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Submitted on 1 Jan 1985

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Estimation of isospin effects in nucleon-\(^4\)He scattering

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(Reçu le 27 novembre 1984, accepté sous forme définitive le 9 juillet 1985)

Résumé. — Dans le cadre d'un modèle de Glauber simple mais calculable sans approximation quant aux contraintes du centre de masse, nous étudions l'effet des différences de densités entre neutrons et protons dans le noyau \(^4\)He, sur la section efficace de diffusion élastique nucléon-\(^4\)He à 800 MeV, à petits transferts. Les résultats suggèrent une méthode de calibration absolue d'un faisceau de neutrons.

Abstract. — Within the framework of a very simple Glauber model treating the centre-of-mass effects exactly, we investigate the effect of neutron-proton density differences in the nucleus \(^4\)He on the 800 MeV nucleon-\(^4\)He elastic scattering cross section at small transfers. From the results, a method of obtaining an absolute calibration of a neutron beam is proposed.

1. Introduction.

Recently, a Russian-French collaboration has begun to study the n-p elastic scattering at intermediate energies (\(E_n = 400\) to \(1\) \(100\) MeV) with the neutron beam available at the Saturne synchrotron of Saclay. One of the problems in this experiment is to measure the intensity of the neutron beam used: we have accurate relative monitors, but we need absolute calibration of them.

Among the various possible methods, one of them does not require a large experimental effort. Using the Saturne neutron beam and the same detector as in the n-p experiment, we can easily obtain relative elastic cross sections at small transfers for the n-\(^4\)He system. Absolute elastic cross sections have been accurately measured at Gatchina for the p-\(^4\)He system. Thus, if the nuclear cross sections are the same in both systems, the comparison of our relative n-\(^4\)He cross sections with the absolute p-\(^4\)He (Coulomb deduced) cross sections gives a direct calibration of the monitors of the neutron beam-line.

The purpose of this paper is to investigate the question: at small transfer \(t\), how far do we have

\[
\frac{d\sigma}{dt}(\text{p-}^4\text{He}) = \frac{d\sigma}{dt}(\text{n-}^4\text{He}) \tag{1}
\]

up to Coulomb corrections?

In first order, we can say the following. \(f_{pp}\) and \(f_{pn}\) being the elementary nucleon-nucleon amplitudes, the p-\(^4\)He and n-\(^4\)He amplitudes are given by:

\[
\begin{align*}
    f_{p-^4\text{He}} &\sim f_{pp} \otimes S_p(\text{p}^4\text{He}) + f_{pn} \otimes S_n(\text{p}^4\text{He}) \\
    f_{n-^4\text{He}} &\sim f_{pp} \otimes S_p(\text{n}^4\text{He}) + f_{pn} \otimes S_n(\text{n}^4\text{He})
\end{align*}
\]

where \(S_p(\text{He}^4)\) and \(S_n(\text{He}^4)\) are the form factors of protons and neutrons in \(\text{He}^4\). (We shall write \(S_p\) and \(S_n\) for simplicity).

If charge symmetry holds, we have \(f_{pp} = f_{nn}\).

Then, by taking the difference between the two previous equations we obtain

\[
\begin{align*}
    f_{p-^4\text{He}} - f_{n-^4\text{He}} &\sim (f_{pp} - f_{pn}) \otimes (S_p - S_n) \tag{2}
\end{align*}
\]

We know from N-N scattering that \(f_{pp}\) and \(f_{pn}\) are not extremely different. Also, in the very symmetric \(\text{He}^4\) nucleus, the form factors \(S_p\) and \(S_n\) which are directly related to the densities \(\rho_p\) and \(\rho_n\) of protons and neutrons in \(\text{He}^4\), are not expected to be very different. Thus, at first sight, equation (1) is satisfied and our calibration method is valid. To study the accuracy of this answer, we have made precise calculations using a simple Glauber model, where proton and neutron densities in \(\text{He}^4\) are taken to be different; this is explained in section 2. In section 3 the results of our calculations are compared with expe-
In section 4, the difference between cross sections of elastic scattering on $^4$He by protons and by neutrons is predicted. In particular, the deviations from equation (1) are studied as a function of the neutron-proton density difference in $^4$He. Notice that we have made these calculations for an incident energy of 800 MeV of the proton (or of the neutron) which is about at the middle of the range of interest for the Saturne neutron beam.

