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Fire propagation in a 2-D random medium

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Résumé. — Nous simulons la propagation d'un front — en prenant l'exemple d'un feu de forêt — dans un milieu aléatoire bidimensionnel et introduisons plusieurs modèles d'interactions. Nous calculons les concentrations critiques correspondantes, puis les exposants critiques à partir des lois d'échelles pour les systèmes finis. Ces exposants sont exprimés en fonction de la dimension fractale de l'amas infini et de la dimension d'étalement. Nous montrons enfin que la structure du front est autosimilaire (fractale) au voisinage du seuil de percolation et calculons la dimension de Hausdorff associée.

Abstract. — The propagation of a front (we use the model of a forest fire) in a bidimensional random lattice is studied for several different types of interactions. We obtain the corresponding critical concentrations and critical exponents calculated by means of the finite size scaling conjecture. These exponents are expressed in terms of the fractal dimension of the infinite cluster and of the spreading dimension. We show that the front structure is fractal and we determine its Hausdorff dimension.

1. Introduction.

We study a dynamic percolative model for the numerical simulation of the propagation of a wavefront in a random bi-dimensional medium. This model proposed by P. Clavin [1] is similar to Grassberger's General Epidemic Model [2] and is partly inspired by the more phenomenological work of Rothermel [3] and of Frandsen and Andrews [4]. It concerns the general field of the propagation of fronts in random media where one or more species, distributed randomly in space, are transformed by a wave front that progresses by finite range contamination. In order to provide a tangible image we will describe the model in terms of a fire propagating through a forest constituted by different sorts of trees, but the model could be applied to many other processes of front propagation in random media including underground fires, deflagration of a suspension of inflammable dust particles, epidemic spreading of a disease, etc. The medium is created at random by populating a given fraction of sites on a regular lattice. However, after ignition, the front propagation is determined only by the configuration of sites and is thus entirely deterministic. This contrasts with a different approach (see for example [5, 6]) in which all sites are equivalent, but where the propagation from site to site has a probability less than unity. This is the difference between bond and site percolation, however in our approach the site percolation model offers a greater flexibility and is closer to physical reality. The differences, however, are not great and both models belong to the same class of universality.

The novel features of the model are the following :

The model is dynamic and its evolution can be observed as a function of time.

The interaction between contaminated and non-contaminated sites (heat transfer in the case of a fire) :

a) depends on the nature of each site (type of « tree »),

b) is not limited to first neighbour interactions,

c) takes into account latency (a « tree » will only ignite when it has received enough « heat »),

d) allows for cooperative contamination (a « tree » may be « heated » by several neighbours simultaneously), and,

e) may be non-isotropic (effect of « wind » or « slope » for example).

In this work the model is restricted to the simplest case of a « forest » containing only two types of sites, combustible sites (occupied by a « tree ») and non-combustible sites, distributed in a random fashion on a regular lattice. The extension to the case of a random medium of more complex structure is trivial but not necessarily more instructive. The propagation of the « fire » will depend on the variable parameters, which are the percentage of combustible sites on the lattice, the characteristics of these sites, the intensity and range of the interaction between sites and also the physical size of the lattice used in the simulation.

The active sites are characterized by the amount of « heat » that the site must receive before the site « ignites » and by the amount of « heat » released whilst the site « burns ».

Two types of lattices have been used (square and triangular) and several different types of interactions have been studied ranging from first neighbour to fifth neighbour. In this work we are mainly interested in the propagation of the front close to the critical concentration of active sites where the front is near to extinction. The properties of the front have been observed for different sized lattices ranging from 10×10 to 300×300 sites. At the critical concentration the correlation lengths in an infinite lattice diverge and the results obtained for a finite lattice are a function of the lattice size (correlation lengths, critical concentration, critical slowing down).

Particular attention is paid to the vicinity of the critical concentration P_c since the divergence of the correlation length should give rise to universal properties [7-8] which do not depend on the exact details of the model. We use the law of similarity to obtain critical exponents (which may be compared with those of other models and in particular with the General Epidemic Model of Grassberger [2]) by means of the theory of finite size scaling proposed by Nightingale [9-11].

