Multiplicative renormalization of continuous polymer theories, in good and $\theta$ solvents, up to critical dimensions

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1. Introduction.

It has been shown by Edwards [1] that polymers in solution can be described as continuous chains with contact interactions. The theory has been developed through perturbation expansions in the coupling constant, and it exhibits short-range divergences which necessitate a regularization procedure (for example with a cut-off which may be related to the size of one link). A « renormalized » theory emerges in the limit where the regularization is removed. The situation is quite similar to that encountered in local quantum field theory. As a matter of fact, there is a complete equivalence [2] between the theory of continuous chains with two-body (respectively two and three body) interactions, and the theory of \( n \) component scalar fields \( \phi \), with \( \phi^4 \) (respectively \( \phi^4 \) and \( \phi^6 \)) self interactions, in the limit of zero components. The mathematical bridge between both theories is the Laplace-de Gennes transform. Thanks to it, all the renormalization machinery of the second theory may be exploited to renormalize the first one. Our purpose here is precisely to show that this program may be completed, up to the critical dimension where the theory is just renormalizable, and to show that polymer partition functions renormalize multiplicatively.

Indeed, this has already been done for polymers in good solvents [3], when the dimension is strictly smaller than the critical one \( d = 4 \). In that case, the associated field theory is super-renormalizable, and only a mass renormalization is needed. Through the inverse Laplace transform, this mass renormalization (a shift in the square mass) turns into a divergent factor : this is the multiplicative renormalization of polymer theory. To be more specific, let \( Z(S, u, \lambda) \) be the partition function of an isolated polymer in a space of dimension \( d < 4 \), where \( S \) is its « area », \( u \) is the coupling constant, and \( \lambda \) is some regulator which takes account of the divergences of the theory \(^{(1)}\). Let \( G(k^2, \lambda^2, u, \lambda) \) be the regularized propagator of the associated Euclidean scalar \( \phi^4 \) theory in the same \( d \) dimension (\( \lambda \) is the bare mass, \( u \) the coupling constant). Then the inverse Laplace-de Gennes transform reads :

\[
Z(S, u, \lambda) = \frac{1}{2 \pi i} \int d\lambda^2 \, e^{\lambda^2} \, G(0, \lambda^2, u, \lambda) .
\]

All divergences of the propagator \( G \) can be reor-

\(^{(1)}\) All these quantities will be defined in details in section 2.
ganized in a shift of $m^2$. More precisely, defining $m^2_c(u, A)$ by the equation

$$G(0, m^2_c; u, A)^{-1} = 0,$$  

(2)

one shows [4] that $G(0, m^2, u, A)$ is a finite function of $m^2 - m^2_c$ and $u$:

$$G(0, m^2, u, A) = F(m^2 - m^2_c, u).$$  

(3)

Thus:

$$Z(S, u, A) = \exp(Sm^2_c(u, A)) Z_R(S, u)$$  

(4)

where all divergences are collected in the exponential factor, and where the renormalized partition function $Z_R$ is given by

$$Z_R(S, u) = \frac{1}{2\pi i} \int dm^2 \ e^{Sm^2} F(m^2, u).$$  

(5)

When the dimension is equal to 4, further divergences appear, which necessitate the renormalization of the coupling constant. Equation (3) is replaced by

$$G(0, m^2, u, A) = \theta(m^2, u, A) G_R(0, m^2_R, u_R)$$  

(6)

where $m^2_R$, $u_R$ and $\theta$ are functions of $m^2$, $u$ and $A$, which diverge when $A \to \infty$, and $G_R$ is a finite function of $m^2_R$ and $u_R$.

Now, since in general both $\theta$ and $G_R$ are functions of $m^2$ at fixed $u$ and $A$, the factorization of the divergences of $G$ in equation (6) does not entail any longer through the Laplace transform (1), the same desired property for $Z$. However, if it so happens that the renormalization constant $\theta$ does not depend on $m^2$, then it factorizes out in equation (1), and one recovers the multiplicative renormalization of $Z(S, u, A)$. As it is well known, this favourable circumstance occurs with the minimal dimensional renormalization procedure of 't Hooft and Veltman [5], which provides mass independent renormalization constants. This is the clue of the problem, and in the following sections, we show that it works for any polymer partition function.

Of course, this result is not unexpected. Indeed, it has been explicitly assumed in the literature [3, 6], on physical grounds.

The initial motivation for proving the multiplicative renormalization property at critical dimension was the following. Polymer $\varepsilon$-expansion calculations are usually performed (in the good solvent case) at dimension $d < 4$, and physical quantities are first expressed in terms of the unrenormalized coupling constant $u$. Of course, the expression one obtains diverge when $\varepsilon = 4 - d$ goes to zero. But if the variables $u$ and $S$ are eliminated to the benefit of more physical quantities like $g$, the dimensionless second virial coefficient, and $R^2$, the end to end distance, then it is observed [3, 7, 8] that the singularities at $\varepsilon = 0$ disappear. This is indeed expected for the coherence of the theory. We will show that it is a consequence of the divergence factorization property, so that the singularities at $\varepsilon = 0$ must disappear at any order of the perturbative expansion.

