Recycling interferometric antennas for periodic gravitational waves (*)

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(Reçu le 25 juillet 1985, accepté sous forme définitive le 21 novembre 1985)

Résumé.— Depuis quelques années, des dispositifs expérimentaux utilisant soit des lignes à retard soit des cavités de Pérot-Fabry comme détecteurs d'ondes de gravitation ont été proposées puis développées. On présente ici des formules générales permettant de comparer ces éléments lorsqu'ils sont associés dans un montage à recyclage synchrone, adapté aux ondes gravitationnelles périodiques.

Abstract.— Experimental devices using either delay lines or Fabry-Pérot cavities in detectors of gravitational waves have been proposed and developed for a few years. In order to make clear the comparison between these elements when arranged in a synchronous recycling setup suitable for periodic waves, general formulae giving the signal amplitudes are proposed.

Introduction.

Interferometric gravitational wave antennas involving either delay lines or Fabry-Pérot cavities in a Michelson configuration are widely known and have given rise to a great deal of experimental as well as theoretical work [1, 2]. These antennas are rather devoted to the detection of short bursts of gravitational radiation. In the case of periodic waves of known period radiated by quasi-stationary sources, this supplementary information about the period theoretically permits enhancement of the signal-to-noise ratio (SNR), provided this information is stored somewhere in the detection device pattern: a few years ago, R. W. P. Drever proposed a novel type of configuration [3] especially suitable for periodic waves — that we shall call for brevity synchronous recycling — in which delay lines or Fabry-Pérot cavities may be introduced as well. The aim of the present study is to derive general formulae which enable us to compare the two different systems. A simple situation will be assumed: the gravitational wave propagates orthogonally with respect to the laboratory plane, the light waves are plane waves, and the mirrors are consequently plane mirrors.

1. Optics in a rippled space-time.

1.1. — An interferometer is essentially made of light and mirrors, and both partners may be thought to be affected by a passing gravitational wave. A possible choice of coordinate system makes the mirrors to be at rest with fixed coordinate values. Then the gravitational wavelength being always much larger than the optical one, the only gravitational effect is contained in the perturbed ds²:

$$\Delta s^2 = c^2 \Delta t^2 - [1 + h(t)] \Delta x^2 - [1 - h(t)] \Delta y^2$$

for a properly directed and polarized GW, and properly chosen x and y axes in the laboratory plane. We have:

$$h(t) = h \cos \phi, \quad \phi = 2 \pi vt + \varphi$$

where h and v are the gravitational amplitude and frequency, and \( \phi \) an arbitrary phase. All the subsequent calculations are at first order with respect to h. An elementary calculation gives the retarded time corresponding to a round trip of length 2l along one of the axes:

$$t_r = t - \frac{2l}{c} - \frac{h}{c} \frac{l \sin \eta}{\eta} \cos(\phi - \eta)$$

with

$$\eta = 2 \pi v l / c$$

(*) This work was carried out within the Groupe de Recherche sur les Ondes de Gravitation (Laboratoire de l'Horloge Atomique, Orsay).
\[ \varepsilon = +1 \text{ (resp. } -1) \text{ for a } x \text{ (resp. } y \text{) propagating light wave.} \]

1.2. — At every point of the optical path the time dependent part of the EM fields will be taken of the form

\[ A(t) = \left( A_0 + \frac{1}{2} h e^{i\phi} A_1 + \frac{1}{2} h e^{-i\phi} A_2 \right) e^{-i\omega t} \]  
(\(\omega = 2\pi \times \text{ optical frequency})\).

Thus all optical elements acting upon the light complex amplitude will be represented by linear operators acting upon vectors \( (A_0, A_1, A_2) \).