In section 2 the model is described in detail, in section 3 results are presented for quantities such as the critical density (analogous to the critical temperature for a second order phase transition), the time dependence of the number of burned sites, the speed and length of the front and the influence of the size of the sample. It will be seen (section 4) that the length of the front increases rapidly close to the propagation threshold and has a fractal behaviour with a critical exponent that has been calculated.

2. The model.

A random medium is created by placing combustible sites on a two-dimensional lattice. A square lattice has been used except, as noted, where a triangular lattice was used. The size of the square domain will be denoted by L and of the $L \times L$ available sites a fraction P are combustible and a fraction $1 - P$ are non combustible. A spatially random distribution is obtained by a Monte Carlo procedure. The combustible sites are characterized by two parameters τ_i, τ_b and by the type of interaction. τ_b is the number of machine time steps that an ignited site remains in the burning state, and is thus a measure of the heat released by a site. τ_i is a measure of the heat required to ignite a site and corresponds to the number of machine time steps needed to ignite a site if one site in interaction is burning. In the example of an isotropic four-neighbour interaction (see Fig. 1), if one neighbour is burning then the site will ignite after τ_i time steps, but if all four neighbours are alight then only $\tau_i/4$ time steps are required. The different types of interaction studied here are shown in figure 1. The centre site is ignited and transfers one unit of heat to each neighbour at each time step. At time $t = 0$ all the combustible sites in the first two rows are ignited. At each successive time step the heat transfer from each burning site to its neighbours is calculated from the type of interaction (see Fig. 1). A search is then made to see if new sites have received enough heat to ignite and to see if burning sites have burned long enough to extinguish. This process is then repeated until either

a) a site on the last row is ignited (penetration) or

b) all sites are extinguished (exhaustion).

Periodic boundary conditions are implemented on the lateral edges of the domain. The model is thus deterministic and is related to the model of site percolation with some complication due to the parameters τ_i and τ_b and incorporating the possibility of non-first neighbour interactions and also co-operative effects. In this work only two species (combustible and non-combustible) are present. The generalization to a mixture of several types of combustible species is quite straightforward. Different types of interaction have been studied ranging from 1st neighbour to 5th neighbour but only for the case of uniform isotropic heat transfer. Again the generalization to the case of more complicated heat transfer patterns (e.g. anisotropic) is trivial. In some of the early work a slightly different and less realistic algorithm was used.

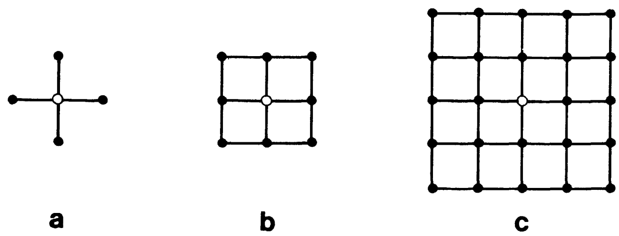

Fig. 1. — a) Von Neumann neighbourhood (4 neighbours); b) Moore neighbourhood (8 neighbours); c) 24 neighbours. In these three interactions schemes, the central site (empty circle) influences (or is influenced by) the neighbours represented by a full circle. The equivalent of a triangular array is easily obtained by the suppression of two diagonally opposite sites. The full circles can be taken non equivalent to represent anisotropic properties of the medium, slopes etc...

In this first (and fastest) algorithm the lattice was scanned from left to right, row by row, and each time a burning site was encountered, the heat transfert was calculated and the affected sites were ignited and extinguished as necessary. Each sweep through the lattice constitutes one time step and in the limiting case of a densely populated lattice with $\tau_i = \tau_b = 1$ the front traverses the lattice in one time step. This algorithm was used to determine the critical densities presented in figure 2. A number of calculations performed with the second algorithm showed that within statistical errors this apparent violation of left-right symmetry did not influence the position of the threshold.