Section 2 deals with polymers in good solvents, that is to say polymers that can be described as continuous chains with two-body interactions. In section 3, we extend the analysis to polymers in $\theta$-solvents [9], that is to say continuous chains with two-body and three body contact interactions.

### 2. Continuous chains with two body interactions.

#### 2.1 POLYMER THEORY.

In the continuous model [1, 3, 10], a polymer in good solvent is viewed as a continuous chain with contact two-body interactions, in a space of dimension $d (= 4 - \varepsilon)$. A point on the chain is given by a vector $r(s) (0 \leq s \leq S)$, where the parameter $s$, which has the dimension of a surface, is the coordinate of the point along the chain. $S$ is the total « area » of the chain.

Consider a set of $N$ polymers, of areas $S_\alpha (\alpha = 1, \ldots, N)$. The statistical weight of a configuration of this set is given by:

$$P[r_1, \ldots, r_N] = \exp \left\{ \sum_{\alpha=1}^{N} I[r_\alpha] + \sum_{\alpha, \beta=1}^{N} J[r_\alpha, r_\beta] \right\},$$  

(7)

where

$$I[r_\alpha] = \frac{1}{2} \int_0^{S_\alpha} ds \left[ \frac{\partial^2 \delta_d[r_\alpha(s)]}{\partial s^2} \right]^2,$$  

(8)

$$J[r_\alpha, r_\beta] = \frac{u}{2} \int_0^{S_\alpha} ds \int_0^{S_\beta} ds' \delta_d[r_\alpha(s) - r_\beta(s')].$$  

(9)

$u$ is the contact interaction coupling constant, and we call $P_0[r_1, \ldots, r_N]$ the weight for $u = 0$ (Gaussian weight).

With the notation of reference [3], we define the mean value $\langle F \rangle$ of the functional $F[r_1, \ldots, r_N]$ by

$$\langle F \rangle = \frac{\int \! d[r_1] \ldots d[r_N] \delta_d[r_1(0)] P[r_1, \ldots, r_N] F[r_1, \ldots, r_N]}{\int \! d[r_1] \ldots d[r_N] \delta_d[r_1(0)] P[r_1, \ldots, r_N]}.$$  

(10)

Among all the partition functions that can be defined, only a few in a first step will be of interest for us,
those which are used to compute the few physical parameters characterizing polymers in good solvents. They are (2)

\[ Z^{(1)}(S, u; \varepsilon) = \int d[r] \delta_{\Delta}(r(0)) P[r] \int d[r] \delta_{\Delta}(r(0)) P_0[r] \]

\[ Z^{(1)}(k^2; S, u; \varepsilon) = Z^{(1)}(S, u; \varepsilon) \langle \exp(\mathbf{k} \cdot [r(S) - r(0)]) \rangle \]

\[ Z^{(1)}(k^2; S_1, S_2, u; \varepsilon) = Z^{(2)}(k; S_1, S_2, u; \varepsilon) \langle \exp(\mathbf{k} \cdot r_1(0) + \mathbf{k}_2 \cdot r_3(S_1) + \mathbf{k}_3 \cdot r_2(0) + \mathbf{k}_4 \cdot r_2(S_2)) \rangle = Z^{(2)}(S_1, S_2, u; \varepsilon) \]

\[ Z^{(1)}(k^2; S_1, S_2, u; \varepsilon) = Z^{(1)}(S_1, S_2, u; \varepsilon) \]

All of them are plagued with divergences, and they have to be regularized. We choose the dimensional regularization, so that it is \( \varepsilon = 4 - d \) which plays the role of the regulator.

\[ A[\varphi_1, ..., \varphi_n] = -\int d^d x \left\{ \frac{1}{2} \sum_{i,j} \left[ \left( \nabla \varphi_i \right)^2 + m^2 \varphi_i^2 \right] + \frac{u}{8} \sum_{i,j} \varphi_i^2 \right\}. \]

Note that i) we have given different masses \( m_a \) to the fields \( \varphi_a \), ii) the interaction term is \( \Theta(\varepsilon n) \) symmetric (\( \varepsilon n \) dimensional orthogonal group), but the complete action is not, precisely because of the mass terms.

To the above defined connected partition functions are associated the following connected Green functions (4):

i) the propagator \( G^{(1)}(k^2; m_a, u; \varepsilon) \) of the field \( \varphi_a \), which in the limit of zero components \( (n = 0) \), depends on only the mass \( m_a \);

ii) the vertex function \( G^{(4)}(k; m_a, m_b, u; \varepsilon) \), Fourier transform of the connected vacuum expectation value \( \langle \varphi_{a}(x) \varphi_{b}(y) \varphi_{b}(z) \varphi_{b}(t) \rangle \); as previously \( k \) stands for the collection \( \{ k_1, ..., k_4 \} \), and \( k_1 + k_2 + k_3 + k_4 = 0 \);

iii) the connected Green function \( G^{(2,1)}(k_1, k_2, q; m_a, m_b, u; \varepsilon) \), Fourier transform of the vacuum expectation value \( \langle \varphi_a(x) \varphi_b(y) \varphi_b(z) \varphi_{b}(t) \rangle \); of a product of two \( \varphi \) fields, and one composite field \( \varphi_{b}(u) \) (\( \varphi_{b}(u) \) is always in the \( n = 0 \) limit); \( k_1, k_2 \), and \( q \) are respectively the conjugate variables of \( x, y, z \) and \( t \), and \( k_1 + k_2 + q_1 + q_2 = 0 \);