1.3. — For instance, by substituting (3) into (4) we find the elementary operator \( D \) corresponding to a round trip of length 2 \( l \):

\[ D = X \begin{bmatrix} 1 & 0 & 0 \\ \frac{i\varepsilon \sin \eta}{\eta} e^{-in} & Y & 0 \\ \frac{i\varepsilon \sin \eta}{\eta} e^{in} & 0 & Y \end{bmatrix} \]  
(5)

where the following notations have been employed:

\[ \varepsilon = \pm 1 \text{ (same rule as above in (1.1))}, \quad X = \exp(2i\xi), \quad Y = \exp(2in). \]

1.4. — The operator associated with any optical part or system is always found to be of the following form:

\[ O = \begin{bmatrix} O_{11} & O_{21} & O_{31} \\ O_{21} & O_{22} & O_{32} \\ O_{31} & O_{32} & O_{33} \end{bmatrix}. \]

These matrices build up a non-commutative algebra. \( O_{11} \) is a complex number which gives the reflectance or transmittance of the system in the absence of gravitational modulation. \( O_{21} \) and \( O_{31} \) give the amplitude of the modulated output light for given unmodulated input light. Therefore, if \( O \) represents the global operator associated with a complete detection system, the relevant coefficients will be \( O_{11} \), which enables one to calculate the noise amplitude, and \( O_{21}, O_{31} \) which give the signal amplitude. Moreover, for an input intensity \( I_0 \) the output intensity \( I \) is actually

\[ I = I_0 |O_{11}|^2 + I_0 \frac{h}{2} [(O_{11} O_{21} + O_{11} O_{31}) e^{-i\phi} + \text{c.c.}]. \]

The SNR thus appears to be proportional to

\[ |O_{21}|^2 + |O_{31}|^2. \]

Therefore, in the subsequent analysis, attention will be focused on the moduli of \( O_{21} \) and \( O_{31} \).

2. Multipass delay lines and Fabry-Perot cavities.

2.1. — Consider a multipass cell of length \( l \) in which \( n \) reflections on the rear mirror have been arranged, so that the total propagation length before output is 2 \( nl \). Let \( ip \) and \( ir \) be the amplitude reflection coefficient of the back and front mirror respectively. It is easy to see that the operator attached to such a multipass delay line (MDL) is

\[ iM = (ir)^{n-1} (ipD)^n \]

where \( D \) is defined by equation (5). Thus

\[ M = (-)^{n-1} \rho^{n-1} D^n \]

so that

\[ M = (-)^{n-1} \rho^{n-1} X^n \begin{bmatrix} 1 & 0 & 0 \\ \frac{i\varepsilon \sin (n\eta)}{\eta} e^{-in} & Y^n & 0 \\ \frac{i\varepsilon \sin (n\eta)}{\eta} e^{in} & 0 & Y^n \end{bmatrix} \]  
(6)

2.2. — Now consider a reflecting Fabry-Perot cavity (FP) of length \( l \), whose back mirror has an amplitude reflection coefficient \( ip \), whereas the parameters of the front mirror are \( t \) (transmission) and \( r \) (reflection). Let

\[ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \]

be the vector amplitudes taken on the front mirror of the four waves involved: \( \mathcal{A} \) is the incoming, \( \mathcal{A}_4 \) the outgoing amplitudes, \( \mathcal{A}_2 \) and \( \mathcal{A}_3 \) are respectively the transmitted and the reflected wave inside the cavity. These amplitudes obey the following system:

\[ \begin{cases} \mathcal{A}_4 = ir\mathcal{A}_1 + t\mathcal{A}_3 \\ \mathcal{A}_2 = it\mathcal{A}_1 + ir\mathcal{A}_3 \\ \mathcal{A}_3 = ipD\mathcal{A}_2 \end{cases} \]

the solution of which is

\[ \mathcal{A}_2 = iF.\mathcal{A}_1 \]

\( F \) being the FP operator:

\[ F = (r + \sigma\rho\rho D)(1 + RD)^{-1} \]  
(7)

where \( \sigma = r^2 + t^2 \) and \( R = rp. \) One can note the formal analogy with the ordinary reflectance of a FP. Evaluation of (7) gives

\[ F = \begin{bmatrix} a & 0 & 0 \\ \frac{i\varepsilon T\rho e^{-in} \sin \eta X}{(1 + RX)(1 + RXY)} & a_- & 0 \\ \frac{i\varepsilon T\rho e^{in} \sin \eta e^{i\eta} X}{(1 + RX)(1 + RXY)} & 0 & a_+ \end{bmatrix} \]  
(8)
2.3. — The responses of classical configurations can be derived from matrices (6) and (8).

