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Equilibrium critical properties of random field systems : new conjectures

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Résumé. — Une conception nouvelle des transitions de phase dans les systèmes avec désordre figé est introduite en vue de tenir compte des propriétés expérimentales des modèles d'Ising en champ aléatoire. Selon cette conception, la mise en équilibre nécessite, près de T_c, des sauts activés entre des puits d'énergie libre dont la distance dans l'espace des phases est grande. Ces sauts impliquent une dynamique très lente, avec un temps de relaxation donné par une loi de Vogel-Fulcher modifiée, $\tau \approx \tau_0 \exp[\mathrm{Cte}/(T - T_c)^Z]$. La théorie dépend de 3 exposants critiques. Le nouvel exposant correspond, comme l'a remarqué Krey, à la renormalisation du champ aléatoire. Les exposants fondamentaux doivent vérifier certaines inégalités. On indique les formules qui donnent les autres exposants. On critique les idées habituelles comme la réduction dimensionnelle. L'exposant η' correspond, si notre description est correcte, à des fluctuations thermiques (inobservables !) entre des puits éloignés, un phénomène particulier à ce type de problème, qui est à notre avis incompatible avec la réduction dimensionnelle.

Abstract. — In order to account for experimentally observed qualitative properties of random field Ising systems, a new picture of transitions in systems with frozen disorder is suggested. According to this picture, equilibration processes near T_c require activated jumps between remote free energy wells in the phase space. These jumps involve very slow dynamics, described by a modified Vogel-Fulcher law for the relaxation time $\tau \sim \exp[\mathrm{Const.}/(T - T_c)^Z]$. The theory depends upon 3 critical exponents. The new exponent corresponds, as remarked by Krey, to random field renormalization. The inequalities satisfied by the exponents are investigated, as well as the equalities which give the other exponents. Classical concepts, such as dimensional reduction, are criticized. The exponent η' corresponds, if our picture is correct, to thermal fluctuations between remote wells, a novel effect which seems to be incompatible with dimensional reduction.

1. Introduction.

It is generally believed [1-4] that the long-range order of the ferromagnetic or antiferromagnetic Ising model is not destroyed by weak random, frozen fields at low temperature if the space dimension D is larger than 2.

However, the behaviour at the transition at thermal equilibrium is not well understood. Metastability effects hinder experimental investigation, computer simulations and also theoretical research, as will be seen.

There are arguments [5-7] which indicate that the critical exponents are given by the same ε-expansions in D dimensions as for the zero field Ising model in $(D - 2)$ dimensions. However, the lower critical dimensions with and without random field are 2 and 1, respectively. The difference is 1, not 2. The reason may be that the ε-expansion is valid only, down to some particular dimension, or that some non-analytic terms should be added to the exponents, or that the expansion is wrong (see Sect. 4.1).

In order to obtain the correct lower critical dimensions, it has been suggested that the so-called « dimensional reduction » occurs, i.e., the critical exponents are the same with a random field in D dimensions as without field in d dimensions, where the relation $d = D - 2$ is replaced, according to Aharony *et al.* [5] and Schwartz [8] by

$$d = D - 2 + \eta(d) \tag{1.1}$$

and, according to Shapir [9] by

$$d = (D + 1/\nu)/2 \quad \text{if} \quad \alpha(d) \leqq 0 \tag{1.2a}$$

$$d = D - 1/\nu \quad \text{if} \quad \alpha \geqq 0\,. \tag{1.2b}$$

On the other hand, high temperature expansions [10] yield a critical exponent $\gamma \simeq 1.4$ (1) and Young [11] has suggested the transition is first order.

(1) The prime indicates critical exponents of the random field Ising model. Exponents without prime correspond to zero field.

The present paper is intended to contribute to the confusion which normally precedes the clarification of difficult problems. We propose new ideas, the main consequence of which is a modified Vogel-Fulcher law for the relaxation time of physical quantities near the transition. The modified law involves a new critical exponent which is related, as noticed by Krey [12], to the random field renormalization.