iv) the connected Green function \( G^{(2,2)}(k_1, \varphi_1, \varphi_2; m_a^2, m_b^2, m_c^2, u; \varepsilon) \), Fourier transform of the vacuum expectation value \( \langle \varphi_a(x) \varphi_b(y) \rangle \); of a product of \( \varphi \) fields, and one composite field \( \varphi_{b}(u) \) (\( \varphi_{b}(u) \) is always in the \( n = 0 \) limit); \( k_1, k_2, q_1, q_2 \) are respectively the conjugate variables of \( x, y, z \) and \( t \), and \( k_1 + k_2 + q_1 + q_2 = 0 \).

All these Green functions are dimensionally regularized.

2. 3 LAPLACE-DE GENNES TRANSFORMS. — The relations (5) between polymer partition functions and Green functions of the field theory read as follows (6):

\[ Z^{(N, L)}(\varphi) = \int \mathcal{D}\varphi \exp\left\{ i S[\varphi] \right\} \]

\[ Z^{(N, L)}(\varphi) = \int \mathcal{D}\varphi \exp\left\{ i S[\varphi] \right\} \frac{1}{2} \left( \sum_{a=1}^{N} \frac{1}{2} \left( \nabla \varphi_a \right)^2 + m^2 \varphi_a^2 \right) \]

\[ \int \mathcal{D}\varphi \exp\left\{ i S[\varphi] \right\} \]

(6) More generally, \( Z^{(N, 1)} \) is the inverse Laplace transform of \( G^{(1, N)} \).
As for the partition functions \( Z^{(1)}(S; u; \epsilon) \) and \( Z^{(2)}(S; S_1, S_2; u; \epsilon) \), they are given by equations (17) and (18), respectively for \( k^2 = 0 \) and \( k = 0 \).

In the above equations, integrals run over the usual inverse Laplace transform contours, that is to say contours that stay at the right of the rightest singularities in the square masses complex planes.

2.4 Dimensional field theory renormalization with minimal-prescription. — As explained in the introduction, we use the minimal renormalization prescription of 't Hooft and Veltman [5] to renormalize the above defined field theory at \( d = 4 \), which insures that the counters terms built at each order of the perturbation expansion do not depend on the masses.

A first important consequence of this mass independence is that in the renormalized action \( A_R \), the \( O(N_n) \) symmetry of the interaction term is preserved, so that we have to introduce only one renormalized coupling constant \( u_R \):

\[
A_R[\phi^R_1, ..., \phi^R_M] = - \int d^d x \left\{ \frac{1}{2} \sum_{\alpha \beta} \left[ \frac{1}{2} (V{\phi^R_\alpha})^2 + m^2_{\beta R} \phi^R_{\alpha} \right] + \mu^d \frac{U_R}{8 \pi} \left[ \sum_{\alpha} \phi^R_{\alpha} \right]^2 \right\}
\]

\( \mu \) is an arbitrary mass introduced in such a way that \( U_R \) be dimensionless irrespective of \( d \), and the \( m_{\beta R} \)'s are the renormalized masses.

To prove the above assertion, one has to show that the vertex counterterms built with the above renormalized action do possess the \( O(N_n) \) symmetry. In fact any given vertex diagram, with two \( \phi_\alpha \) and two \( \phi_\beta \) legs, contributes a counterterm of the form \( c(\mu_R, \epsilon) \phi^2_\alpha \phi^2_\beta \) if \( \alpha \neq \beta \), and \( c(\mu_R, \epsilon) \phi^4_\alpha/2 \) otherwise, the last factor 1/2 coming from an extra \( S_2 \) symmetry of the diagram when \( \alpha = \beta \). For all \( \alpha \) and \( \beta \)'s, this gives a total counterterm \((1/2) c(\mu_R, \epsilon) \left( \sum_{\alpha} \phi^2_\alpha \right)^2 \), which is \( O(N_n) \) symmetric.

The relations between bare and renormalized quantities read [12], in the zero component limit:

\[
\phi^R_\alpha = \phi(\mu_R, \epsilon)^{1/2} \phi^R_\alpha,
\]

\[
\phi^R_\alpha \phi^R_\beta = \phi_x(\mu_R, \epsilon)(\phi^R_\alpha \phi^R_\beta)^R,
\]

\[
m^2_{\beta R} = m(\mu_R, \epsilon) m_{\beta R},
\]

\[
u = \nu(\mu_R, \epsilon) \mu^d \nu_R,
\]

where \( \phi_x, \phi_x^2, \mu_x \) and \( \nu_x \) functions of \( \mu_R \) and \( \epsilon \) are respectively the \( \phi \) field, composite \( \phi \phi \) field, masses and coupling constant renormalization constants, independent of the masses.

