functions are finite in terms of the dimensionally renormalized quantities. In section 4, we define in details the direct renormalization scheme for semi-attractive chains near the theta-point. The renormalization equations are established, and it is stated that this direct renormalization yields quantities which are finite when \( \varepsilon \to 0 \), to all orders in \( \varepsilon \) and in the coupling constants, when they are expressed in terms of physical polymer quantities. As a consequence, the renormalization functions defined in section 3 are shown to be also regular functions (for \( \varepsilon \to 0 \)) of the physical variables describing the polymer chains near the theta-point. The relation between the dimensional and the direct tricritical renormalizations is described in appendix. The proof of the finiteness of the tricritical direct renormalization to all orders is established. We also give some explicit results of the analytical calculation of universal laws governing the polymer chains at the theta point (for \( d < 3 \)), obtained by the direct method.

2. Field and polymer theories.

We consider the Hamiltonian :

\[
H = \frac{1}{2} \int dx \sum_{a=1}^{N} \left\{ \frac{1}{2} (\nabla \varphi_a(x))^2 + m_a^2 \varphi_a(x)^2 \right\} + \frac{b}{2!} \int dx \left\{ \frac{1}{2} \sum_{a=1}^{N} \left[ \nabla \varphi_a(x) \right]^2 \right\}^2 + \frac{w}{3!} \int dx \left\{ \frac{1}{2} \sum_{a=1}^{N} \left[ \nabla \varphi_a(x) \right]^2 \right\}^3
\]

of the \( N \) fields \( \varphi_a(x) \), \( a = 1, \ldots, N \) with \( n \) components \( l = 1, \ldots, n \). \( N \) is an arbitrary number which will actually correspond to the number of different polymer chains. The space dimension \( d \) satisfies \( d \leq 3 \), and we set \( d = 3 - \varepsilon \), \( \varepsilon > 0 \). The \( b \) interaction term corresponds to two-body interactions, while the \( w \) term is the three-body one, which is also relevant for \( d < 3 \). In dimensional regularization, the tricritical point will correspond to \( b = 0 \) (and \( m_a \to 0 \)), and around it, \( b \leq 0 \). But naturally one assumes \( w \) to be strictly positive. In the \( n = 0 \) limit, the Hamiltonian (2.1) will describe a system of \( N \) polymer chains, after a Laplace transform with respect to all the independent masses \( m_a, a = 1, \ldots, N \). More precisely, the connected Green functions \( G^{2N} \), defined in Fourier space by :

\[
(2\pi)^d \delta^{d}(k_1 + \cdots + k_{2N}) \mathcal{G}^{2N}(\{k\}, \{m_a\}, b, w, \varepsilon) = \left\langle \varphi_{1,1}(k_1) \varphi_{1,1}(k_2) \cdots \varphi_{N,1}(k_{2N-1}) \varphi_{N,1}(k_{2N}) \right\rangle_{\text{connected}}
\]

where \( \varphi_{a,1}(k) = \int d^d x e^{ik \cdot x} \varphi_{a,1}(x) \) are the inverse Laplace transforms of the partition functions \( \mathcal{F}_N \) of a set of \( N \) polymer chains [10], i.e.

\[
\mathcal{F}_N(\{k\}, S_1, \ldots, S_N, b, w, \varepsilon) = \int_{-\infty}^{+i\infty+\eta} \prod_{a=1}^{N} \frac{d^2 m_a}{2\pi i} \exp \left( \sum_{a=1}^{N} m_a^2 S_a \right) \mathcal{G}^{2N}(\{k\}, \{m_a^2\}, b, w, \varepsilon) \bigg|_{\varepsilon=0}
\]

Here the \( S_a \) represent the Brownian areas of the \( N \) polymer chains. The partition functions \( \mathcal{F} \) (in Fourier space) of these polymer chains are those of the continuous model [7, 16] described by the probability density \( P \) for a
configuration \( \{ r_a(s_a) \} \) of the \( N \) chains \( a = 1, \ldots, N \):

\[
P \{ \mathbf{r} \} = \exp \left\{ -\frac{1}{2} \sum_{a=1}^{N} \int_{0}^{s_a} ds \left( \frac{d^2 \mathbf{r}_a(s)}{ds^2} \right)^2 - b \sum_{a=1}^{N} \sum_{a' = 1}^{N} \int_{0}^{s_a} ds \int_{0}^{s_{a'}} ds' \delta(\mathbf{r}_a(s) - \mathbf{r}_{a'}(s')) \right\}
\]

The connected partition functions are defined for \( N \) chains \([12]\) by:

\[
\text{where } P_0 \text{ stands for the probability density (2.4) of Brownian chains, i.e. for } b = 0, w = 0. \text{ The connected partition function of } N \text{ chains reads accordingly: }
\]

\[
\mathcal{Z}_N(\{ S_a \}, b, w, \varepsilon) = \int \mathcal{D} \{ \mathbf{r} \} P_0 \{ \mathbf{r} \} \exp \left\{ i \sum_{a=1}^{N} \left[ \mathbf{k}_{2a-1} \cdot \mathbf{r}_a(S_a) + \mathbf{k}_{2a} \cdot \mathbf{r}_a(0) \right] \right\}
\]

where \( P_0 \) stands for the probability density (2.4) of Brownian chains, i.e. for \( b = 0, w = 0 \). The connected partition function of \( N \) chains reads accordingly:

\[
\mathcal{Z}_N(\{ S_a \}, b, w, \varepsilon) = \mathcal{Z}_N(\{ 0 \}, \{ S_a \}, b, w, \varepsilon).
\]