2.3.1. — If the system consists of a single MDL (The Michelson configuration has two of them), the signal amplitude will be given by

\[ M = iM_{21} = (-)^2 \rho \pi r^{-1} X^n \frac{\sin(n\eta)}{\eta} e^{-i\eta}. \]

Introduce the storage time \( \tau_s = 2n/c \), choose \( X = -1 \). We obtain

\[ M(v) = \frac{1}{2} \rho^n r^{-1} \sin \left( \frac{\pi \nu \tau_s}{\nu \pi} \right) e^{-i\nu \tau_s}. \quad (9) \]

The converse Fourier transform of (9) gives the impulse response

\[ M(t) = \frac{1}{2} \rho^n r^{-1} \sin \left[ Y(t) - Y(t + \tau_s) \right] \]

\( Y(t) \) being the Heaviside function.

2.3.2. If the system now consists of a single FP cavity (The Michelson configuration has two of them), the signal amplitude will be:

\[ F(v) = iF_{21} = -\varepsilon XT \rho \frac{\sin \eta}{\eta} \times \]

\[ \times e^{-i\eta}(1 + RX)^{-1} (1 + RXY)^{-1} \]

with the resonance condition \( X = -1 \) and approximation \( \eta \ll 1 \), this becomes

\[ F(v) = \frac{\varepsilon}{2} \frac{T}{1 - R} \omega \tau_s (1 + 2i\nu \tau_s)^{-1} \quad (10) \]

with the time constant \( \tau_s = 2R/lc(1 - R) \). The impulse response is now:

\[ F(t) = \frac{\varepsilon}{2} \frac{T}{1 - R} \omega Y(t) e^{-i\tau_s}. \quad (11) \]

3. Synchronous recycling.

3.1. — The idea of synchronous recycling (SR), due to R. W. P. Drever is summarized in figure 1. A light ray transmitted by the recycling mirror \( r_1 \) is reflected by a detecting device \( G \), then by a transfer mirror \( r_2 \) a second detecting device \( G' \), and finally by the recycling mirror \( r_1 \). The whole SR setup may be regarded as a reflecting ring cavity with generalized reflectors \( r, r_2, G, G' \). Actually, two counterrotating light waves will be launched in this ring cavity, and if the storage time \( \tau \) of the detecting devices is suitably chosen with respect to the gravitational period, one can expect the phase difference between the two light waves to increase until a steady state imposed by the losses is reached. In other words, the effective storage time of the SR device is expected to be \( \tau \) multiplied by a factor of the same order as the finesse of the ring cavity.

3.2. — Let \( l_1, l_2, l_3, l_4 \) be the connecting lengths between the four generalized reflectors. The whole setup is itself a generalized reflector, and we can compute the operator is attached to it. We find

\[ S = \left( r_1 - \sigma_1 r_2 ZG'.G \right) \left( 1 - r_1 r_2 ZG'.G \right)^{-1} \quad (12) \]

where

\[ \sigma_1 = r_1^2 + l_1^2, Z = \exp[i\omega(l_1 + l_2 + l_3 + l_4)/c], \quad T_1 = r_1^2. \]