Now, the connected Green function \( G^{(M,N)}(\ldots) \), vacuum expectation value of a product of \( M \phi \) fields and \( N \phi \phi \) composite fields, renormalizes according to the general formula:

\[
G^{(M,N)}(k_q, q; \{ m^2_{\alpha q} \}; \epsilon) = \theta^{M/2}_\phi \theta^{N}_{\phi x} G^{(M,N)}(k_q, q; \{ m^2_{\alpha q} \}; \mu_R, \epsilon).
\]

The renormalized Green function \( G^{(M,N)} \), as a function of \( \epsilon \), is free from singularities at \( \epsilon = 0 \).

Of course, the renormalization of the mass, and that of the \( \phi \phi \) composite field are not independent, and this implies a relation between \( \mu_x \) and \( \nu_x^2 \). Let us prove that

\[
\theta_{\mu_x} \theta_{\nu_x^2} = 1.
\]

One knows that the function \( G^{(2,1)}(k_q, -k_q; \ldots) \) with one \( \phi \phi \) insertion of vanishing momentum is related to the propagator by:

\[
G^{(2,1)}(k_q, -k_q; m^2, \epsilon) = \frac{\partial}{\partial m^2} G^{(2)}(k_q; m^2, \epsilon).
\]

This entails for the corresponding renormalized functions the relation

\[
\theta_{\mu_x} \theta_{\nu_x^2} G^{(2,1)}(k_q, -k_q; \ldots) = \theta_{\mu_x} \theta_{\nu_x^2} \theta^2_{\mu_x} \theta_{\nu_x^2} G^{(2)}(k_q; m^2, \epsilon) = \frac{\partial}{\partial m^2} G^{(2)}(k_q; m^2, \epsilon).
\]

Taking account of equation (23), one obtains

\[
\theta_{\mu_x} \theta_{\nu_x^2} = \left[ \frac{\partial}{\partial m^2} G^{(2)}(\ldots) \right] G^{(2,1)}(\ldots).
\]
Due to the minimal prescription, the left hand-side, as a function of \( \epsilon \), has the structure \( 1 + \sum_{m=0}^{\infty} a_m(\mu) \epsilon^m \).

The right hand side is regular at \( \epsilon = 0 \). Consequently both sides are equal to one.

### 2.5 Dimensional Polymer Renormalization with Minimal Prescription

It remains now to translate equations (17) \( \rightarrow \) (19) written for bare functions, into relations between renormalized ones. The inverse Laplace transforms have to be performed on the bare variables \( m_2^2 \), at fixed \( k^2, k, \) and \( \epsilon \). Thanks to equations (23) and (24), they can be changed into integrals on the renormalized variables \( m_2^2 \) with \( \theta_m \) as Jacobian, at fixed \( k^2, k, \) and \( \mu, \epsilon \).

where the renormalized partition function \( Z_R \)'s, regular at \( \epsilon = 0 \), can be expressed as inverse Laplace transforms of renormalized Green functions:

\[
Z_R^{(1)}(k^2; S, \mu, \epsilon) = \frac{1}{2 \pi i} \int \text{d}m_2^2 \exp(S_R m_2^2) G_1^{(2)}(k^2; m_2^2, \mu, \epsilon) \tag{33}
\]

\[
Z_R^{(2)}(k^2; S_{1R}, S_{2R}, \mu, \epsilon) = \frac{1}{2 \pi i^2} \int \text{d}m_{1R}^2 \text{d}m_{2R}^2 \exp(S_{1R} m_{1R}^2 + S_{2R} m_{2R}^2) G_2^{(3)}(k^2; m_{1R}^2, m_{2R}^2, \mu, \epsilon) \tag{34}
\]

\[
Z_R^{(1,2)}(k^2; s_{1R}, s_{2R}, s_{3R}, \mu, \epsilon) = \frac{1}{2 \pi i^3} \int \text{d}m_{1R}^2 \text{d}m_{2R}^2 \text{d}m_{3R}^2 \times \exp(s_{1R} m_{1R}^2 + s_{2R} m_{2R}^2 + s_{3R} m_{3R}^2) G_2^{(2,3)}(k, \mu, \epsilon) \tag{35}
\]

The renormalized polymer areas \( S_R \) and \( s_R \) are defined by \( S_R m_2^2 = S_R^{(2)} m_R^2 \), that is to say

\[
S_R = \theta_m S, \tag{36}
\]

and similarly for the \( s_R \)'s (note that in Eq. (35), the total renormalized area \( S_R \) of the chain is given by \( S_R = S_{1R} + S_{2R} + S_{3R} \)). Areas are multiplicatively renormalized, like the masses in equation (23).

Formulæ for the partition functions \( Z^{(1)}(S, \mu, \epsilon) \) and \( Z^{(2)}(S_1, S_2, \mu, \epsilon) \) are obtained simply by setting \( k^2 = 0 \) and \( k = 0 \) respectively in equations (30), (33) and (31), (34).

Obviously the above treatment may be generalized to any polymer partition function. The general result which emerges is the following. Let \( Z^{(N,1)}(k; S_1, \ldots, S_N, \mu, \epsilon) \) be any connected partition function of \( N \) polymers, with \( Z \) insertions of wave numbers \( k \).