We must now define an order parameter. As in a static percolation problem we have chosen the same as Grassberger's [2], that is to say in our language of forest fire, the proportion of burned trees in a row far from the initially ignited row (t and $n \gg 1$, where t is the number of sweeps executed by the computer and n the distance from the row ignited at $t = 0$). The notation for the order parameter is

$$Z(n, t, \varepsilon) = \lim_{\substack{n,t \gg 1 \\ n \leqslant L}} \frac{\text{nb of burned trees in the } n\text{th row}}{P.L} \tag{2.1}$$

where $\varepsilon = (P - P_c)/P_c$.

Our order parameter $Z(n, t, \varepsilon)$ is essentially the same as the order parameter in static percolation theory, defined by the proportion of sites belonging to the

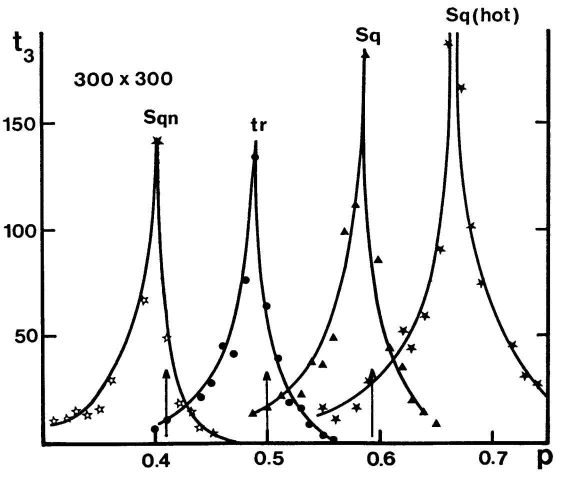

Fig. 2. — Average termination time t_3 (penetration or exhaustion) of fire in square lattice (Sq), triangular lattice (tr), square lattice with nearest and next-nearest neighbour interaction (Sqn), and Sqn lattice with intermediate hot stage ($\tau_i = 2\,\tau_b$) before ignition (Sqn hot). A fraction P of lattice sites is randomly occupied with trees. Each point is based on at least three samples of size 300 x 300. The arrows give the known percolation thresholds for Sqn, tr and Sq infinite lattices. As expected these thresholds coincide with ours for the case $\tau_i = \tau_b$. However it can be seen that introducing cooperative effects drastically affects the threshold. For the case Sqn with $\tau_i = 2\,\tau_b$ (two of the 8 neighbours must burn for the fire to propagate) the threshold is raised from $P_c = 0.407$ to $P_c = 0.67$.

infinite cluster. Using the universality postulate [8], we assume that Z is a generalized homogeneous function near the threshold ($\varepsilon \ll 1$) and then, $\forall \lambda$:

$$\lambda Z(n, t, \varepsilon) = Z(\lambda^{a_n} n, \lambda^{a_t} t, \lambda^{a_\varepsilon} \varepsilon) . \tag{2.2}$$

If we define two quantities, a correlation length : $\xi \propto \varepsilon^{-\nu}$ and a characteristic time : $\theta \propto \varepsilon^{-\tau}$, the three critical exponents a_n, a_t, a_ε can be expressed as functions of ν, τ and β with $Z(\infty, \infty, \varepsilon) \propto \varepsilon^\beta$:

$$a_n = \frac{-\nu}{\beta}, \quad a_t = \frac{-\tau}{\beta}, \quad a_\varepsilon = \frac{1}{\beta} . \tag{2.3}$$

With these relations (2.2) takes the form of a finite size scaling law [9-11] with the three variables n, t, ε :

$$Z(n, t, \varepsilon) = n^{-\beta/\nu} f(n/\xi, t/\theta) . \tag{2.4}$$

However, in our calculation, it is easier to calculate the time dependency of the mean advance and of the total number of burned trees. If we were exactly at the critical density, we would have for the order parameter :