In a similar way one defines partition functions with insertions of wave vectors at fixed points along the chains. We shall not consider all of them in detail here, but our results also apply to them. We shall consider here the form factor of a single chain \([15]\):

\[
H(\mathbf{q}, S, b, w, \varepsilon) = \frac{1}{S^2} \int_{0}^{S} ds \int_{0}^{S} ds' \langle e^{i \mathbf{q} \cdot (\mathbf{r}(s) - \mathbf{r}(s'))} \rangle
\]

where the average is taken with respect to the weight (2.4). One has:

\[
H(\mathbf{q}, S, b, w, \varepsilon) = \frac{\mathcal{Z}_2^{(2)}(\mathbf{q} - \mathbf{q}, S, b, w, \varepsilon)}{\mathcal{Z}_1(S, b, w, \varepsilon)}
\]

where \( \mathcal{Z}_1^{(2)} \) is the partition function of a single chain with two insertions of wave vectors \( \mathbf{q} - \mathbf{q} \), integrated along the chain, and where \( \mathcal{Z}_1 \) is the partition function of the chain. In terms of the Green functions of the \( n = 0 \) field theory one has \([15]\) a formula analogous to (2.3):

\[
\mathcal{Z}_1^{(2)}(\mathbf{q} - \mathbf{q}, S, b, w, \varepsilon) = \frac{1}{S^2} \int_{\eta = -i \infty}^{\eta = i \infty} \frac{dm^2}{2\pi i} e^{m^2 S} \mathcal{G}_2^{(2)}(\mathbf{q} - \mathbf{m}^2, b, w, \varepsilon)
\]

where \( \mathcal{G}_2^{(2)} \) is the connected Green function with two mass insertions defined by \([15]\):}

\[
\mathcal{G}_2^{(2)}(\mathbf{q} - \mathbf{m}^2, b, w, \varepsilon) = \int d^d r \int d^d r' e^{\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} \frac{\partial}{\partial m^2(r)} \frac{\partial}{\partial m^2(r')} \mathcal{G}_2(0, 0, \{ m^2(\mathbf{r}) \}, b, w, \varepsilon)
\]

where the mass is allowed to depend on the position \( \mathbf{r} \), and where \( \mathcal{G}_2 \) is the two-leg Green function defined in (2.2).
regularization, the tricritical point of the field theoretic Hamiltonian (2.1) corresponds to:

\[ b = 0, \quad m_a^2 = 0 \quad \forall a = 1, \ldots, N. \]  

\[ (2.11) \]

For a theory with a cut-off, this is not true since there are additive renormalizations of the masses and of the coupling constant \( b \) [7, 19].

Now, the dimensional renormalization scheme, or minimal subtraction scheme, for the field theory described by the Hamiltonian (2.1) consists in the change of variables:

\[
\begin{align*}
  w &= \mu^{2e} w_R Z_a(w_R, e) \\
  m_a^2 &= m_{a,R}^2 Z_a(w_R, e) + \left[ b_R Z_d(w_R, e) \right]^2 \zeta \\
  \varphi &= Z^{1/2}(w_R, e) \varphi_R \\
  \zeta &= \mu^{-2e} \zeta'(w_R, e).
\end{align*}
\]

\[ (2.12) \]

Here \( \mu \) is the usual arbitrary \( t' \) Hooft’s mass which gives the scale. In (2.12), all the renormalization factors \( Z_a, Z_m, Z_d, \zeta \) have the minimal form:

\[ \mathcal{G}^{2N}(\{ k \}, \{ m_2 \}, b, w, e) = Z^{N}(w_R, e) \mathcal{G}^{2N}_R(\{ k \}, \{ m_2^{2N}_a \}, b_R, w_R, \mu, e) \]  

\[ (2.15) \]

where \( \mathcal{G}^{2N}_R \) is a function which stays finite when \( e = 3 - d \to 0^+ \).

Inserting now this renormalization equation into the inverse Laplace transform (2.3), we find immediately:

\[
\overline{\mathcal{F}}_N(\{ k \}, \{ S_a \}, b, w, e) = (ZZ_m)^N \exp \left[ \left( b_R Z_d \right)^2 \zeta \sum_a S_a \right] \times \overline{\mathcal{F}}_{N,R}(\{ k \}, \{ S_{a,R} \}, b_R, w_R, \mu, e)
\]

\[ (2.16) \]

where the renormalized partition function \( \overline{\mathcal{F}}_{N,R} \) is defined by the inverse Laplace transform:

\[
\overline{\mathcal{F}}_{N,R}(\{ k \}, \{ S_{a,R} \}, b_R, w_R, \mu, e) = \int_{-\infty}^{+\infty} N \prod_{a=1}^N \frac{dm_a^2}{2\pi i} \exp \left( \sum_{a=1}^N m_{a,R}^2 S_{a,R} \right) \mathcal{G}^{2N}_R(\{ k \}, \{ m_2^{2N}_a \}, b_R, w_R, \mu, e)
\]

\[ (2.17) \]

together with the relation:

\[
S_{a,R} = Z_m(w_R, e) S_a.
\]

\[ (2.18) \]

The renormalized partition functions have now a finite limit when \( e \to 0^+ \), as functions of the renormalized variables, \( S_{a,R}, b_R, w_R \). Thus, we see that the dimensional renormalization of the field theory (2.3) can be trivially applied to the polymer theory, yielding the renormalization equation (2.16). We also observe that the renormalization of the partition functions is, as always, multiplicative with factors \( ZZ_m \), which correspond to the renormalization at the tricritical point. However, away from this tricritical point, \( b \neq 0 \), we note the appearance of a supplementary renormalization factor for the partition functions, namely \( \exp \left( b^2 \zeta \sum S_a \right) \), which corresponds to the appearance of new divergences induced by the \( \phi^4 \) insertions away from the tricritical point [7, 19]. The existence of this factor will be essential [7, 11] for the calculation of the non normalized partition functions, or of the free energy \( \ln \mathcal{F} \) and specific heat. Another fact must be emphasized: in the dimensional renormalization scheme (2.12) (also called mass independent, or soft-mass renormalization scheme) the renormalization factors are independent of the masses and it is precisely this point which makes the passage to polymer theory by an inverse Laplace transform so trivial [14, 15]. Let us finally note that the renormalization factors \( Z_m, Z_a, \zeta, Z \) depend only on \( w_R \) and \( e \). This makes the relation (2.16) very useful in polymer theory as we shall see in section 3.
A similar equation exists for $H$ and $\mathcal{F}^{(2)}_1(2.8)$. We first consider $\mathcal{F}^{(2)}_2(2.10)$ which, owing to equation (2.12b) (2.15) reads:

$$\mathcal{F}^{(2)}_2(q, -q, m^2, b, w, \epsilon) = Z(w, \epsilon) \left[ Z_m(w, \epsilon) \right]^{-2} \mathcal{F}^{(2)}_{2,(k)}(q, -q, m^2, b, w, \mu, \epsilon)$$

where the renormalized $\mathcal{F}^{(2)}_{2,(k)}$ is defined by:

$$\mathcal{F}^{(2)}_{2,(k)}(q, -q, m^2, b, w, \mu, \epsilon) = \int d^{2}r d^{2}r' e^{q(r-r')} \frac{\partial}{\partial m^2(r')} \mathcal{F}^{(2)}_{2,(k)}(0, 0, \mu, \epsilon),$$