Then

\[
Z^{(N,1)}(k; S_1, \ldots, S_N, \mu, \epsilon) = \theta_m^N Z^{(N,1)}_R(k; S_{1R}, \ldots, S_{NR}, \mu, \epsilon), \tag{37}
\]

where the renormalized partition function \( Z^{(N,1)}_R \) is regular at \( \epsilon = 0 \). Thus each continuous chain contributes a short range divergent factor \( \theta_m^N \). (Note that the divergent factor in Eq. (37) does not depend on the number \( L \) of insertions, a result which was already pointed out in Ref. [3], see Eq. (5.13) there).

This demonstrates that the continuous polymer theory is multiplicatively renormalizable up to dimension \( d = 4 \) (in the good solvent case).

We now turn to the consequences for

### 2.6 Polymer Physical Parameters

The mean square end to end distance \( R^2 \), and the radius of gyration \( R_g^2 \) are defined by

\[
R^2 = -2 \frac{d}{dk^2} \ln Z^{(1)}(k^2; S, \mu, \epsilon) \bigg|_{k^2=0} \tag{38}
\]

\[
R_g^2 = -2 \frac{d}{S^2} \int \text{d}S_1 \text{d}S_2 \text{d}S_3 \delta(S - s_1 - s_2 - s_3) \frac{d}{dk^2} \ln Z^{(1,2)}(k^2; s_1, s_2, s_3, \mu, \epsilon) \bigg|_{k^2=0} \tag{39}
\]

Using equations (30) and (32), one obtains

\[
R^2 = -2 \frac{d}{dk^2} \ln Z_R^{(1)}(k^2; S_R, \mu, \epsilon) \bigg|_{k^2=0} \tag{40}
\]
This expresses $R_2^2$ and $R_0^2$ as functions, finite up to $d = 4$, of the renormalized parameters $S_R$ and $\mu_R$, and of the mass $\mu$.

One also defines the dimensionless physical parameter $g$, which is essentially the second virial coefficient [3]:

$$g = -(2\pi)^{-d/2} e^S \tilde{S}^{(2)}(S, S, u; \epsilon)/Z^{(1)}(S, u; \epsilon)^2.$$  

The « effective » area $\tilde{S}$ is defined by $R_2^2 = d e\tilde{S}$. From equations (30) and (31) one obtains

$$g = -(2\pi)^{-d/2} e^S \tilde{S}^{(2)}(S_R, \mu_R; \mu, \epsilon)/Z^{(1)}(S_R, u_R; \mu, \epsilon)^2$$  

which shows that $g$, as a function of $S_R$, $\mu_R$ and $\mu$, stays finite when $\epsilon \to 0$.

We can take advantage of equations (40) and (43) to eliminate $S_R$ and $\mu_R$ to the benefit of $e\tilde{S}$ and $g$. In that way, $S_R \mu^2$ and $\mu_R$ become dimensionless functions of $e\tilde{S}$, $g$ and $\epsilon$:

$$S_R \mu^2 = f(e\tilde{S}, g, \epsilon),$$  

$$u_R = u_R(e\tilde{S}, g, \epsilon).$$  

Now let $Q(q^2, S, u; \epsilon)$ be any dimensionless physical quantity, which renormalizes without multiplicative divergences, that is to say

$$Q(q^2, S, u; \epsilon) = Q_R(q^2, S_R \mu^2, u_R; \epsilon),$$  

where $Q_R$ stays finite when $\epsilon \to 0$, at fixed $S_R Q^2$, $S_R \mu^2$ and $u_R$. Thanks to equations (44) and (45), $Q_R$ can be expressed as a function of $e\tilde{S} q^2$, $e\tilde{S} s^2$, $g$ and $\epsilon$.

According to equations (30) and (32)

$$H(q^2; S, u; \epsilon) = \frac{1}{S^2} \int ds_1 ds_2 ds_3 \delta(S - s_1 - s_2 - s_3) Z^{(1,2)}(q^2; s_1, s_2, s_3, u; \epsilon)/Z^{(1)}(S, u; \epsilon).$$  

Thus the form factor can be written as a function $H_R(e\tilde{S} q^2, g, \epsilon)$ which is independent of the mass $\mu$ and stays finite when $\epsilon \to 0$, at fixed $e\tilde{S} q^2$ and $g$.

Consider now a monodisperse polymer solution. The osmotic pressure $\Pi$ and the concentration $C$, in terms of the fugacity $f$ read

$$\Pi = \sum_{N=1}^{\infty} \frac{f^{N}}{(N-1)!} Z^{(N)}(S, \ldots, S, u; \epsilon),$$  

$$C = \sum_{N=1}^{\infty} \frac{f^{N}}{(N-1)!} Z^{(N)}(S, \ldots, S, u; \epsilon).$$  

Defining the renormalized fugacity $f_R$ by

$$f_R = \theta_{\mu} \theta_m^2 f,$$  

and using equation (37), one obtains the relations

$$\Pi \beta = \sum_{N=1}^{\infty} \frac{f_{R}^{N}}{(N-1)!} Z^{(N)}_{R}(S_R, \ldots, S_R, u_R; \mu, \epsilon),$$  

$$C = \sum_{N=1}^{\infty} \frac{f_{R}^{N}}{(N-1)!} Z^{(N)}_{R}(S_R, \ldots, S_R, u_R; \mu, \epsilon),$$  

which express $\Pi \beta$ and $C$ as $\epsilon = 0$ regular functions of the renormalized parameters $f_R$, $S_R$, $u_R$ and of the mass $\mu$, or equivalently of the variables $f_R$, $e\tilde{S}$, $g$ and $\mu$. Eliminating $f_R$, and arguing as previously, one obtains that $\Pi \beta/C$ is a dimensionless function, regular at $\epsilon = 0$, of $e\tilde{S} q^{2/2}$ and $g$.