$$Z(n, t, 0) = n^{-\beta/\nu} z_1(n^{-\tau/\nu} t) . \tag{2.5}$$

Consequently Z varies as $n^{-\beta/\nu}$ if $n^{-\tau/\nu} t$ = constant, i.e. n scales as $t^{\nu/\tau}$, this gives the mean propagation speed. We can easily determine this exponent ν/τ by measuring the mean position of the front (mean abscissa of the burning trees at a given time). It is also possible to calculate accurately another critical exponent, noticing that the total number of burned tree at given t is obtained by

$$N_b(t, \varepsilon) = P.L \int_n \mathrm{d}n\, Z(n, t, \varepsilon) \tag{2.6}$$

some straightforward algebra gives the result

$$N_b(t, \varepsilon) \propto t^{(\nu - \beta)/\tau} . \tag{2.7}$$

We have obtained two expressions to calculate the critical exponents from log-log curves :

$$n \propto t^{\nu/\tau} \quad \text{and} \quad N_b(t, \varepsilon) \propto t^{(\nu - \beta)/\tau} .$$

It is possible to obtain a third relation to determine all the critical exponents :

$$Z(n, t, \varepsilon) = \varepsilon^\beta g(\varepsilon^\nu n, \varepsilon^\tau t) . \tag{2.8}$$

For large t, we again find the static problem (we are looking at the forest after the exhaustion of the fire) and (2.8) gives :

$$Z(n, \infty, \varepsilon) = \varepsilon^\beta z_2(n\varepsilon^\nu) . \tag{2.9}$$

Using the same argument as before, if we examine the particular case of $n = L$, then z_2 is independent of L if $L\varepsilon^\nu$ is constant. When $L \to \infty$, $z_2 \to$ constant because $Z(\infty, \infty, \varepsilon) \propto \varepsilon^\beta$. So ε must be equal to 0 and the threshold is exactly at P_c. For finite L, we find the threshold for $P \neq P_c$ with the law :

$$\varepsilon \propto L^{-1/\nu} . \qquad (2.10)$$

Figure 3 shows the evolution of the percentage of penetrating fires *versus* P, for forests of different sizes, and with $\tau_i = \tau_b = 1$. The averages are taken over 200 trials and for networks up to 100 × 100. Larger dimensions gave results too close to the discontinuity associated with an infinite medium. This method permits an exact calculation of P_c, associated with an infinite system, from results given by finite systems [14]. For $P < P_c$, the probability of penetration increases for decreasing sizes and it becomes different from 0 when the size of the system is of the order of the correlation length. For $P > P_c$ we have a similar phenomenon. But strictly at $P = P_c$ the correlation length is infinite and the size of the system is no longer a relevant variable, so all the curves must intersect at the critical density. The family of curves represented on figure 3 gives a very good approximation for the threshold (Von Neumann neighbourhood) : $0.593 < P_c < 0.594$ for sizes in the interval $20 < L < 100$. Near P_c it is possible to calculate the critical exponent ν from (2.10) : however the results are not very precise and we find :

$$1.2 < \nu < 1.4 \quad (\text{instead of } 4/3) .$$

3. Results and discussion.

To calculate ν/τ et $(\nu - \beta)/\tau$, we have used a systematic but empirical criterion : array 200 × 200 and 300 trials

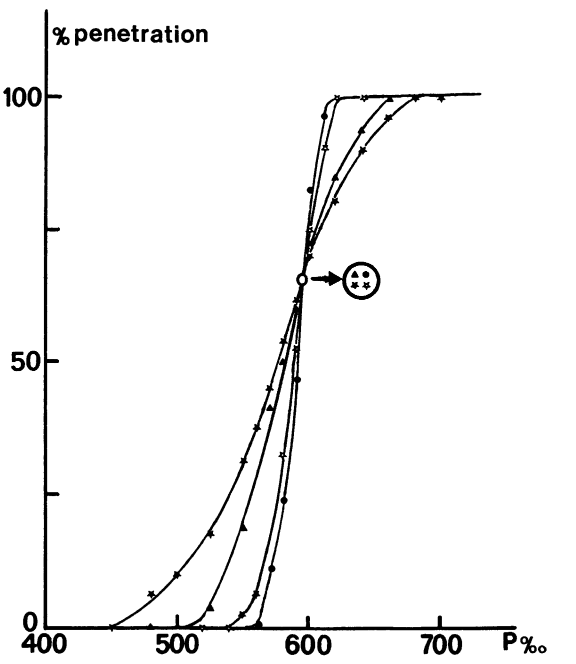

Fig. 3. — Probability of penetration *versus* P for several sizes of the network and for $\tau_i = \tau_b = 1$: ★ (20 × 20), ▲ (40 × 40), ☆ (80 × 80), ● (100 × 100). All curves converge near the point : probability = 66 %, $P = 0.593$.