The basic hypotheses will be stated without justifications in section 2, and the consequences will be derived in section 3. The hypotheses made in section 2 will be justified (as well as possible) in section 4. This unnatural procedure is intended to make reading easier, since the reader may be more motivated to learn the argument once he knows the conclusion.

2. Basic assumptions.

At low temperatures, the ferromagnetic Ising model in a weak random field has many metastable states which may be described [13] in terms of domains. However, only the two ferromagnetically ordered states, with respectively + and − magnetization, have an appreciable Boltzmann factor for $D > 2$. In fact, one may even be tempted to consider only one state, since the energy difference between both states is of order $HL^{D/2}$ and goes to infinity with the size L of the sample. However, it *is* necessary to consider both + and − states because the energy difference may be overcompensated by a weak uniform field, or by an exchange field if the sample is « glued » together with another one. Our first assumption is that the situation is similar at T_c.

Assumption A. At T_c, at thermal equilibrium, a given sample of a random field Ising model may have either one of two domain configurations (or « states »), which have respectively positive and negative order parameter. Other possible metastable states have negligible Boltzmann factor for a big sample. The system is normally in its state of lower free energy, but the other state may become accessible when the sample is glued together with other ones and becomes a member of a bigger system.

Assumption B. Just above T_c, a random field Ising model is made of blocks of size

$$\xi \approx \left(\frac{T - T_c}{T_c}\right)^{-\nu'} \tag{2.1}$$

which have one of the two structures (+ or −) relevant at T_c. The decomposition into blocks is unique for a given sample.

Assumption C. When T goes to T_c, ξ increases by uncorrelated flipping of these blocks. At a given temperature T, only blocks of size given by (2.1) are allowed to flip.

Assumption D. When flipping, a block of size R must jump over a free energy barrier of height

$$\omega(R) \approx R^{D/2}\, H(R) \tag{2.2}$$

where

$$H(R) \approx HR^{-\sigma} \tag{2.3}$$

and σ is an exponent.

Assumption E. The free energy difference between the + and − states of a block of size R is normally also of order (2.2) in the critical region (but, of course, vanishes at the temperature T at which the block flips).

Assumption A has the merit of simplicity, and therefore it is the hypothesis which should be considered first. Assumptions B and C are natural scaling hypotheses. Assumptions D and E are extensions of low temperature properties [1, 13], completed by the natural scaling assumption (2.3).

The concept of « blocks » and « domains », which was introduced in this section, is different. In the critical region, domains have a fairly well-defined size ξ_0, while the block size ξ depends on T. Domains have a fractal structure (tentatively described in section 6, Fig. 2), while blocks are compact objects. Domains can in principle be seen by an appropriate instrument, if their size is large enough. Blocks are not geometrically defined, and cannot be seen, they are just the volume units which become unstable at a given temperature. They contain a large number interpenetrating domain tubes, so that their average magnetization $\xi^{-D} M_R$ (see formula (3.4) below) is much less than the domain magnetization m_0.

More detailed explanations and justifications will be given in sections 4 and 6.

3. Critical behaviour of the random field Ising model at equilibrium.

3.1 Relaxation time above T_c. — At a given temperature $T > T_c$, the relaxation time of most of the characteristic quantities is the time necessary to flip the blocks of size ξ mentioned in assumption B. According to (2.2) and (2.3), this implies jumping over a free energy barrier of height

$$\omega(\xi) \approx H\xi^{\frac{D}{2}-\sigma}.$$

According to the Arrhenius law, the relaxation time is

$$\tau = \tau_0 \exp\left[\frac{K}{(T - T_c)^{(D/2-\sigma)\nu'}}\right] \tag{3.1}$$

where the constants K and τ_0 depend on H. Equation (3.1) is a generalized Vogel-Fulcher law. The ordinary Vogel-Fulcher law [14-21] corresponds to