2. The frame of our calculations.

The calculations presented in this paper have been made using the Glauber model in the eikonal form [1]. This model is known to be of practical use and rather accurate in nucleon-nucleus scattering at intermediate energies, especially at small transfers.

Let us start from the basic formula giving the amplitude $f_{N\alpha}(q)$ of the scattering of a nucleon on a mass $A$ target [1].

$$f_{N\alpha}(q) = \frac{iK}{2\pi} \int d^2b \ e^{i\mathbf{q}\cdot\mathbf{b}} \int \psi^{\alpha}(\mathbf{r}_1, \ldots, \mathbf{r}_A) \frac{1}{A} \left( \sum_{j=1}^{A} \mathbf{r}_j \cdot \mathbf{n} \right) \psi(\mathbf{r}_1, \ldots, \mathbf{r}_A) \times$$

$$\times \left[ 1 - \frac{A}{A-1} \int d^2q' \ e^{i\mathbf{q'}\cdot(\mathbf{b}-\mathbf{s})} f_N(\mathbf{q'}) \right] d\mathbf{r}_1 \ldots d\mathbf{r}_A. \quad (3)$$

In this formula, the symbols have the following meanings:

- $q$: Momentum transferred.
- $K$: Relative momentum between the projectile and the nucleus.
- $k$: Relative momentum between the projectile and the $j$th nucleon of the target.
- $f_{Nj}$: Scattering amplitude of the incident nucleon $N$ with the $j$th nucleon of the target.
- $r_j$: Position of the $j$th nucleon in the $A$ nucleus.
- $b$: Impact parameter of the projectile.
- $s_j$: Projection of $r_j$ on a plane perpendicular to the beam axis.
- $\psi_i$ and $\psi_f$: Wave functions of the initial and final states of the nucleus.

In our case (elastic scattering on the nucleus $^4$He), the product $\psi_i \psi_f^*$ in equation (3) is then the normalized density $\rho_{He}(\mathbf{r}_1, \ldots, \mathbf{r}_A)$, so that we can rewrite equation (3) as:

$$f_{N\alpha}(q) = \frac{iK}{2\pi} \int d^2b \ e^{i\mathbf{q}\cdot\mathbf{b}} \int \rho_{He}(\mathbf{r}_1, \ldots, \mathbf{r}_A) \left( \frac{\mathbf{r}_1 + \ldots + \mathbf{r}_A}{4} \right) \times$$

$$\times \left[ 1 - \frac{A}{A-1} \int d^2q' \ e^{i\mathbf{q'}\cdot(\mathbf{b}-\mathbf{s})} f_{N}(\mathbf{q'}) \right] d\mathbf{r}_1 \ldots d\mathbf{r}_A. \quad (4)$$

To evaluate this expression, we need now to choose the nucleon-nucleon amplitudes $f_{NJ}$ and the helium density $\rho_{He}$.

2.1 Choice of the nucleon-nucleon amplitudes $f_{NJ}$ — For proton-proton as well as for neutron-proton, the general expression of the N-N amplitude contains a central term, a spin-orbit term and spin-spin terms as can be seen in the classical general expression:

$$f_{NN}(q) = A(q) + C(q)(\sigma_1 + \sigma_2) \cdot \mathbf{n} +$$

$$+ B(q)(\sigma_1 \cdot \mathbf{n})(\sigma_2 \cdot \mathbf{n}) + D(q)(\sigma_1 \cdot \mathbf{m})(\sigma_2 \cdot \mathbf{m}) + E(q)(\sigma_1 \cdot \mathbf{l})(\sigma_2 \cdot \mathbf{l}).$$

(Here, $l$, $m$ and $n$ are the unit vectors in the longitudinal, perpendicular to the beam in the scattering plane, and perpendicular to the scattering plane directions).

One knows from time reversal invariance that $C(0) = 0$: the spin-orbit term will not play any role for the small-transfer that we are studying. The spin-spin terms are small at intermediate energies, but they are not completely negligible: an extensive study [2] of small transfer elastic scattering has shown that, for the proton-proton case, the cross section can be written in the center of mass frame as:

$$\frac{d\sigma}{d\Omega} = \frac{k^2 \sigma_{tot}^2}{16\pi^2(hc)^2} \left( 1 + \rho_0^2 + \beta_0 \right) e^{-bq^2} +$$

Coulomb terms \(5\)

where $\sigma_{tot}$ is the total p-p cross section,

$\rho_0$ is the ratio of the real to imaginary part of the central amplitude at $q^2 = 0$,

$\beta_0$ expresses the contribution of the spin-spin terms at $q^2 = 0$ also (a few percent), and

$b$ is the slope of the nuclear cross section.