and is regular when $\epsilon \to 0$. Then using equations (2.9), (2.12b, c) and (2.19) we find immediately for the partition function with two insertions $\mathcal{F}^{(2)}_1(q, -q, S, b, w, \epsilon)$:

$$\mathcal{F}^{(2)}_1(q, -q, S, b, w, \epsilon) = ZZ_m(w, \epsilon) e^{b\mu} \mathcal{F}^{(2)}_{1,(k)}(q, -q, S, b, w, \mu, \epsilon)$$

where $\mathcal{F}^{(2)}_{1,(k)}$ is a dimensionally renormalized partition function, which is a simple inverse Laplace transform with respect to the renormalized mass:

$$\mathcal{F}^{(2)}_{1,(k)}(q, -q, S, b, w, \mu, \epsilon) = \frac{1}{S_k} \int_{-i\infty}^{i\infty} \frac{dm^2}{2\pi i} e^{s m^2} \mathcal{F}^{(2)}_{2,(k)}(q, -q, m^2, b, w, \mu, \epsilon).$$

We immediately obtain an interesting consequence for the form factor $H(2.8)$ by using the fundamental renormalization equations (2.16) and (2.20):

$$H(q, S, b, w, \epsilon) = \frac{\mathcal{F}^{(2)}_1(q, -q, S, b, w, \mu, \epsilon)}{\mathcal{F}^{(2)}_{1,(k)}(S, b, w, \mu, \epsilon)} = H_R(q, S, b, w, \mu, \epsilon)$$

where $H_R$ is a regular function of $\epsilon$, when expressed in terms of the renormalized quantities $S_R, b_R, w_R$. We notice the disappearance in $H$ of the renormalization factor $e^{b\mu}$, as expected, since $H$ is a ratio of partition functions, i.e. a physical quantity.

Let us consider also another important physical quantity for a polymer solution, i.e. the osmotic pressure $\Pi$. We consider here a monodisperse system for simplicity, but we could generalize our results to polydisperse solutions. In the grand canonical formalism, the osmotic pressure reads:

$$\Pi = \beta \mathcal{Q}(f)$$

$$\mathcal{Q}(f) = \frac{f \partial}{\partial f} \mathcal{Q}(f)$$

where $C$ is the chain concentration. Inserting into (2.22) the renormalization equation (2.16) we get immediately:

$$\mathcal{Q}(f) = \mathcal{Q}_R(f_R)$$

$$= \sum_{n \geq 1} \frac{f_R^n}{n!} \mathcal{F}^{(n)}_R(N \times S, b, w, \epsilon)$$

$$= f_R \frac{\partial}{\partial f_R} \mathcal{Q}_R(f_R)$$

where $f_R$ is the renormalized fugacity:

$$f_R = fZ_m(w, \epsilon) e^{b\mu}.$$ (2.24)

Eliminating $f_R$ in favour of $C$, we therefore find at this stage that $\Pi$ has the renormalized form:

$$\Pi = \Pi_R(C, S, b, w, \mu, \epsilon).$$ (2.25)

where this function $\Pi_R$ is regular with respect to $\epsilon$, when $\epsilon \to 0$. We shall later on obtain an universal form of $\Pi$ in terms of renormalized physical quantities only (Sect. 4), once we have introduced the tricritical direct renormalization method.

3. Renormalization functions for polymer theory.

Let us note first that the minimal renormalization factors $Z_m, Z_m, Z_m, \zeta, \zeta$ could be calculated as series in $w_R$, by using only the polymer theory and the finiteness of the $\mathcal{F}_R$ partition functions in (2.16). Doing this, one would necessarily find the same factors as in field theory for $n = 0$, as we have checked for instance in the critical case [15].

Here, we shall deduce from the general relation (2.16), another result which has proven to be very useful for explicit calculations in the tricritical polymer theory [11].

Let us consider the set of polymer chains and
write:

\[ S_a = x_a S \]  

(3.1)

where \( S \) is a Brownian area which gives the common scale of the areas \( S_a \), the numbers \( x_a \) being all finite. We then define a parameter \( z_0 \) by:

\[ z_0 = bS^{1/2} \]  

(3.2)

For \( d = 3 \), \( z_0 \) is dimensionless and is similar to the dimensionless \( z \) parameter of polymer theory [20]:

\[ z = (2\pi)^{-d/2} bS^{2-d/2} \]  

For \( d < 3 \) these parameters differ and \( z_0 \) acquires a positive mass dimension, whereas \( z \) stays dimensionless. So, like the dimensional parameter \( w \), \( z_0 \) plays the role of the coupling constant of an operator which becomes relevant at \( d \leq 3 \).

We introduce then the logarithmic derivatives:

\[ \sigma_n = S \frac{\partial}{\partial S} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon) \]  

(3.3)

taken at fixed \( w \) and \( z_0 \), and (trivially) \( \mu \). Owing to equation (2.16), we have:

\[ \sigma_n = S \frac{\partial}{\partial S} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon) \]  

(3.4)

But \( w_R \) depends only on \( w \) and \( \mu \), and not on \( S \). Thus we have immediately:

\[ S \frac{\partial}{\partial S} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon) = 0, \quad S \frac{\partial}{\partial S} (z_0^2) = 0. \]  

(3.5)

The only quantities which depend on \( S \), at fixed \( w, z_0, \mu \) are thus \( b, S_R, \) and \( b_R \). One has, according to (2.12), (3.2):

\[ S \frac{\partial}{\partial S} S_R = S_R, \quad S \frac{\partial b}{\partial S} = -\frac{1}{2} b, \quad S \frac{\partial b_R}{\partial S} = -\frac{1}{2} b_R. \]

Therefore we find:

\[ \sigma_n = S \frac{\partial}{\partial S} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon) = S_R \frac{\partial}{\partial S} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon) - \frac{1}{2} b_R \frac{\partial}{\partial b_R} \ln \mathcal{F}_n(\{ k \}, \{ S_a \}, b, w, \varepsilon). \]  

(3.6)

This shows the important result:

**Theorem I:**

*The renormalization functions of polymer theory:*

\[ \sigma_n = S \frac{\partial}{\partial S} \ln \mathcal{F}_n \]  

are regular in \( \varepsilon \), and have a finite limit for \( \varepsilon \to 0 \), when expressed in terms of the renormalized variables \( S_R, w_R, b_R, \mu \).