(forests created randomly). Attempts with more trials (up to 500) did not give significant differences for the mean quantities calculated.

At $\varepsilon = 0$ the $n(t)$ curves (see Fig. 4) present three typical regions labelled I, II, III. In region I, we are in a transitory state where the ignition of the whole of the first two rows gives a quick advance of the front (the system « remembers » the initial conditions). Region II is associated to a steady propagation where the scaling laws work : this is the interesting region for us. In zone III, the finite dimension of the system is the prevailing factor : the propagation is restricted by the border of the forest, giving a decrease of the speed of the mean position of the front.

We have calculated the critical exponents τ and β with the exact value of ν given by the literature, our value for this last parameter being not very precise.

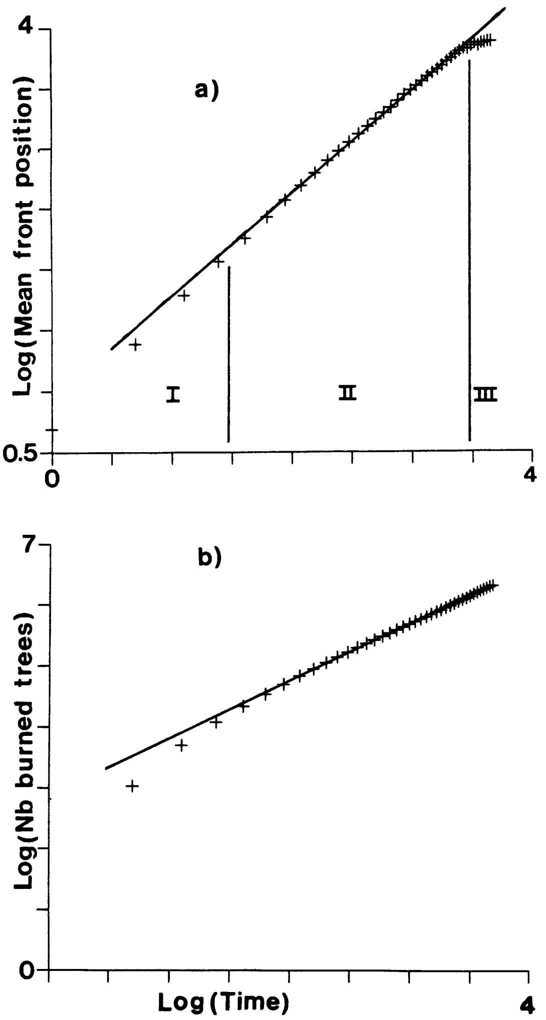

Fig. 4. — Calculation for a forest (200 × 200) and with $\tau_i = \tau_b = 1$: a) Mean position of the front; b) Total number of burned trees *versus* time plotted in log-log coordinates. Region I : region of high speed (transitory state) where the ignition conditions are playing a noticeable role. Region II : the universality hypothesis works. Region III : the finite size of the system is experienced by the front.

Table I.

Neighbourhood	$\frac{\nu}{\tau}$	$\frac{\nu - \beta}{\tau}$	τ	β	f	P_c
Von Neumann	0.87 ± 0.02	0.79 ± 0.02	1.55 ± 0.04	0.12 ± 0.03	1.87 ± 0.03	0.593
Moore	0.87 ± 0.02	0.78 ± 0.02	1.55 ± 0.04	0.13 ± 0.03	1.88 ± 0.03	0.407
24	0.86 ± 0.02	0.77 ± 0.02	1.55 ± 0.04	0.11 ± 0.03	1.88 ± 0.03	0.168

This has been done for all the different interactions that we have introduced in our model and our results are summarized in table I.