Our preliminary results of the Saturne measurement of the n-p elastic scattering also show an exponential behaviour of the cross sections.

Thus, both for pp and pn, we have adopted for our calculations the usually used [3, 4] simple form
of the N-N interaction:
\[
f_{NN}(q) = \frac{k\sigma}{4\pi \hbar c}(i + \rho) e^{-\frac{4q^2}{\sigma}}
\]
where, to keep the optical theorem valid, we are led to adopt
\[
\sigma = \sigma_\text{tot}; \quad \rho = \rho_\text{eff} = \pm \sqrt{\rho_0^2 + \beta_0^2}.
\]
For the p-p system, at 800 MeV, we have adopted the values given in the table 3 of [2], i.e.
\[
\sigma = \sigma_\text{tot} = 47.1 \text{ mb}, \quad b = 5.83 \text{ (GeV/c)}^{-2},
\]
\[
\rho_0 = 0 \quad \text{and} \quad \beta_0 = 0.063
\]
which lead to \( \rho = 0.251 \).
For the n-p system, \( \sigma_{n-n} \) is more or less known [5]. \( \rho_{n} \) has been measured at 3 energies and is well accounted for by the forward dispersion relations calculations made by Grein and Kroll [6]; thus, we have used their predicted value. \( \beta_0 \) is not known at all but is expected to be small in n-p also. At 800 MeV, it could be determined from the only measurement existing for n-p at small transfers [9]. But the inaccuracy of the \( \sigma_{n-n} \) measurements makes its determination very unprecise, but compatible with 0: we have taken 0. Thus, at 800 MeV for the np system, the parameters that we have used are:
\[
\sigma = \sigma_{n-n} = 39 \text{ mb}
\]
(average between several measurements)
\[
b = 5.24 \text{ (GeV/c)}^{-2}
\]
\[
\rho = -0.3.
\]
2.2 Choice of the Neutron and Proton Densities in \( ^4\text{He} \). — The charge form factor of the \( ^4\text{He} \) nucleus is now quite well-known from electron scattering experiments [7, 8].
It is related to the normalized density of proton in \( ^4\text{He} \) by:
\[
F_{ch}(q) = \int e^{iq\rho_p(r)} dr
\]
where \( F_{ch}(q) \) is the internal charge form factor of the proton that we have taken as usual in the form [10]:
\[
f_{im}(q) = \left(1 + \frac{q^2}{0.71}\right)^{-2} \quad \text{(where} \ q \ \text{is in (GeV/c)} \right)
\]
The density \( \rho_p \) of protons in \( ^4\text{He} \) is itself related to the wave function of protons in \( ^4\text{He} \) and it can easily be shown that equation (8) can be rewritten as:
\[
F_{ch}(q) = f_{im}(q) \left(\frac{e^{iqr_1} + e^{iqr_2}}{2}\right) \times
\]
\[
\delta\left(\frac{r_1 + \ldots + r_k}{4}\right) |\psi(r_1 \ldots r_k)|^2 dr_1 \ldots dr_k
\]
where \( r_1 \) and \( r_2 \) are the coordinates of the 2 protons of \( ^4\text{He} \) and \( \psi \) is the wave-function of this nucleus.
The presence of a dip at \( \approx 10 \text{ fm}^{-2} \) in the \( ^4\text{He} \) charge form factor [8] prevents from using a simple Gaussian shape for \( \rho_p \) or \( \psi_p \) since it is not able to reproduce such a structure. Various more complicated analytic forms have been proposed which all give a good fit to the \( ^4\text{He} \) charge form factor. We use here the double-Gaussian expression of the wave-function already used previously [11] and, since neutron and proton densities are expected to be similar in the very symmetric \( ^4\text{He} \) nucleus, we have adopted the same analytical shape for both; thus we use:
\[
\psi(r_1 \ldots r_4) = N \prod_{i=1}^4 \exp\left(-\frac{r_i^2}{2R_i^2}\right) \times
\]
\[
\left[1 - D_i \exp\left(-\frac{r_i^2}{\gamma_i^2 R_i^2}\right)\right].
\]
where \( R_1 = R_2 = R_p \), \( D_1 = D_2 \), \( D_3 = D_4 \) and \( \gamma_1 = \gamma_2 \) are the parameters related to protons and \( \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 \) correspond to neutrons. The factor \( N \) allows us to normalize the wave-function \( \psi \).
Since we have no precise idea about the neutron form factor, we have adopted (for simplicity) \( D_1 = D_2 = D_3 = D_4 = D \) and \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma \). But, since the purpose of this paper is to study the effect of neutron-proton density differences, we have taken \( R_n \neq R_p \) contrary to previous calculations [3, 4].
Since we have chosen a Gaussian form for \( \psi \) (Eq. (10)), we have been able to evaluate the charge form factor \( F_{ch}(q) \) (Eq. (9)) exactly, without using any approximate prescription to evaluate the effect of the \( \delta \) in the integral (see Appendix for details). We have then adjusted the parameters \( D, \gamma, R_p \) and \( R_n \) to obtain the best fit to the data of electron scattering [8]. Notice that these data fix the charge density only and do not say anything about the root mean square radius of the neutron \( \langle r_{\text{rms}} \rangle \). But, in our model, the presence of the \( \delta \) in equation (9) very slightly couples the parameters \( R_p \) and \( R_n \) in such a way that \( \langle r_{\text{rms}} \rangle_p \) can be kept practically constant and the fit almost equally good under large (small) variations of \( R_n (R_p) \) correlated in the way indicated in the bidimensional representation of the \( \chi^2 \) (Fig. 1). Approximately, around \( R_p = R_n \), the correlation is linear
\[
R_n + 6.7 R_p \approx 10 \text{ fm}.
\]
Notice that, in figure 1, we have restricted ourselves to the region of the plane centered around \( R_p = R_p \). To study too large deviations of \( R_n \) from \( R_p \) would have no meaning.
The corresponding invariant proton \( \langle \text{r.m.s.} \rangle \) ra-
Fig. 1. — 1a) Iso-$\chi^2$ curves of the fit of the $^4$He charge form factor data [8] in our model (Eqs. (9), (10) and Table I). 1b) The experimental and calculated (with $R_n = R_p = 1.295$ fm) $^4$He charge form factor.