We note that differentiating at fixed \( z_0 \) is essential for \( \sigma_n \) to be finite. Taking naively the \( S \)-derivatives at fixed \( b \) would yield a supplementary term in (3.4) proportional to \( \zeta \), which still diverges when \( \varepsilon \to 0 \).

It remains to precise the nature of the renormalized parameters. For this we shall define a certain number of physical quantities, say \( \{ X \} \), describing our polymer chains near the theta point. Then we shall show that the change of variables relating these quantities \( \{ X \} \) to the field-theoretic parameters \( w_R, b_R, S_R \) and \( \mu \), is regular in \( \varepsilon \), and finite when \( \varepsilon \to 0 \).

As a consequence, all other physical quantities have regular expressions in \( \varepsilon \), in terms of the primary quantities \( \{ X \} \). This is very useful, since the physical quantities \( \{ X \} \) will be experimentally measurable. These \( \{ X \} \) will then be called the « directly renormalized » variables. This is the philosophy of the « direct renormalization method » introduced in reference [12], for the excluded volume or critical case and which can be proven to work to all orders [14, 15].

Here we define a « direct renormalization method » for the tricritical case, i.e. semi-attractive polymer chains. We used extensively elsewhere [11] this method for doing explicit calculations, and we shall prove here that it works actually to all orders.

4. **Direct renormalization for semi-attractive polymer chains.**

For simplicity, we consider here chains which have the same Brownian area \( S \), but the extension of the analysis to the polydisperse case (3.1) is straightforward.

Let us consider the general connected partition function \( \mathcal{F}_n(2.5) \) of the system of polymer chains. It will be convenient to consider its formal series expansion with respect to the \( \delta \) parameter:

\[ \mathcal{F}_n = \mathcal{F}_n + \sum_{p=1}^\infty \frac{\beta^p}{p!} p \mathcal{F}_n, \]  

(4.1)

where the dot denotes the value of the partition.
functions at the value $b = 0$, i.e. at the theta-temperature of infinite chains, and where the partition functions $\hat{P}_N$ depend only on $w$ (and $S$, $\{k\}$, $\varepsilon$).

The general connected partition functions (2.5) $\hat{P}_N$ yield, when taken at zero momenta $\{k\} = \{0\}$, the regularized partition functions of the chains:

$$\hat{P}_N(\{k\} = \{0\}, \ldots) \equiv \mathcal{Z}(N \times S, b, w, e).$$  (4.2)

By dimensional analysis, one has immediately (see Eq. (2.5)):

$$\mathcal{Z}_N(\{k\}, N \times S, b, w, e) = \mathcal{Z}_N(\{kS^{1/2}\}, z, y, e)$$  (4.3)

where the dimensionless quantities $z$ and $y$ are defined by:

$$z = (2\pi)^{-d/2} b S^{2-d/2}, \quad y = (2\pi)^{-d} w S^{3-d}$$  (4.4)

$z$ is the usual $z$-parameter [20], and $y$ is the analogous of the latter for ternary interactions. In particular, one sees that $y$ does not depend on $S$ when the space dimension is equal to the upper tricritical dimension $d^* = 3$.

The end-to-end distance of a single polymer chain is given by:

$$R^2 = \langle \| r(S) - r(0) \|^2 \rangle = dX^2$$  (4.5)

where

$$X^2 = -2 \frac{\partial}{\partial k^2} \frac{\mathcal{Z}(k, -k, S, b, w, e)}{\mathcal{Z}(S, b, w, e)}\bigg|_{k=0}.$$  (4.6)

By dimensional analysis, one has immediately:

$$X^2 = S \mathcal{X}_0(y, z, \varepsilon)$$  (4.7)

where $\mathcal{X}_0$ is the swelling factor of an isolated semi-attractive chain. Incidentally, let us remark that $\mathcal{X}_0$, when expanded in series of $y$ and $z$ has an infinite series of poles in $1/\varepsilon$.

We shall also consider the square radius of gyration $R_G^2$ defined by:

$$R_G^2 = \frac{1}{2S^2} \int_0^S \int_0^S ds \, ds' \langle \| r(s) - r(s') \|^2 \rangle$$  (4.8)

which reads in terms of the form factor (2.7)

$$R_G^2 = -\int \frac{d}{dq^2} H(q, S, b, w, e)\bigg|_{q=0}.$$  (4.9)

By dimensional analysis, one may set:

$$R_G^2 = \frac{dS}{6} \mathcal{X}_0(y, z, \varepsilon)$$  (4.10)

where $0R_G^2 = dS/6$ is the Brownian result, and where, here again, $\mathcal{X}_0$ contains an infinite series of poles in $1/\varepsilon$. We shall also consider the universal ratio:

$$\mathcal{N} = 6 \frac{R_G^2}{R^2} = \mathcal{X}_0/\mathcal{X}_0.$$  (4.11)

The osmotic pressure $\Pi$ (2.22) of the polymer solution has the virial expansion [7]:

$$\Pi \beta = C - \frac{1}{2} C^2 \frac{\mathcal{Z}(S, S)}{[\mathcal{Z}(S)]^2} -$$

$$- \frac{C^3}{3} \left( \frac{\mathcal{Z}(S, S)}{[\mathcal{Z}(S)]^3} - \frac{3}{2}[\mathcal{Z}(S)]^2 \right) + \ldots$$  (4.12)

where $\beta$ is the inverse temperature, and $C$ the number of chains per unit of volume. For simplicity the explicit dependence of the $\mathcal{Z}$ partition functions on $b, w$ and $\varepsilon$ has not been indicated.