One can note the formal analogy of equation (12) with the ordinary reflectance of a ring cavity. With \( P = G'.G \) we obtain

\[
\begin{align*}
S_{11} &= (r_1 \sigma_1 r_2 ZP_{11})(1-r_1 r_2 ZP_{11}) \\
S_{22} &= (r_1 - \sigma_1 r_2 ZP_{22})(1-r_1 r_2 ZP_{22}) \\
S_{33} &= (r_1 - \sigma_1 r_2 ZP_{33})(1-r_1 r_2 ZP_{33}) \\
S_{21} &= -T_1 r_2 ZP_{31}(1-r_1 r_2 ZP_{11})(1-r_1 r_2 ZP_{22}) \\
S_{31} &= -T_1 r_2 ZP_{31}(1-r_1 r_2 ZP_{11})(1-r_1 r_2 ZP_{33})
\end{align*}
\]

(13)

The relevant coefficients being \( S_{11}, S_{21}, S_{31} \) as remarked in paragraph 1.3.

3.3 Synchronous recycling with MDLs. — Assume that \( G \) refers to a \( x \) directed MDL, and \( G' \) to a \( y \) directed identical MDL. Thus \( G = M(e = 1) \) and

![Diagram of Synchronous Recycling](image)
\( G' = M(e = -1) \), and we find with equations (6) and (13)

\[
S_{21} = -2 T_1 T_2 Z \xi \frac{\sin^2(n \eta)}{\eta} \times \\
\frac{b^2}{(1 - r_1 r_2 Z b^2)^{-1}} (1 - r_1 r_2 b^2 \bar{Y}^2 b^2)^{-1} (1 - r_1 r_2 Z b^2) \bar{Y}^2 b^2 \eta^{-1}
\]

(14) with \( b = (-r)^{n-1} \rho^2 \xi \). With \( Z = 1 \) (resonance of the ring cavity) and \( X = 1 \) we obtain

\[
S_{21} = -2 r_2 b^2 \frac{T_1}{1 - r_1 r_2 b^2 \bar{Y}^2} \times \\
\frac{\xi}{\eta} \sin^2(n \eta) \bar{Y}^2 (1 - r_1 r_2 b^2 \bar{Y}^2 b^2)^{-1}.
\]

(15)

The overall frequency response is shown in figure 2. Clearly, a resonance occurs for \( n \eta_0 = \frac{\pi}{2} \) (i.e. \( v_0 = c/4 n \) or \( v_0 = 1/2 \). With \( n \eta = \frac{\pi}{2} + \pi(v - v_0) \tau_s \) and \( v \) close to \( v_0 \), equation (15) becomes

\[
S_{21}(v) = (-)^{n-1} \frac{T_1}{1 - r_1 r_2 b^2 \bar{Y}^2} \times \\
\frac{\omega \xi}{\eta} \times (1 + 2i m(v - v_0) \tau_s)^{-1}.
\]

(16)

where \( \tau_s \) is the expanded time constant:

\[
\tau_s = 2 n l r_1 r_2 b^2 / (1 - r_1 r_2 b^2).
\]

(17)

The impulse response is

\[
S(t) = (-)^{n-1} \frac{T_1}{1 - r_1 r_2 b^2 \bar{Y}^2} \times \\
\frac{\omega}{\eta} \times Y(t) e^{2i \omega n t} e^{-\eta t}.
\]

(18)

Assume an antiresonant middle cavity \( (Z = -1) \). For each eigenfrequency \( \omega_0 \) of one FP of time constant \( \tau'_s \) the SR system exhibits two split eigenfrequencies

\[
\omega_g = \omega_0 - 1/\tau'_s, \quad \omega_A = \omega_0 + 1/\tau'_s
\]

corresponding to symmetrical and antisymmetrical modes, respectively. A FP alone at optical resonance \( (X = -1) \) is known to introduce a phase difference of \( \pi \) between the incident and the reflected waves. Suppose that

\[
1 - R \ll 1, \quad 1 - \rho \ll 1 - R, \quad 1 - \sigma \ll 1 - R
\]

\[
a = a_0 = (r - \sigma \rho)(1 - R) \neq -1.
\]

For the two eigenmodes of frequencies \( \omega_g, \omega_A \), the phase shifts induced by the cavity happen to be \( \pi/2 \) and \( 3 \pi/2 \) respectively, which provides one a convenient experimental criterion.