The values found for τ and β are close to those found by Grassberger ($\tau = 1.494$) or obtained theoretically ($\beta = 5/36 \simeq 0.14$).

However, the error in our result for β is nearly of the order of magnitude of the parameter. It is more interesting to use the theoretical value of β, instead of that of ν [12]. We then obtain : $\tau = 1.54 \pm 0.03$ and $\nu = 1.34 \pm 0.03$, this last result agrees with the theoretical one ($\nu = 4/3$) [12], our errors being statistical only.

For the exponent ν/τ, we verified our calculation by another method : for $\varepsilon > 0$, we plot the mean time necessary for the fire to go through the forest, for different forest sizes ($L = 50$ to 200) discarding the trials giving an exhaustion. We then plot log t *versus* $\log L (t = L^{\tau/\nu})$ and we find a straight line with a slope $\tau/\nu = 1.15$: this is the same result as the one obtained previously (see Fig. 5).

As expected, we found the same critical exponents for all models (see Table I) and our results agree well with those given in the literature [2, 12], in particular the value of the exponent τ is very close to the value obtained by Grassberger [2] with his general epidemic model. These exponents can be related to general ones such as :

— $\bar{d}$, the fractal dimension of the percolating cluster of burned trees, defined such that the total « mass » of trees and the size of the array obeys :

$$M \propto L^{\bar{d}} \tag{3.1}$$

— $\hat{d}$, the spreading dimension [2, 15-17]. If l_m is the mean number of steps required to cross the forest, $\hat{d}$ is defined as :

$$M \propto (l_m)^{\hat{d}} . \tag{3.2}$$

By equating (3.1) and (3.2) :

$$l_m \propto L^{\bar{d}/\hat{d}} \tag{3.3}$$

where $\bar{d}/\hat{d}$ is the δ exponent of Vannimenus [15]. Now l_m is simply t, the mean time for penetration, and so : $t \propto L^{\bar{d}/\hat{d}} \propto L^{\tau/\nu}$. It is possible to calculate $\bar{d}$ from [7, 20] :

$$\bar{d} = d - \beta/\nu \tag{3.4}$$

where d is the dimensionality of the system (here $d = 2$), giving :

$$\bar{d} = 1.89 \quad \text{and} \quad \hat{d} = \bar{d} . \nu/\tau = 1.64 .$$

These values are very close to those found in references [16] (1.896 and 1.65-1.70 respectively) and [2] where the very accurate value $\hat{d} = 1.675$ is given.

In figure 6 we show typical « snapshots » of two forests with differents types of interactions. In both cases the population of active sites is close to the critical concentration. It can be seen that in both cases the number of trees actually burning at a given time is small. This is a physical explanation of the critical slowing down. In the case of the 4 neighbour (Von Neumann) environment the front is extremely rough and there are many pockets of unburned trees left behind the front. In the case of the 24 neighbour environment there are very few pockets (one on this example, often there were no pockets left in a finite size sample) and the front appears somewhat smoother. In the case of the 8 neighbour (Moore) environment, not shown, the results are intermediate between the two types of behaviour shown here. The image for 24 neighbour interaction is closer to the observed form of fronts such as in forest fires and we may conclude that cooperative effects are important in determining the form of such fronts.