The dimensionless second and third virial coefficients $g$ and $h$ are then defined by:

$$g = g(y, z, \varepsilon) = -\left(2\pi X^2\right)^{-d/2} \frac{\mathcal{Z}(S, S, b, w, \varepsilon)}{[\mathcal{Z}(S, b, w, \varepsilon)]^2}$$  (4.13)

$$h = h(y, z, \varepsilon) = -\left(2\pi X^2\right)^{-d} \left[\mathcal{Z}(S, S, b, w, \varepsilon) - \mathcal{Z}(S, b, w, \varepsilon)\right] -$$

$$- \frac{3}{2}[\mathcal{Z}(S, b, w, \varepsilon)]^2 \frac{\mathcal{Z}(S, S, b, w, \varepsilon) - \mathcal{Z}(S, b, w, \varepsilon)}{[\mathcal{Z}(S, b, w, \varepsilon)]^3}$$  (4.14)

and the virial expansion of the osmotic pressure reads:

$$\Pi \beta / C = 1 + \frac{\beta^2}{2} C \mathcal{X}^4 + \frac{(\varepsilon)}{3} (C \mathcal{X}^4)^2 + \ldots$$  (4.15)

$$\Pi \beta / C = C \mathcal{F}(C \mathcal{X}^4, g, h, e)$$

where the universal function $\mathcal{F}$ is regular and finite when $\varepsilon \to 0$. In the same way, the ratio $\mathcal{N}$ (4.11) is a universal function $\mathcal{N}(h, g, \varepsilon)$ which is regular when $\varepsilon \to 0$ (see below).

We shall prove below, with the help of the direct renormalization method that the osmotic pressure $\Pi$ has a universal tricritical form in terms of $X, g, h$:

$$\Pi \beta / C = C \mathcal{F}(C \mathcal{X}^4, g, h, e)$$

where the universal function $\mathcal{F}$ is regular and finite when $\varepsilon \to 0$. In the same way, the ratio $\mathcal{N}$ (4.11) is a universal function $\mathcal{N}(h, g, \varepsilon)$ which is regular when $\varepsilon \to 0$ (see below).

The tricritical theta-point corresponds to the two conditions [7]:

$$\begin{align*}
g &= 0 \iff b = 0 \\
S &\to \infty \iff S \to \infty.
\end{align*}$$  (4.17)

Let us now introduce the direct renormalization scheme for semi-attractive chains. Using the expansion (4.1), we define successively (the dots refer to the special point $b = 0$):

$$X^2 = -2 \frac{\partial}{\partial k^2} \frac{\mathcal{Z}(k, -k, S, b, w, e)}{\mathcal{Z}(S, b, w, e)}\bigg|_{k=0} =$$

$$\mathcal{X}_0(y, \varepsilon) S = \mathcal{X}_0(y, z = 0, \varepsilon) S.$$  (4.18)
We also define a renormalized binary interaction \( \beta \) by:

\[
\beta(y, e) = -(2\pi X^2)^{-d} \frac{\mathcal{F}(S, S, W, e)}{[\mathcal{F}(S, W, e)]^2} \tag{4.19}
\]

Incidentally let us note that the quantity \( \mathcal{A} \) (4.19) is defined at the infinite chain theta-point \( b = 0 \). However, it does not coincide with the third virial coefficient \( \mathcal{A} \) for \( b = 0 \). One has indeed, owing to equation (4.15):

\[
\mathcal{A}(y, e) \equiv \mathcal{A}(\beta, \mathcal{A}) = \mathcal{F}(S, W, e) = -(X^2)^{-2} \frac{\mathcal{F}(S, S, W, e)}{[\mathcal{F}(S, W, e)]^2} \tag{4.20}
\]

We also define a renormalized binary interaction \( \beta \) by:

\[
\beta = \beta \cdot \mathcal{E}(y, e). \tag{4.22}
\]

At the \( \beta \)-point of infinite chains, \( b = 0 \), but the second virial coefficient \( \gamma \) of finite chains does not vanish

\[
\gamma(y, z = 0, e) \neq 0,
\]

since there are residual ternary interactions \( y \), which in turn induce an effective two-chain interaction, for finite chains.

We consider \( \beta \) instead of \( h \), since it is a simpler quantity to calculate. In the tricritical limit, \( b = 0 \), \( S \to \infty \), both \( \beta \) and \( h \) reach for \( d < 3 \), \( d = 3 - \varepsilon \) the fixed point value \([11]\) \( h^* = \varepsilon / 44 \pi + \ldots \), or, for \( d = 3 \), have the same asymptotic limit \( \beta = 1 / (44 \pi \ln S) \) \([7,11]\).

Since \( g \) does not vanish at \( b = 0 \) for finite chains, it will be convenient also to define a linearized dimensionless second virial coefficient \( g \) by:

\[
g = (2\pi)^{-d/2} \beta(X^2)^{2-d/2} = (2\pi)^{-d/2} \beta \cdot \mathcal{E}(X^2)^{2-d/2}. \tag{4.24}
\]

This quantity \( g \) vanishes at the theta-point \( b = 0 \).

As in field theory, there is a fifth renormalization term in polymer theory corresponding to \( \zeta \). \( \zeta \) is the renormalization factor of the additional renormalization of the mass due to \((g^2)^2\) insertions (Eq. (2.12b)). We have (see Eq. (2.12c)) \( \zeta = \zeta(w_R, e) \mu^{-2\varepsilon} \). It will be convenient to use (2.12a) and write \( \zeta \) in the form:

\[
\zeta = \mu^{-2\varepsilon} \zeta(w_R - 2\varepsilon, e) \tag{4.25}
\]

where \( \zeta \) is diverging when \( \varepsilon \to 0 \), but is not of the minimal form. All these quantities are defined at the point \( b = 0 \).

Now we state that, once expressed in terms of these physical quantities \( \{ X, \beta, b \} \), the other physical quantities describing the polymer system, will be finite, i.e. will have a well-defined series expansion in power of \( \varepsilon \). These quantities \( \{ X, \beta, b \} \) will be the « directly renormalized » variables whereas \( \mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_4 \) are renormalization factors.

Let us give then the multiplicative renormalization equation, which, in the direct renormalization method, corresponds to the multiplicative renormalization equation (2.16) and is deduced from it (see the Appendix).

**Theorem II.** The dimensionally regularized connected partition functions \( \mathcal{F}_n \) of the polymer tricritical theory satisfy a multiplicative renormalization equation:

\[
\mathcal{F}_n \{\{ k \} \times S, b, w, e, \mu \} = \mathcal{F}_n \{\{ k \} \mathcal{X}, \beta, b, \mu \}. \tag{4.26}
\]

where \( \mathcal{F}_n \) is a renormalized partition function, which is a regular function of \( \varepsilon = 3 - d \), when expressed in terms of the physical quantities \( \mathcal{X}, \beta, b \) defined in (4.18), (4.19), (4.21), (4.22). \( \mathcal{F}_n \) is the partition function of a single polymer chain, at \( b = 0 \).

\( \zeta \) is a renormalization factor, which diverges when \( \varepsilon \to 0 \). \( z_0 \) is the dimensional \( z \)-like parameter \( z_0 = bS^{1/2} \).