Owing to its symmetry, the gravitational wave will be able to transfer energy from one type of mode to the other, provided that the resonance condition

\[
v_s = (\omega_A - \omega_g)/2 \pi = 1/\pi \tau'_s
\]

(20)

is satisfied. The same idea in the microwave frequency range has been developed in [4].

Computer investigation of \( S_{21} \) for \( Z = -1 \) shows the resonance peak at \( v_0 = 1/\pi \tau'_s \) (see Fig. 3).

The resonance condition (20) corresponds to

\[
\eta = \eta_0 = 2 \pi v_0 l/c = 2 \pi (1 - R)/2 (m \in \mathbb{N}).
\]

Let us consider a symmetrical mode, and assume

\[
1 - R \ll 1, \quad 1 - \rho \ll 1 - R, \quad 1 - \sigma \ll 1 - R
\]
The reflectance of each FP cavity is for this mode

\[ a = i a_1 e^{-i\chi} \]

with

\[ a_1 = \left[ (a_0^2 + \sigma^2 \rho^2)(1 + R^2) \right]^{1/2} \]

\[ \chi \approx \chi = (a_0 + R \sigma \rho)(\sigma \rho - R a_0) \ll 1 \]

note that

\[ a_1^2 \approx |a_0|. \]

The neighbourhood of the gravitational resonance will be scanned by

\[ \eta = (1 - R)(1 + \kappa/2) \quad |\kappa| \ll 1. \]

The reflectance of each FP for the upper sideband of the modulated light (i.e. the antisymmetrical mode) is

\[ a_+ = -i a_1 e^{i\chi'} \]

\[ \chi' = \tan \chi' = \frac{(a_0 + R \sigma \rho)(1 + 2 \kappa)}{(\sigma \rho - R a_0)(1 + \kappa)} \ll 1 \]

now, with \( Z = -1 \), we get

\[
(1 - r_1 r_2 a_1^2 Z)(1 - r_1 r_2 a_1^2 Z) = \\
= (1 - r_1 r_2 a_1^2 z)\left(1 + 2 i r_1 r_2 a_1^2 \frac{k}{1 - r_1 r_2 a_1^2}\right).
\]

The signal amplitude is then

\[
S_{31} = \frac{T_1}{1 + r_1 r_2 a_0} \left[ \frac{T}{2(1 - R)} \right]^2 \frac{\omega r_2}{1 + r_1 r_2 a_0} \times \\
\times \left[1 + 2 i \kappa r_1 r_2 \frac{|a_0|}{(1 + r_1 r_2 a_0)^{-1}} \right]^{1/2} \\
or (T \neq 2(1 - R)).
\]

with

\[ v_0 = 1/\pi \kappa', \quad \kappa = 2 r_1 r_2 |a_0| \tau/\left(1 + r_1 r_2 a_0\right) \]

(equivalent time constant).

The system being resonant for both unmodulated light and upper sideband, it unfortunately cannot also be resonant for the lower sideband. (There are only pairs of neighbouring eigenfrequencies), thus the coefficients \( S_{21} \) is not of the same order of magnitude as \( S_{31} \) and for a strong recycling

\[ (1 + r_1 r_2 a_0 \ll 1), \]

may be neglected.

4. Conclusion.

The synchronous recycling has been shown to be effective with multipass delay lines as well as with Fabry-Pérot cavities, the resonant amplitudes of output modulated light being comparable. The signal-to-noise ratio however is better by a factor \( \sqrt{2} \) in the MDL case because the two sidebands of the modulated light are resonant.

Acknowledgments.

The author wishes to thank R. W. P. Drever for valuable advice concerning the resonance condition of 3-4, and L. Schnupp for useful discussions.

References