$$(D/2 - \sigma)\, \nu' = 1\,. \tag{3.2}$$

3.2 ORDER-PARAMETER SUSCEPTIBILITY. — The idea is that some regions, of size about ξ given by (2.1), have almost the same free energy in the + and in the − state, and can be flipped by a very weak field h. In the ferromagnetic case ($J > 0$), h is a uniform field, while in the antiferromagnetic case h is a staggered field. The calculation requires the introduction of an exponent ρ which describes the correlations of the order parameters $m(\mathbf{r})$ at T_c

$$\overline{\langle m(0) \rangle \langle m(\mathbf{r}) \rangle} \sim m_0^2 \, r^{-(D-2+\rho)} . \qquad (3.3)$$

The brackets denote thermal averages and the bar denotes an average over the random field. The average total magnetization (or staggered magnetization) M_R in a volume of linear size R is given by

$$M_R^2 \approx m_0^2 \, R^D \int_0^R \mathrm{d}r \, r^{2-\rho-1} \approx m_0^2 \, R^{2-\rho+D} \qquad (3.4)$$

and the corresponding Zeeman energy in a field h is

$$\delta W \approx h M_R \approx h m_0 \, R^{1+(D-\rho)/2} . \qquad (3.5)$$

Assumption E of section 2 yields an evaluation of probability, for a given block of size ξ, that the energy difference, between its + and − states, is less than δW. This probability should approximately be

$$P \approx \delta W / \omega(\xi) \approx (\delta W / H) \, \xi^{\sigma - D/2}$$

or, inserting (3.5) with $R = \xi$,

$$P \approx (h/H) \, m_0 \, \xi^{1+\sigma-\rho/2} .$$

The magnetization m in a field h is obtained by multiplying by M_ξ / ξ^D, where M_ξ is given by (3.5). One obtains

$$m \approx m_0^2 (h/H) \, \xi^{2-\rho+\sigma-D/2}$$

or

$$\chi \equiv m/h \sim (T - T_c)^{-\gamma'}$$

with

$$\gamma' = (2 + \sigma - \rho - D/2) \, \nu' . \qquad (3.6)$$

The divergence of the susceptibility is probably very difficult to observe, since the field produces a complete reorganization of the domain wall array. In other words, the system should go from one « valley » of the phase space to another one which is very distant near T_c. This requires a huge time, given by (3.1).

3.3 THERMAL FLUCTUATIONS. — Strictly speaking, the thermal fluctuations $\delta m(\mathbf{r}) = m(\mathbf{r}) - \langle m(\mathbf{r}) \rangle$ should satisfy the fluctuation-dissipation theorem :

$$\chi = \beta \iint \mathrm{d}^D r \, \overline{\langle \delta m(0) \, \delta m(\mathbf{r}) \rangle} . \qquad (3.7)$$

Introducing the critical exponent η' defined by

$$\overline{\langle \delta m(0) \, \delta m(\mathbf{r}) \rangle} \sim 1/r^{D-2+\eta'} \, (T = T_c) , \qquad (3.8)$$

relation (3.7) yields $\gamma' = (2 - \eta) \, \nu'$, hence, according to (3.6),

$$\eta' = \rho - \sigma + D/2 . \qquad (3.9)$$

However, the « thermal fluctuations » which appear in (3.7) and (3.8) are not of the usual type. The divergence of (3.8) indicates that the system hesitates between various « valleys » of the phase space which, again, are very distant. Strictly speaking, the system does fluctuate between those valleys, but the life time of each valley is so long that we do not see any way to check the divergence of (3.8). Neutron scattering can see no divergence because, near T_c, a neutron has enough time to cross the sample many times before the system moves to another valley. Of course, neutron scattering should be sensitive to the divergence of (3.3), provided the system is at equilibrium.

In our picture, thermal fluctuations within a valley are *not* divergent at T_c.