\( \mu \) is an arbitrary mass scale, which is necessary for defining \( \zeta \) and \( \mathcal{F}_n \), but of which \( \mathcal{F}_n \) is naturally independent.

This result is proven in the Appendix by starting from equation (2.16). In particular, for proving that the direct renormalization method yields a well-defined renormalization, to all orders in perturbation theory, it is sufficient to show that the « change of variables »:

\[
\{ X, \beta, b \} \leftrightarrow \{ S_R, w_R, b_R, \mu \} \tag{4.27}
\]

between the physical quantities and the dimensionally renormalized quantities considered before, is a finite change of variables, regular in \( \varepsilon \). Of course, \( \mu \) does not appear in any polymer physical quantity (it recombines necessarily with \( w_R \)). This is the reason why the number of physical variables \( X, \beta, b \) is reduced by one unit, with respect to the renormalized variables \( S_R, w_R, b_R, \mu \).

As a consequence, the theorem I, formulated in terms of the renormalized variables \( S_R, w_R, b_R, \mu \), yields, when the change of variables \( \{ S_R, w_R, b_R, \mu \} \to \{ X, \beta, b \} \) is performed, the new result.

Of course, since \( g \) is related to \( \beta \) by the regular equation (4.24), the substitution:

\[
\{ X, g, \beta \} \leftrightarrow \{ S_R, w_R, b_R, \mu \} \tag{4.28}
\]

in the physical quantities describing the polymer chains, is also regular in \( \varepsilon \). Thus, one can use the dimensionless \( g \) instead of \( \beta \) in the renormalization equations (4.26).

Now, we can discard the mass scale \( \mu \) appearing in the dimensional renormalization scheme. Let us set by convention in our direct renormalization scheme:

\[
\mu \cdot X = 1. 
\]
We obtain easily from definition (3.2) for the renormalization argument \( \exp(\xi / \mu) \)

\[
A \equiv z_0^2 \mu^{-2\epsilon} \xi(w\mu^{-2\epsilon}, \epsilon) = b^2 S \cdot X^{2\epsilon} \xi,
\]

and owing to (4.18)

\[
A = b^2 S^\Delta \cdot X_0^{2\epsilon} \xi = (2\pi)^d z^2 \cdot X_0^{2\epsilon} \xi.
\]

Finally \( \xi(w\mu^{-2\epsilon}, \epsilon) \) can be rewritten as a function \( \xi' \) of \( y, \epsilon \) alone, owing to the choice of \( \mu \). Therefore we have finally:

\[
A = (2\pi)^d z^2 \cdot X_0^{2\epsilon} \xi = z^2 \xi'(y, \epsilon).
\]

where \( \xi' \) is a renormalization factor diverging when \( \epsilon \to 0 \) (and not of the minimal form). Thus we may rewrite the direct renormalization equation (4.26) in the most useful form for polymer theory:

\[
\mathcal{F}_N(\{ k \}, N \times S, b, w, \epsilon) = [\mathcal{F}_i(y, \epsilon)]^N \mathcal{F}_0(y, \epsilon) \cdot X \cdot \xi'(y, \epsilon) \times \mathcal{F}_N R(\{ k \}, \cdot X, \cdot b, \cdot h, \epsilon)
\]

(4.26bis)

where the directly renormalized polymer partition functions \( \mathcal{F}_N^R \) are regular functions of \( \epsilon \).

Important results can be inferred from the multiplicative renormalization equation (4.26) (4.26bis) and from the regularity of the substitutions (4.27) (4.28). Let us first consider the generating function for the moments of the end-to-end distance of a single chain:

\[
G(q, S, b, w, \epsilon) = \langle e^{iq \cdot (\vec{r}_N)} \rangle
= \frac{\mathcal{F}_i(q - q, 1 \times S, b, w, \epsilon)}{\mathcal{F}_i(1 \times S, b, w, \epsilon)}.
\]

(4.29)

According to (4.26bis), we get immediately:

\[
G(q, S, b, w, \epsilon) = \frac{\mathcal{F}_i(q - q, \cdot X, \cdot b, \cdot h, \epsilon)}{\mathcal{F}_i(\cdot X, \cdot b, \cdot h, \epsilon)}
\]

(4.30)

which is regular in \( \epsilon \). Using now equation (4.6) and dimensional analysis we get:

\[
X^2 = X^2 F_0(y, h, \epsilon)
\]

(4.31)

where the function \( F_0 \) of the physical variables \( y, h \) is regular with respect to \( \epsilon \). Thus the size \( X \) of the chain away from the theta point and the size \( X \) exactly at the theta point are related by a finite factor, as expected. Let us now turn to the form factor of a single chain \( H(q) \) defined in equation (2.7). According to equation (2.21) it has a finite renormalized expression \( H_R \) in terms of the variables \( S_R, b_R, w_R, \mu \), of the dimensional tricritical renormalization scheme.

Thus, using the regularity of the substitutions (4.27), or (4.28), and dimensional analysis, we state that the form factor \( H \) can be written in the form:

\[
H(q, S, b, w, \epsilon) = H\left(\frac{1}{2} q^2 \cdot X^2, y, h, \epsilon\right)
\]

(4.32)

where the function \( H \) of the dimensionless physical quantities \( q, X, h, \epsilon \) is regular in \( \epsilon \), when \( \epsilon \to 0^+ \).

As a consequence, the squared radius of gyration \( R_g^2 \) (4.9) has the form:

\[
R_g^2 = d \cdot X^2 F_G(y, h, \epsilon)
\]

(4.33)

and is a function of \( y, h \), regular when \( \epsilon \to 0^+ \). In particular, at the theta-point \( y = 0 \), we have obtained [11]:

\[
\mathcal{N} = 6 F_G / F_0 = \mathcal{N}(y, h, \epsilon)
\]

(4.34)

Then in the limit of very long chains one has \( y \to h^* = c/447 \pi + \cdots \) and the ratio \( \mathcal{N} \) reaches the universal value at the theta-point:

\[
\mathcal{N}^* = 1 + \frac{103}{3 \times 2^4 \times 11} \epsilon + \cdots
\]

Let us finally consider the osmotic pressure \( \Pi \) given by equation (2.25) in its renormalized form in terms of the minimal subtraction scheme variables \( S_R, b_R, w_R, \mu \). Using simple dimensional analysis, and, again, the regularity of the substitution (4.28), we obtain the result that \( \Pi \) is given by a universal function \( \mathcal{F} \):

\[
\Pi \beta = \mathcal{F}(C \cdot X^d, y, h, \epsilon)
\]