\[
\langle \text{r.m.s.} \rangle_p = \int \frac{r_1^2 + r_2^2}{2} \delta \left( \frac{r_1 + \cdots + r_4}{4} \right) \times \prod_{i=1}^{4} \exp \left( -\frac{r_i^2}{R_i^2} \right) \left[ 1 - D_i \exp \left( -\frac{r_i^2}{\gamma_i^2 R_i^2} \right) \right]^2 \, dr_1 \cdots dr_4
\]

is, in the region $R_n = R_p$ of the correlation line of figure 1

\[
\langle \text{r.m.s.} \rangle_p = 1.427 \text{ fm}
\]

which, combined with the r.m.s. of the internal form factor of the proton $f_{\text{in}}(q)$, leads to a value of the r.m.s. charge radius of $^4$He

\[
\langle \text{r.m.s.} \rangle_{\text{ch}} = 1.641 \text{ fm}.
\]

This value is quite compatible with the value of $1.644 \pm 0.005$ fm measured by the muonic atoms method [12].

Table I summarizes all the parameters discussed in this section, that we are now going to use to compute the p-$^4$He and n-$^4$He scattering amplitudes expressed in equation (4).

3. Results of the calculation of the proton-$^4$He elastic scattering.

Before going into the investigation of the validity of equation 1, which is the main question that we want to solve in this paper, let us test if our model is realistic by comparing results of calculation to experimental values.

Using the density of equation (10)

\[
\rho_{\text{He}}(r_1 \ldots r_4) = \left| \psi(r_1 \ldots r_4) \right|^2 = N^2 \prod_{i=1}^{4} \exp \left( -\frac{r_i^2}{R_i^2} \right) \left[ 1 - D_i \exp \left( -\frac{r_i^2}{\gamma_i^2 R_i^2} \right) \right]^2
\]