3.4 SPECIFIC HEAT. — As usual, it will be assumed that all typical free energies near T_c have the same order. Then, the singular par $N f_{\text{sing}}$ of the free energy (where N is the number of sites) is obtained, in order of magnitude, by multiplying $\omega(\xi)$ (given by (2.2)) by the number N/ξ^D of the blocks considered in assumption B of section 2. The singular part of the free energy per site is, according to assumption E,

$$f_{\text{sing}} \approx H \xi^{-\sigma-D/2} .$$

The singular part of the specific heat

$$C = \frac{\mathrm{d}}{\mathrm{d}T} \frac{\mathrm{d}}{\mathrm{d}(1/T)} f_{\text{sing}} \sim (T - T_c)^{-\alpha'}$$

corresponds to a critical exponent

$$\alpha' = 2 - \left(\frac{D}{2} + \sigma \right) \nu' . \qquad (3.10)$$

3.5 MAGNETIZATION IN EXTERNAL FIELD AT T_c. — It is now assumed that the temperature is $T = T_c$, and that the field h associated to the order parameter is not zero. As in subsection 3.2, the field is able to reverse blocks of sufficiently large size R. This can occur if the energy gain $h M_R$ given by (3.4) is larger than the typical energy difference (2.2). Thus, the reversed blocks have a size

$$R_h \approx (H/h m_0)^{1+\sigma-\rho/2} .$$

The corresponding magnetization per site is $m \approx M_{R_h} R_h^{-D}$, or, according to (3.4),

$$m \approx m_0 \, R_h^{1-\frac{D}{2}-\frac{\rho}{2}} \sim h^{1/\delta'}$$

where

$$\delta' = \frac{2 + 2\sigma - \rho}{D - 2 + \rho} . \qquad (3.11)$$

3.6 REMANENT MAGNETIZATION OF A RANDOM FIELD ISING ANTIFERROMAGNET AT T_c. — If a random field Ising antiferromagnet is cooled down to T_c in a uniform external field, it is not in one of the two states allowed at equilibrium when the field is switched off. However, after a waiting time t, the system should be made of blocks, each of which is at equilibrium. The average block size $R(t)$ is such that the typical free energy barrier (2.2) can be jumped over in a time t. The Arrhenius law yields

$$t \approx \tau_0 \exp[\beta H R^{D/2-\sigma}(t)]$$

or

$$R(t) \approx \left(\frac{T}{H} \ln \frac{t}{\tau_0}\right)^{2/(D-2\sigma)} \tag{3.12}$$

What kind of function of R should the remanent magnetization be ? This is not quite clear, but we make the following speculation. At the beginning, blocks of size ξ_0 flip, where ξ_0 is an elementary length to be defined in the next section. Then, blocks of size $2\,\xi_0$ flip, then blocks of size $4\,\xi_0$, etc., so that after a time t, each spin has flipped a number of times proportional to $nR(t)$. It is reasonable to believe that each flip reduces the magnetization by a constant factor, so that the remanent magnetization μ is an exponential of $\ln R(t)$, i.e. a power of $R(t)$. From (3.12) one deduces

$$\mu(t) \sim (\ln t)^{-\Gamma} \tag{3.13}$$

where Γ is some constant. For $T > T_c$, the remanent magnetization is given by (3.13) for short times, then crosses over to an exponential behaviour with a relaxation time given by (3.1).