(4.35)

where \( \mathcal{F} \) is a regular function of \( \epsilon \) which can be calculated by perturbation theory [11]. For instance, we have obtained [11], for \( y = 0 \) (theta-point), for dilute solutions of very long chains:

\[
\Pi \beta = C \left[ 1 - (2\pi)^d \cdot \frac{h}{3} \cdot C \cdot X^d + \cdots \right] =
\]

\[
= C \left[ 1 - (2\pi)^d \frac{11\pi}{447} \cdot C \cdot X^d + \cdots \right]
\]
with \( h \rightarrow h^* = \varepsilon/44 \pi + \cdots \), and for semi-dilute solutions at the theta-point [11]

\[
\Pi\beta = \frac{C}{3} \beta^* \left[ C^2(2 \pi^2 \chi^2) \right]^{1 + \varepsilon} + \cdots
\]

Incidentally, the latter expression of \( \Pi \) has the expected scaling form for semi-dilute solutions

\[
\Pi\beta \propto C^{\nu/d + \varepsilon - 1},
\]

where \( C = CS \) and with \([1, 11]\)

\[
v = \frac{1}{2} + \frac{2}{3} \left( \frac{e^2}{11} \right) + \cdots.
\]

It is now the appropriate place to state that all the results just given hold also true if, instead of using the dotted variables \( \cdot X, \cdot h, \cdot b \) or \( \cdot g \) defined at the theta-point \( b = 0 \), or close to it (i.e. \( g \)), we use the full physical variables \( X, h, g \) away from the theta-point. We show indeed in appendix B that the full second and third dimensionless virial coefficients \( g \) and \( h \) are given in terms of the partial ones \( g, h, b \) by relations:

\[
g = g(g, h, \varepsilon)
\]

\[
h = h(g, h, \varepsilon)
\]

which are naturally regular in \( \varepsilon \). Moreover, as we have seen in equation (4.31), the chain sizes \( X \) (for \( b \neq 0 \)) and \( \cdot X \) (for \( b = 0 \)) are related by a finite function of \( g, h, \varepsilon \). Therefore, in all the above results, one can replace freely the partial virial coefficient \( g, h \) by the exact virial coefficients \( g, h \) defined in equation (4.16). In particular, the generating function \( G \) (4.29), the form factor (4.32) of a single chain have finite expressions

\[
G = \frac{1}{2} q^2 X^2, g, h, \varepsilon \\
H = \frac{1}{2} q^2 X^2, g, h, \varepsilon
\]

In the same way, the universal ratio \( \Xi \) (4.34) is a function \( \Xi = \Xi(g, h, \varepsilon) \) regular when \( \varepsilon \rightarrow 0^+ \). And finally the osmotic pressure (4.35) is given by a universal law in terms of \( g, h, CX^4 \):

\[
\Pi\beta = C\Xi(CX^4, g, h, \varepsilon)
\]

which again is well-defined when \( \varepsilon \rightarrow 0^+ \).

Let us however stress again that for actual computations, the use of \( g \) and \( h \) seems to be more convenient than that of \( g, h \).

Let us now complete the description of the direct renormalization method by giving the analogue of theorem I for our direct physical scheme. We have the theorem III:

**Theorem III:**

*The logarithmic derivatives*

\[
S \frac{\partial}{\partial S} \ln \bar{\Xi}_n(\{ k \}, S, b, w, \varepsilon) \bigg|_{w, 0} = \sigma_n(\{ k \}, S, b, w, \varepsilon) = \sigma_n(\{ k \}, \cdot X, \cdot b, \cdot h, \varepsilon)
\]

are regular functions of \( \varepsilon = 3 - d \), having a finite limit for \( \varepsilon \rightarrow 0 \), when they are expressed in terms of the physical quantities \( \cdot X, \cdot h, \cdot b \).

Let us recall again that it is essential to take \( z_0 = bS^{1/2} \) fixed, for obtaining a finite \( \sigma_n \). The multiplicative renormalization equation (2.16), (4.26bis), and the theorems I, II, III constitute our main results.

Let us also note that the theorem III may be given in another form, which we used for explicit calculations [11]. The polymer partition functions \( \bar{\Xi}_n \) can always be written in the form (see (4.3), (4.4)):

\[
\bar{\Xi}_n = S^{\frac{d}{2} (n - 1)} \bar{\Xi}_n
\]

where \( \bar{\Xi}_n \) is dimensionless and depends only on \( \{ kS^{1/2} \}, z, y \) (and \( \varepsilon \)). Then, the quantity:

\[
\sigma_n(\{ kS^{1/2} \}, z, y, \varepsilon) = S \frac{\partial}{\partial S} \ln \bar{\Xi}_n \bigg|_{w, 0}
\]

can also be written as:

\[
\sigma_n = \left( S \frac{\partial}{\partial S} \right)_{z, y} \sigma_n + \frac{e}{2} z \frac{\partial}{\partial z} + e y \frac{\partial}{\partial y} \ln \bar{\Xi}_n
\]

\[= \sigma_n(\{ k \cdot X \}, \cdot g, \cdot h, \varepsilon) \]

Its expression in terms of \( \cdot X, \cdot g, \cdot h \) is, according to theorem III, finite when \( \varepsilon = 3 - d \rightarrow 0^+ \).

In summary, we have established, from the dimensional renormalization of field theory, the validity to all orders of a direct renormalization scheme, applicable to polymer solutions near the theta tricritical point. This proof completes explicit calculations which we had previously done and which will be published elsewhere.