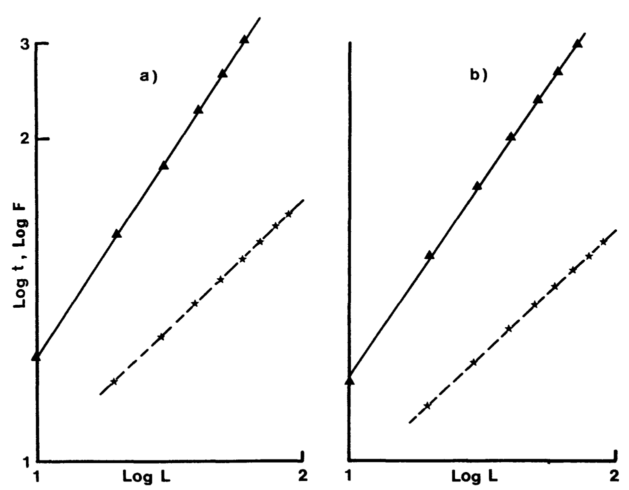

Fig. 5. — The penetration times (broken lines) are plotted *versus* the forest dimensions (log-log diagram) for the Von Neumann (a) and the Moore (b) neighbourhoods in the case $\tau_i = \tau_b = 1$. We also represent the length of the frontier *versus* the forest dimensions (full lines).