3.7 INEQUALITIES. — There is no obvious equality relating ν', ρ, σ, but certain inequalities should be fulfilled. Schwartz and Soffer [22] derived the following exact inequality

$$\partial \langle m_q \rangle / \partial h_q \leqq H^{-1} \sqrt{\overline{|\langle m_q \rangle_h|^2}}\,.$$

In terms of the exponents η' and ρ defined by (2.7) and (2.12), this inequality implies

$$1/q^{2-\eta'} \leqq \sqrt{1/q^{2-\rho}}$$

or $\eta' > 1 + \rho/2$. According to (3.9), this implies

$$\rho - 2\,\sigma + D \geqq 2\,. \tag{3.14}$$

The behaviour of energy differences at T_c is described by (2.2) and (2.3). This behaviour is expected to be intermediate between low temperatures ($\sigma = 0$) and high temperatures ($\sigma = D/2$). Thus, we assume

$$0 < \sigma < D/2\,. \tag{3.15}$$

In relation (3.3), the exponent $(D - 2 + \rho)$ should obviously be positive, as well as $(D - 2 + \eta')$ in (3.8). On the other hand, the Fourier transforms

$$\overline{|\langle m_q \rangle|^2} \sim 1/q^{2-\rho}\,, \quad \overline{\langle |\delta m_q|^2 \rangle} \sim 1/q^{2-\eta'}$$

are normally expected to diverge at $q = 0$. This implies

$$\begin{aligned} 2 - D < \rho < 2 \\ 2 - D < \eta' < 2 \end{aligned} \tag{3.16}$$

Using (3.9), the latter relation reads

$$2 - D < \rho - \sigma + D/2 < 2\,. \tag{3.17}$$

The exponent (3.11) should be positive. Combining with (3.14), one finds

$$2 - D < \rho - 2\,\sigma < 2\,. \tag{3.18}$$

The law (3.12) is to be compared with that obtained below T_c [13], namely

$$R(t) \approx (gT/H^2) \ln t\,. \tag{3.19}$$

This relation implies that, for a given, long time t, $R(t)$ decreases when approaching T_c from below, because g decreases (as seen below, formula (4.12)). Equation (3.12) is then expected to be smaller than (3.19) for any long time. This implies

$$2 - D + 2\,\sigma \leqq 0\,. \tag{3.20}$$

In 3 dimensions, the point (ρ, σ) should lie within a quadrangle (Fig. 1). In two dimensions, conditions (3.20) and (3.15) imply $\sigma = 0$.

3.8 CRITICAL PROPERTIES BELOW T_c AND ORDER PARAMETER. — The situation below T_c is similar. Note that ξ represents the order of magnitude of the domain size *and* of the distance between domains (see section 6 and Fig. 3). The exponents γ', α', ρ, η' which describe the susceptibility are easily seen to be the same above and below T_c.

The zero-field value of the order parameter, m, is easily found from (3.3), which should hold for $r \lesssim \xi$, while the left-hand side is equal to m^2 for

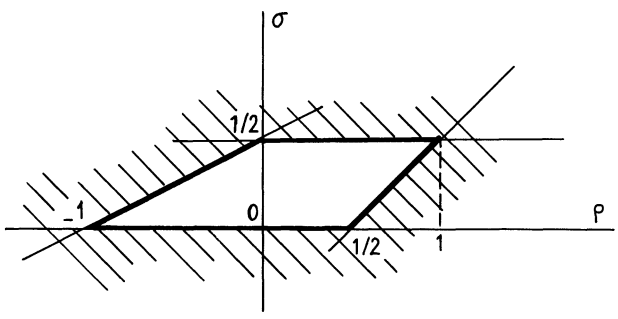

Fig. 1. — Domain of the (σ, ρ) plane allowed by the inequalities of subsection 3.7 in 3 dimensions.

$r \gtrsim \xi$. The matching condition is

$$m \sim m_0\, \xi^{-(D+\rho-2)/2} \sim (T_c - T)^{\beta'}$$
$$\beta' = \left(\frac{D+\rho}{2} - 1\right) \nu' . \tag{3.21}$$

Relations (3.21), (3.10) and (3.16) yield the usual relation

$$\alpha' + 2\,\beta' + \gamma' = 2 . \tag{3.22}$$

3.9 Conclusion of this section. — The picture presented here is an alternative to the suggestion of a first-order transition made by Young [11] and by Andelman [23]. The dynamics implied by domain reorganization is slow enough to explain the hysteresis observed experimentally and in computer simulations, and referred to in the next section.