**Appendix A.**

**The relation between direct and dimensional tricritical renormalizations.** — We shall consider here the dimensionally renormalized partition functions \( \bar{\Xi}_R \) defined in (2.16) and we expand them formally in powers of \( b_R \), as in (4.1):

\[
\bar{\Xi}_R = \bar{\Xi}_R + \sum_{p=1}^{\infty} \frac{b_R^p}{p!} \bar{\Xi}_R.
\]
Now let us calculate the four physical quantities (4.18)-(4.21). From the multiplicative dimensional renormalization equation (2.16) applied to definitions (4.18), (4.19) we find trivially:

\[ X^2 = -2 \frac{\partial}{\partial k^2} \frac{\mathcal{X}_R(k_r - k, S_R, w_R, \mu, \epsilon)}{\mathcal{X}_R(S_R, w_R, \mu, \epsilon)} \bigg|_{k = 0} \]  
(A.2)

\[ h = -(2 \pi X^2)^{-\frac{1}{2}} \frac{\mathcal{X}_R(3 \times S_R, w_R, \mu, \epsilon)}{\mathcal{X}_R(1 \times S_R, w_R, \mu, \epsilon)} \]  
(A.3)

By dimensional analysis, this shows that \( X, h \) can be written in terms of dimensionless variables:

\[ X^2 = S_R F_0(w_R, \mu^2 S_R, \epsilon) \]
\[ h = F_3(w_R, \mu^2 S_R, \epsilon) \]  
(A.4)

where the \( F_n, F \) are easily defined in terms of the renormalized \( \mathcal{X}_R \) and, as the latter, are regular functions of \( \epsilon \). These equations can be inverted to give formally:

\[ w_R = F_{3}^{-1}(h, \mu \cdot X, \epsilon) \]
\[ S_R = X^2 F_{0}^{-1}(h, \mu \cdot X, \epsilon) \]  
(A.5)

where, again \( F_{3}^{-1}, F_{0}^{-1} \) are regular in \( \epsilon \). This is an essential result since we can now substitute freely \( h, X \) to \( w_R, S_R \).

\[ \mathcal{X}_4[h, \epsilon] = Z_4^{-1}(w_R, \epsilon) \frac{1}{\mathcal{X}_R(2 \times S_R, w_R, \mu, \epsilon)} \mathcal{X}_R(S_R, w_R, \mu, \epsilon) \]  
(A.8)

By homogeneity property of \( \mathcal{X}_R \), we may set:

\[ \mathcal{X}_R(2 \times S_R, w_R, \mu, \epsilon) \equiv S_R^{1} \mathcal{X}_R(2, w_R, \mu^2 S_R, \epsilon) \]  
(A.9)

and we finally find:

\[ \mathcal{X}_4[h, \epsilon] = Z_4^{-1}(w_R, \epsilon) \mathcal{X}_R(2, w_R, \mu^2 S_R, \epsilon) F_{0}^{-2}(w_R, \mu^2 S_R, \epsilon) \]  
(A.10)

Thus, up to a regular factor, \( \mathcal{X}_4 \) and the minimal field theoretic renormalization factor \( Z_4^{-1} \) coincide. Here again the relation between them is further simplified if one sets \( \mu^2 S_R = 1 \). The important result is the relation between the directly renormalized parameter \( b \) (4.22) and the dimensionally renormalized one \( b_R \) (2.12c). Because of (A.10), it exists a finite function \( F_4 = b_R F_{4}^{-2} \) such that:

\[ b = b_R F_4(w_R, \mu^2 S_R, \epsilon) \]  
(A.11)

or equivalently:

\[ b_R = \frac{b}{F_4^{-1}(h, \mu \cdot X, \epsilon)} \]  
(A.12)

where \( F_4^{-1} \) is regular in \( \epsilon \).

Now, collecting equations (A.5) and (A.12), which describe the regular change of variables:

\[ \{ S_R, b_R, w_R, \mu \} \rightarrow \{ X, b, h \} \]  
(A.13)

we can substitute in theorem I of section 3 the variables \( X, b, h \) to the variables \( S_R, b_R, w_R \) and we get the announced theorem III of section 4. Q.E.D.

For finally establishing the multiplicative renormalization equation of theorem II, it is then sufficient to consider the renormalization equation (2.16). One uses the finite relation (A.6) between \( ZZ_m \) and \( \mathcal{X}_R \). One recalls the identities (2.12c) relating \( b \) and \( b_R \), (3.2) defining \( z_0 \), (4.25) giving \( \zeta \). One uses finally the fact that the substitution (A.13) is a regular change of variables with respect to \( \epsilon \), and one gets the multiplicative equation of theorem II. Q.E.D.
Appendix B.

THE RELATION BETWEEN THE PARTIAL VIRIAL COEFFICIENTS \( h, g \) AND THE COMPLETE VIRIAL COEFFICIENTS \( h, g \). — We have already seen in (4.31) that the sizes \( X, \dot{X} \equiv X(b=0) \), are related by the equation:

\[
X^2 = \dot{X}^2 F_\partial(h, g, \epsilon) \tag{B.1}
\]

regular with respect to \( \epsilon \).

Now, consider the general directly renormalized polymer partition function \( \mathcal{Z}_{N,R} \) appearing in (4.26bis). By homogeneity we may set:

\[
\mathcal{Z}_{N,R}(X, b, h, g, \epsilon) = (2\pi X^2)^{d(2(N-1))} \mathcal{Z}_{N,R}(g, h, \epsilon) \tag{B.2}
\]

where the \( \mathcal{Z}_{N,R} \) are dimensionless renormalized partition functions, regular with respect to \( \epsilon \). Considering the definition (4.13), (4.14) of \( g \) and \( h \), and using the fundamental equation (4.26bis), we have then:

\[
g = -(2\pi X^2)^{-d/2} \frac{\mathcal{Z}_{1,R}(X, b, h, \epsilon)}{[\mathcal{Z}_{1,R}(X, b, h, \epsilon)]^2} \tag{B.3}
\]

\[
h = -(2\pi X^2)^{-d} \frac{\mathcal{Z}_{1,R}(X, b, h, \epsilon)}{[\mathcal{Z}_{1,R}(X, b, h, \epsilon)]^3} + 3g^2. \tag{B.4}
\]

Using now (B.1) and (B.2) we get immediately:

\[
g = -(F_\partial)^{-d/2} \frac{\mathcal{Z}_{2,R}(g, h, \epsilon)}{[\mathcal{Z}_{1,R}(g, h, \epsilon)]^2} \tag{B.5}
\]

\[
h = -(F_\partial)^{-d} \frac{\mathcal{Z}_{3,R}(g, h, \epsilon)}{[\mathcal{Z}_{1,R}(g, h, \epsilon)]^3} + 3g^2. \tag{B.6}
\]

Therefore, \( F_\partial, \mathcal{Z}_{N,R} \) being regular functions of \( h, g, \epsilon \), we obtain the desired result (4.36):

\[
g = g[g, h, \epsilon]
\]

\[
h = h[g, h, \epsilon]
\]

where \( g[... \) and \( h[...] \) are regular functions of \( \epsilon \). Q.E.D.

References