(12)

and the expression given by equation (6) for the N-N amplitudes, we have computed the elastic scattering proton-$^4$He amplitude $f_{p-^4\text{He}}$ given by equation (4). Again, the only functions appearing in integrals are Gaussians, and the expression given by equation (4) which contains all orders of the multiple scattering, can be calculated exactly (see Appendix for details; of course, the analytic calculation of equation (4) has been performed by computer, since the number of non-identical terms is large and it is a cumbersome task to do by hand). Adding to this nuclear amplitude the Coulomb amplitude given by [13] $f_c e^{i\phi}$ where

\[
f_c = \frac{2}{\beta_{\text{lab}}} Z_p Z_{\text{He}} \frac{2\hbar K}{\beta_{\text{lab}}} G_p G_{\text{He}}
\]

\[
\phi = -\frac{Z_p Z_{\text{He}}}{\beta_{\text{lab}}} \left[ \ln (B \mid t) + 0.577 \right]
\]

(see [13] for the meaning of symbols and values), one obtains the differential cross section:

\[
\frac{d\sigma}{d\Omega}_{p-^4\text{He}} = \left| f_c e^{i\phi} + f_{p-^4\text{He}} \right|^2
\]

or, in terms of relativistically invariant momentum
The results of the calculations made with the parameters given in table I and \( R_p = R_n = 1.295 \text{ fm} \) are presented in the upper part of figure 2, together with the recent data measured at 793 MeV at Gatchina [14] with an experimental set-up especially designed for high accuracy measurements of elastic scattering on very light nuclei, at small transfers. In this experiment, the overall normalization error is of the order of \( \pm 2\% \). The data of argonne [15] also appear in figure 2. For this last set of data, the normalization uncertainty is \( \pm 15\% \). The two experiments are compatible within errors, but the Gatchina results are more precise.

In figure 2 one can see that only a very slight renormalization and slope change would be required for a perfect agreement between experimental and calculated cross-sections. The quality of the fit appears to be extremely good with respect to the simplicity of our model, which thus appears to be realistic enough to treat our problem.

### 4. Comparison of n-4He and p-4He elastic scattering.

Let us then come back to the initial question of comparing proton and neutron induced elastic scattering by \(^4\text{He}\) at small transfers. Let us define:

\[
\Delta \left( \frac{d\sigma}{dt} \right)_{np} = \left[ \frac{d\sigma}{dt}_{n-4\text{He}} - \frac{d\sigma}{dt}_{p-4\text{He}} \right]_{\text{Nuc.}} - \left[ \frac{d\sigma}{dt}_{n-4\text{He}} + \frac{d\sigma}{dt}_{p-4\text{He}} \right]_{\text{Nuc.}}.
\]

In this definition, the upper bar means an average of the quantity in brackets, over the small transfer region, namely \( t = 0.07 \) (GeV/c)\(^2\) in the actual calculations presented in the following. The label "Nuc." in \( \left( \frac{d\sigma}{dt} \right)_{p-4\text{He}} \) means that we consider the nuclear part only (no Coulomb term in the amplitude).

Within the framework of our model, we can give a prediction for the quantity \( \Delta \left( \frac{d\sigma}{dt} \right)_{np} \). Indeed, we have for proton as well as for neutron:

\[
\left( \frac{d\sigma}{dt} \right)_{\text{Nuc.}} = \frac{\pi}{K^2} |f_{n-4\text{He}}|^2
\]

where \( f_{n-4\text{He}} \) (Eq. (4)) can be computed using equation (6), (10) and the parameters of table I. If we allow \( R_n \) and \( R_p \) to vary, as described in section 2, along the correlation line of figure 1, we can then compute cross-sections for different values of \( \Delta \langle \text{r.m.s.} \rangle_{np} = \langle \text{r.m.s.} \rangle_n - \langle \text{r.m.s.} \rangle_p \). Averaging each time as written in equation (14), we obtain the mean value of the difference of the differential cross-section at small \( t \) for p-\(^4\text{He}\) and n-\(^4\text{He}\) elastic scattering \( \Delta \left( \frac{d\sigma}{dt} \right)_{np} \), as a function of \( \Delta \langle \text{r.m.s.} \rangle_{np} \). The results of the calculations are presented in figure 3. It is immediately apparent that, even for large differences of n and p radii in \(^4\text{He}\), n and p induced cross-sections exhibit very small differences.
Fig. 3. — Calculated difference between differential cross sections of neutron and proton elastic scattering by \(^4\)He, averaged over the \(t = 0-0.7\) (GeV/c)^2 region, versus the neutron-proton \(< r.m.s. >\) radii difference in the \(^4\)He nucleus.