Note that the multiplicative renormalization of the
fugacity implies an additive renormalization of the chemical potential. Finally, let us give the relations between the two renormalization constants $\theta$ and $\theta_m$, and those defined in reference [3], the swelling factor $X_0 = \mathrm{S} / \mathrm{S}$ and $X_2 = \mathrm{Z}^{(4)}(S, u; \varepsilon)$:

\[ X_0 = -2 \frac{\theta_m}{\partial S} \left[ \frac{1}{2} \ln Z_{\mathrm{R}}^{(4)}(S, k^2, S_R \mu^2, u_R; \varepsilon) \right] \left|_{k^2 = 0} \right. \]

(56)

\[ X_2 = \theta_m^2 \frac{1}{2} \ln Z_{\mathrm{R}}^{(4)}(S_R \mu^2, u_R; \varepsilon). \]

(57)

Our result (37) can now be compared to equation (5.12) of reference [3]. It appears that the renormalized partition functions defined by des Cloizeaux differ from ours by a factor (regular at $\varepsilon = 0$) which is a product of $Z_{\mathrm{R}}^{(4)}(S_R \mu^2, u_R; \varepsilon)$.

Note also that the renormalization constant $X_2$ of reference [3] is nothing but $\theta_m^2$. 

3. Continuous chains with two and three-body interactions.

3.1 POLYMER AND FIELD THEORIES, LAPLACE-DE GENNES TRANSFORMS. — A set of $N$ polymers is now defined as a set of $N$ chains, of areas $S_a$, with the statistical weight [3, 9]:

\[ P[r_1, ..., r_N] = \exp \left\{ \sum_{a=1}^{N} I[r_a] + \sum_{a,b=1}^{N} J[r_a, r_b] + \sum_{a,b,c=1}^{N} K[r_a, r_b, r_c] \right\} \]

(58)

where $I$ and $J$ are as in the previous section, and where the three body interaction $K$ has the form:

\[ K[r_a, r_b, r_c] = \frac{v}{6} \delta_a \delta_b \delta_c \delta_a[r_a] \delta_b[r_b] \delta_c[r_c] + \delta_a[r_a] \delta_b[r_b] \delta_c[r_c] - \delta_a[r_a] \delta_b[r_b] - \delta_a[r_a] \delta_c[r_c] - \delta_b[r_b] \delta_c[r_c]. \]

(59)

The critical dimension is $d = 3$, so we set $\varepsilon = 3 - d$.

Besides the partition functions that we define as in equations (11) – (15) but with the new weight (58), and which now depend on the extra variable $v$, we need two new ones which are connected partition functions of three polymers:

\[ Z^{(3)}(S_1, S_2, S_3, u, v; \varepsilon) = \int d[r_1] d[r_2] d[r_3] \delta_a[r_1(0)] P[r_1, r_2, r_3] \]

(60)

\[ Z^{(3)}(k; S_1, S_2, S_3, u, v; \varepsilon) = Z^{(3)}(S_1, S_2, S_3, u, v; \varepsilon) \left( \exp(|k_1 \cdot r_1(0)| + k_2 \cdot r_1(0) + k_3 \cdot r_2(0) + k_4 \cdot r_2(0) + k_5 \cdot r_3(0) + k_6 \cdot r_3(0)) \right). \]

(61)

This last function is defined for $k_1 + ... + k_6 = 0$.

The associated Euclidean scalar field theory is defined by the bare action

\[ A[\phi_1, ..., \phi_N] = \int d^d x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left( \frac{1}{2} \phi_i(x) + m_i^2 \phi_i(x) \right)^2 + \frac{u}{8} \left( \sum_{i=1}^{N} \phi_i(x) \right)^2 + \frac{v}{48} \left( \sum_{i=1}^{N} \phi_i(x) \right)^3 \right\}. \]

(62)

At critical dimension $d = 3$, $u$ and $v$ have respectively mass dimensions 1 and 0 : $u$ is a super-renormalizable coupling, $v$ is just renormalizable.