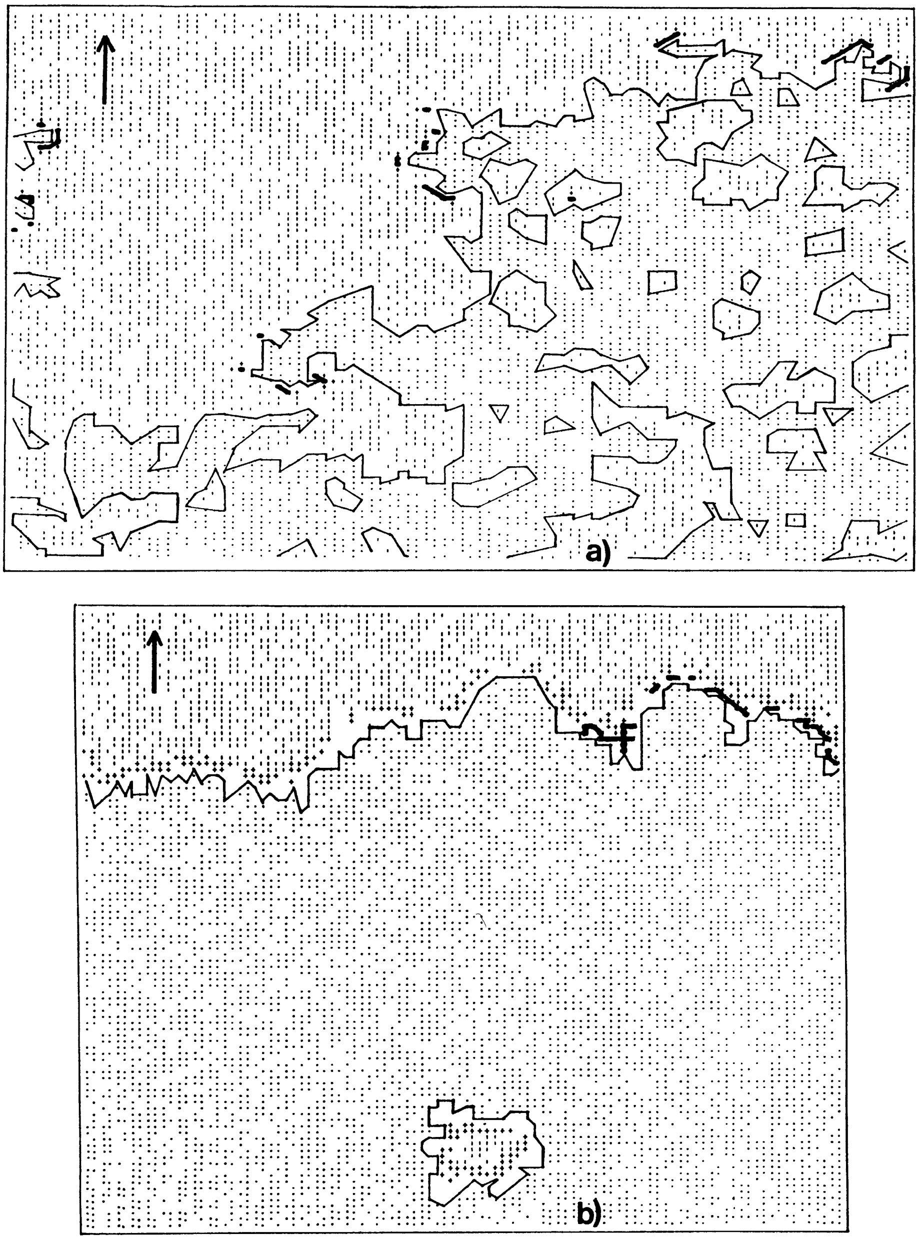

Fig. 6. — « Snapshots » of burning forests size 100 × 100 with populations close to the critical concentration. a) Von Neumann environment (4 neighbours), $\tau_i = \tau_b$, $P = 0.6$,($P_c = 0.593$) ; b) 24 neighbours environment, $\tau_i = 6\ \tau_b$ (6 neighbours must burn) $P = 0.64$ ($P_c = 0.63$).
The direction of propagation of the fire is indicated by the arrows. At time $t = 0$ all trees in the two bottom rows are ignited. The symbols used are : | : fresh tree ; + : hot tree ; .: : burned tree ; **|** : burning tree. The frontier between burned and unburned trees is indicated by a continuous line.

For 8 and 24 neighbours we have also computed the threshold evolution *versus* τ_i/τ_b. These results are given in figure 7 where we have also plotted the threshold given by a simple effective medium model (EMM) : this simple model supposes that in an environment of N neighbours, each site interacts with exactly $N \times P$ surrounding sites. For the two types of interactions, the curves obtained by EMM and by simulation cross ; at low density the simulation threshold is higher : this is easy to understand because some paths of propagation are cut by the random process. We obtain the opposite situation at high density.

Finally we introduce an algorithm to calculate the length of the front or, more precisely, the number F of burned sites which are in contact with unburned sites when the fire has just penetrated the forest. For all models we have plotted $\log F$ *versus* $\log L$ at the threshold and we obtain straight lines (see Fig. 5) with slopes $f = 1.80 \pm 0.03$, value slightly greater than

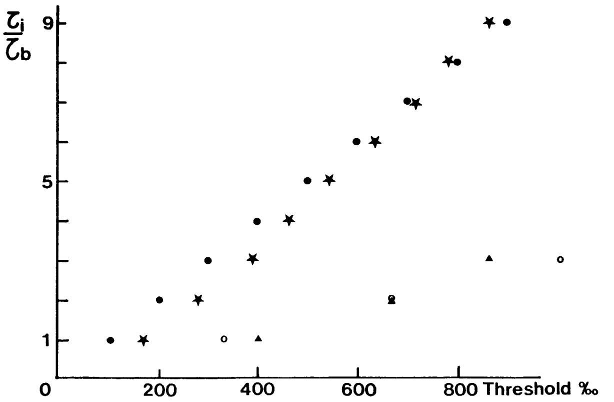

Fig. 7. — Evolution of the threshold *versus* τ_i/τ_b calculated for a forest 150 × 150, for two neighbourhoods : 8 neighbours : ○ for EMM, ▲ for our model; 24 neighbours : ● for EMM, ★ for our model (EMM = Effective Medium Model).

that obtained by Sapoval *et al.* [13] : $f = 1.76$. This discrepancy probably originates from the fact that in our model « pockets » of unburned trees behind the front are counted as belonging to the front, giving an apparent increase in the length of the front. The critical exponent f characterizes the self similar (fractal) structure of the front and its large value indicates a considerable « rugosity » of the fire border at the threshold.

In summary, various models for propagation of fronts in disordered media have been simulated numerically. Different types of interactions can be easily introduced : 4, 6, 8,..., 24 neighbours, anisotropy, slope of the ground in the forest fire example (gradient of a relevant field), different sorts of active sites. Here attention has been focussed on a simple case in order to compare our calculations to known results. We obtain critical exponents which are found to verify the universality conjecture. However the model is very flexible and can be generalized to other problems such as : propagation of a flame, deflagration of a dust suspension, spreading of a disease, etc. Some extensions of the present work are planned.

Acknowledgments.

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