As a matter of fact, we can say something about the actual value of \(\Delta r_{np}\) in \(^4\)He, both from experiment and from theory. Indeed, for \(N = Z\) nuclei heavier than \(^4\)He, the neutron and proton densities have been extracted from intermediate energy electron and proton [16] scattering, or calculated in Hartree-Fock approximation [17]. In figure 4 we have plotted the corresponding calculated values of \(\Delta <\text{r.m.s.}>_{np}\). Extrapolating the curve to \(A = 4\), one sees that \(\Delta <\text{r.m.s.}>_{np}\) for \(^4\)He is probably slightly negative and smaller than 0.10 fm in absolute value. Experimental values of \(\Delta r_{np}\) are compatible with calculated H.F. values, but they exist for few nuclei and are not precise enough to make any extrapolation to \(^4\)He. Considering 0.10 fm as an upper limit for the neutron-proton \(<\text{r.m.s.}>\) difference in \(^4\)He, we then have, from figure 3.

\[|\Delta <\text{d}\sigma/\text{d}t>_{np}| < 1.5\% .\]

Thus, we can now answer the question asked at the beginning of this paper: yes, in the small transfer region, cross sections of elastic scattering by \(^4\)He induced by neutrons and by protons can be considered as equal within an accuracy of the order of 1.5\%.

Therefore we consider that the method of absolute calibration of neutron beams explained in the introduction of this paper can be believed, at 800 MeV, within an "of principle" accuracy of 1.5\% about.

Such estimations will be repeated for other energies in the future.

5. Conclusion.

Within the framework of a very simple but exactly calculable Glauber model, we have been able to reproduce differential cross sections of proton-\(^4\)He elastic scattering at 800 MeV rather well.

Taking into account both experimental results and theoretical calculations on \(N = Z\) nuclei which suggest that the neutron \(<\text{r.m.s.}>\) in \(^4\)He does not differ from the proton one by more than 0.1 fm, we have shown that elastic scattering of 800 MeV protons and neutrons by \(^4\)He have the same nuclear cross section at small transfers, with an accuracy of about 1.5\%. Thus, this number can be considered as an \textit{a priori} error in the method of absolute calibration of neutron beams by comparison of absolute p-\(^4\)He to relative n-\(^4\)He cross sections.

We want to thank Miss B. Louis for her participation to the early stage of the calculations, and Dr. R. J. Lombard for stimulating discussions and communication of the values presented in figure 4.

Appendix

In this paper, a certain number of quantities have been computed, namely

— The nucleon-\(^4\)He scattering amplitude (Eq. (4)).
— The charge form factor of \(^4\)He (Eq. (9)).
— The normalization factor \(N\) of the wavefunction (Eq. (10)) defined by:

\[N^2 \int \delta \left( \frac{r_1 + \cdots + r_s}{4} \right)^4 \exp \left( -\frac{r_i^2}{R_i^2} \right) \times \left[ 1 - D_i \exp \left( -\frac{r_i^2}{\gamma_i^2 R_i^2} \right) \right]^2 = 1 .\]

— The \(<\text{r.m.s.}>\) of proton and neutron radii defined by equation of type (11).

After integration over one of the \(r_i\) to eliminate the \(\delta\), when expanding the squared parenthesis coming from the \(\left[ 1 - D_i \exp \left( -\frac{r_i^2}{\gamma_i^2 R_i^2} \right) \right]^2\) term, all expres-
sions to be computed reduce to a linear combination of integrals of the type:

\[ I = \int e^{-\langle \mathbf{r}_i - \mathbf{r}_A \rangle} M_{A+1} \prod_i dr_i \]

where \(M_{A+1}\) is a symmetric \((A + 1)\)-rank matrix, in which the \(A\)-rank matrix located in the upper left corner is the symmetric matrix \(M_A\). The \(\mathbf{r}_i\) are vectors of a \(n\)-dimensional space. \((A = 3\) and \(n = 2\) or 3 in our calculations). Then with standard linear algebra techniques it is easy to show that such an integral is given by:

\[ I = \frac{x^{A/2}}{||M_A||^2} \exp \left( - \frac{||M_{A+1}||}{||M_A||} q^2 \right) . \]

Notice that the above formula is also valid in the particular case where \(q = 0\), which occurs in the calculation of \(\langle\text{r.m.s.}\rangle\) radii or of the wave function normalization factor \(N\).

### References