Besides the Green functions introduced in the previous section, and which now depend on the extra variable $v$, we need the six point connected Green function $G^{(6)}(k; m_1^2, m_2^2, m_3^2, u, v; \varepsilon)$, Fourier transform of the vacuum expectation value

\[ \left\langle \phi_a(x_1) \phi_b(x_2) \phi_c(x_3) \phi_d(x_4) \phi_e(x_5) \phi_f(x_6) \right\rangle \]

(63)

This equation completes equations (17) – (19), which have to be written with the extra $v$ dependence.
3.2 Renormalization. — Here again we use the minimal renormalization prescription of ’t Hooft and Veltman to renormalize the field theory at \( d = 3 \). As in the previous section, the \( O(Nn) \) symmetry of the two interaction terms is preserved in this renormalization scheme, so that the renormalized action is written in terms of only two renormalized coupling constants \( u_R \) and \( v_R \). We choose \([12]\) \( u_R \) and \( v_R \) to have mass dimensions 1 and 0, irrespective of \( d \):

\[
A_R[\varphi_R^1, ..., \varphi_R^n] = - \int d^d x \left\{ \frac{1}{2} \sum_{i,j} \left[ \frac{1}{2} (\nabla \varphi_R^j)^2 + m_R^2 \varphi_R^j \right] + \frac{u_R}{8} \mu^4 \left[ \sum_{i,j} \varphi_R^{i,j} \right]^2 + \frac{v_R}{48} \mu^2 \left[ \sum_{i,j} \varphi_R^{i,j} \right]^3 \right\}. \tag{64}
\]

The renormalized theory depends on the parameters \( m_R^2, u_R \) and \( v_R \), which have mass dimensions 2, 1 and 0. Thus, by an easy extension of the results of reference [12] to the case where the critical dimension is 3, the relations between bare and renormalized parameters take the form (in the limit \( n = 0 \)):

\[
\varphi_R^j = \vartheta_R(v_R, \mu) \varphi^R, \tag{65}
\]

\[
\varphi_{al} \varphi_{bj} = \vartheta_{al}(v_R, \mu) \varphi_{al} \varphi_{bj}^R, \tag{66}
\]

\[
m_R^2 = \vartheta_{al}(v_R, \mu) m_R^2 + \vartheta_{al}(v_R, \mu) u_R^2, \tag{67}
\]

\[
u = \vartheta_{al}(v_R, \mu) \mu^{2} u_R, \tag{68}
\]

\[
v = \vartheta_{al}(v_R, \mu) \mu^{2} v_R. \tag{69}
\]

Note that the six renormalization constants \( \vartheta_R \)'s are independent not only of the masses, but also of the super-renormalizable coupling constant \( u_R \), and that \( u_R \) appears only polynomially in the mass renormalization equation (67). The coefficient \( \vartheta_{al}(v_R, \mu) \) in this equation is built from the logarithmic divergences of those propagator diagrams that are second order in the coupling constant \( u_R \).

The extension of equation (26) to the present case is immediate, and the Green functions \( G^{(M,N)}(k, q; \{ m^2 \}, u, v; \mu, \epsilon) \) renormalize, mutatis mutandis, as in equation (25).

We now have all the elements to renormalize the polymer partition functions. Straightforward calculations lead to the following relations:

\[
Z^{(3)}(k; S_1, S_2, S_3, u, v; \epsilon) = (\vartheta_{al}(v_R, \mu))^3 \exp(S_{1R} + S_{2R} + S_{3R}) \varphi_R^2(\vartheta_{al}(v_R, \mu))^3) Z^{(3)}_R(k; S_1R, S_2R, S_3R, u_R, v_R; \mu, \epsilon), \tag{73}
\]

where the renormalized partition functions \( Z_R \)'s, regular at \( \epsilon = 0 \), can be expressed once again as inverse Laplace transforms of renormalized Green functions. For example

\[
Z_R^{(3)}(k; S_{1R}, S_{2R}, S_{3R}, u_R, v_R; \mu, \epsilon) = \frac{1}{(2 \pi i)^3} \int dm_{1R} dm_{2R} dm_{3R} \times
\]

\[
\exp(S_{1R} m_{1R}^2 + S_{2R} m_{2R}^2 + S_{3R} m_{3R}^2). G_R^{(6)}(k; m_{1R}, m_{2R}, m_{3R}, u_R, v_R; \mu, \epsilon). \tag{74}
\]

The renormalized polymer areas are defined as in section 2 (Eq. (36)), and in equation (71), \( S_R = S_{1R} + S_{2R} + S_{3R} \).

We note in the above formulae the presence of exponentials with arguments linear in the parameters \( S_{al} \), which are absent in the analogous formulae (30) \( \rightarrow (32) \) of section 2. They obviously contribute to the so-called « attrition » coefficient. Let us comment on them.

Consider the case of vanishing three-body interactions at \( d = 3 \), which describes the physics of polymers in good solvents, in the physical 3-dimensional space. The theory is now super-renormalizable, all the \( \vartheta_R \)'s are equal to 1, and \( \vartheta_{al} \), which comes from the logarithmic divergences of the only divergent diagram of the theory, is equal to 1/4 \( \pi^2 \). Consequently, the area \( S_R \) and the coupling constant \( \mu \) are not renormalized : \( S_R = S_0 \) and each polymer contributes to the partition functions an exponential factor \( \exp(Su^2/4 n^2 \epsilon) \). We thus recover the exponentiation of divergences linear in \( S \), shown in references [3, 4]. Note also that the renormalized partition functions \( Z_R \)'s do not depend any longer on the mass scale \( \mu \).
The generalization of the above formulae to any partition function goes as follows. Let 
\[ Z^{(N,1)}_{R}(\mathbf{k}; S_{1}, \ldots, S_{N}, u; \varepsilon) \]
be defined as in section 2. Then
\[
Z^{(N,1)}_{R}(\mathbf{k}; S_{1}, \ldots, S_{N}, u; \varepsilon) = (\theta_{\mu} \theta_{\nu})^{N} \times \exp((S_{1R} + \cdots + S_{NR}) u_{R}^{2}(\theta_{m}/\theta_{m})) \times 
\times Z^{(N,1)}_{R}(\mathbf{k}; S_{1R}, \ldots, S_{NR}, u_{R}; \mu; \varepsilon) \quad (75)
\]
where the renormalized partition function \( Z^{(N,1)}_{R} \) is regular at \( \varepsilon = 0 \). Thus each continuous chain of renormalized area \( S_{k} \) contributes a short range divergent factor \( \theta_{\mu} \theta_{\nu}^{2} \exp(S_{R} u_{R}^{2}(\theta_{m}/\theta_{m})) \), which now depends on \( S_{k} \).

3.3 POLYMER PHYSICAL PARAMETERS. — The mean square end to end distance \( R_{2} \), the radius of gyration \( R_{g} \), and the parameter \( g \) are still given by equations (40), (41) and (43), with the only change of the extra \( v_{R} \) dependence.

Furthermore, in the same way \( u_{R} \) is related to the second virial coefficient, \( v_{R} \) is related to the third virial coefficient, more precisely to that part of it which is proportional to \( Z^{(3)}(S; S; S)/Z(S)^{3} \). So we define the dimensionless physical parameter \( g' \), as follows:

\[ g' = -2(\pi)^{-d/2} S^{-d} Z^{(3)}(S; S; S; u; v; \varepsilon)/Z^{(1)}(S; u; v; \varepsilon)^{3} \quad (76) \]

In terms of renormalized quantities, it becomes

\[ g' = -2(\pi)^{-d/2} S^{-d} Z^{(3)}(S_{R}; S_{R}; S_{R}; u_{R}; v_{R}; \mu; \varepsilon)/Z^{(1)}(S_{R}; u_{R}; v_{R}; \mu; \varepsilon)^{3} \quad (77) \]

Now, as previously, one can eliminate \( S_{R}, u_{R} \) and \( v_{R} \) to the benefit of \( S, g, \) and \( g' \), and express all physical quantities in terms of these last three variables. Arguing as in section 2, one shows that the ratio \( R_{g}^{2}/R^{2} \) is a function \( \rho(g, g', \varepsilon) \), regular at \( \varepsilon = 0 \), and that the form factor \( H(q^{2}; S, u; v; \varepsilon) \) can be written as a function \( H_{R}(q^{2}; g, g'; \varepsilon) \), also regular at \( \varepsilon = 0 \).

As for the fugacity \( f \), it now renormalizes as follows:

\[ f_{R} = \theta_{\mu} \theta_{\nu}^{2} \exp(S_{R} u_{R}^{2}(\theta_{m}/\theta_{m})).f \quad (78) \]

This implies an additive renormalization of the chemical potential, which now is linear in the polymer area (it was a constant in section 2). Of course, one also shows that \( \Pi\beta/C \) is a dimensionless function, regular at \( \varepsilon = 0 \), of \( S^{d/2} C, g \) and \( g' \).


Equations (37) and (75) constitute the central result of this paper. They prove that in the standard continuous models of polymers the short range divergences of the partition functions factorize (multiplicative renormalization), up to the critical dimension \( (d = 4 \text{ or } 3) \).

A first consequence is that renormalized polymer partition functions are themselves inverse Laplace transforms of renormalized Green functions of the scalar field theory (when the number \( n \) of field components vanishes). This extends the validity de Gennes' relations to renormalized quantities.

A second consequence is that physical quantities, expressed in terms of the effective area \( S \) and the second virial coefficient \( g \) (or \( S, g \) and \( g' \)) are regular when \( \varepsilon \to 0 \). This was indeed required for the coherence of the theory, and was repeatedly checked in \( \varepsilon \)-expansion calculations.

It also appears that the divergent factors \( \theta_{\mu} \) and \( \theta_{\nu} \), or equivalently \( \theta_{\mu} \) and \( \theta_{\nu} \), which renormalize polymer theories, are the same which renormalize scalar zero component field theories. Consequently, when looking for the long range (infra-red) behaviour of polymer physical quantities (the Kuhnian limit of Ref. [3]) one evidently finds the same critical indices one has in field theory.

As a last comment, let us turn back to the 't Hooft-Veltman dimensional renormalization scheme. As a matter of fact, it seems that the only sensible prescriptions for renormalizing polymer theories, are those where the counterterms do not depend on the masses \( (\cdot) \). With prescriptions which do not fulfill this condition, the \( O(N\mu) \) symmetry of the interaction would be broken in the renormalization process, and one would be led to introduce, instead of a unique renormalized coupling constant \( u_{R} \), a matrix \( (u_{R})_{\mu\nu} \), which is physically meaningless.

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\(^{(\cdot)} \) The massless renormalization scheme of Brezin et al. [13] provides renormalization constants which are mass independent, and which renormalize also the massive theory. It is thus convenient for renormalizing polymer theories, and indeed it has been used in reference [14] to derive scaling properties of polydisperse polymer solutions. We are indebted to L. Schäfer for having drawn our attention to that point.
References

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