The present description depends on 3 critical exponents ν', ρ, σ instead of 2. This is in agreement with the well-known fact that the random-field Ising model violates hyperscaling (i.e., the scaling relations which contain the dimension D). Introducing the parameter $d = \sigma - D/2$, relations (3.10) and (3.11) read

$$\alpha' = 2 - d\nu' \tag{3.23}$$
$$\delta' = (2 + d - \eta')/(2 - d + \eta') \tag{3.24}$$

and this is not in contradiction with the « dimensional reduction » hypothesis, according to which the new critical exponent is defined by $\eta' = \eta(d)$, the usual critical exponent in d dimension. However, this idea may be misleading. For instance, the dynamical behaviour described by (3.1) is unusual.

According to Belanger *et al.* [24], the measured critical exponents for $D = 3$ are $\alpha' = 0$ and $\nu' = 1$. Relation (3.10) then implies

$$\sigma = 1/2$$

and

$$0 < \rho < 1$$

i.e. the upper edge of the quadrangle (Fig. 1) is allowed. Insertion of these values into (3.1) yield a proper Vogel-Fulcher law

$$\tau \approx \tau_0 \exp[K/(T - T_c)] .$$

On the other hand, according to Schwartz [8] the system should be represented by a point on the left-hand edge of the quadrangle (Fig. 1).

4. Discussion.

The results of section 3 are based on the assumptions made in section 2, which will now be discussed.

4.1 General considerations (weak random field case). — The program one would like to carry out is the determination of all the minima α of the Landau free energy functional F with their Boltzmann weight $e^{-\beta W} \Big/ \left(\sum_\alpha e^{-\beta W}\right)$. F is given by

$$F = \int d^D r \left[\frac{1}{2} A m^2 + \frac{1}{4} B m^4 + \frac{1}{2} J_R (\nabla m)^2 - H(\mathbf{r})\, m\right] \tag{4.1}$$

where $m(\mathbf{r})$ is the magnetization at site $\mathbf{r}$, B, J_R and $(-A)$ are positive coefficients.

The minima of (4.1) are given by the Euler-Lagrange equation

$$A m(\mathbf{r}) + B m^3(\mathbf{r}) - J_R\, \nabla^2 m(\mathbf{r}) = h(\mathbf{r}) . \tag{4.2}$$

As usual [6, 7] we do not worry about the thermal fluctuations around the solutions of (4.2), which are believed to be less important than fluctuations induced by random fields. In the perturbation expansion, this implies [6, 7] that only tree-diagrams are retained. Although this is a kind of mean-field approximation, A, B and J_R are expressed by the correct scaling relations (derived below) appropriate for the zero field system, provided $m(\mathbf{r})$ has non-vanishing Fourier components m_q only if $q \lesssim 1/\xi_0$. Here,

$$\xi_0 \approx \left(\frac{T_{c0} - T_c}{T_{c0}}\right)^{-\nu} \tag{4.3}$$

is the correlation length of the zero-field system at the transition temperature $T_c(H)$ of the random field system, while $T_{c0} = T_c(H = 0)$. Another quantity of interest is the spontaneous magnetization m_0 of the zero field at T_c,

$$m_0 \approx \left(\frac{T_{c0} - T_c}{T_{c0}}\right)^{\beta} \approx \xi_0^{-(D+\eta-2)/2} \tag{4.4}$$

as well as its differential susceptibility

$$\chi = 1/|A| \approx J\left(\frac{T_{c0} - T_c}{T_{c0}}\right)^{-\gamma} \approx J\xi_0^{2-\eta} \tag{4.5}$$

m_0, ξ_0, the domain wall thickness λ and the domain wall surface tension g are related to A, B, J_R by the mean field relations

$$m_0 = \sqrt{|A|/B} \tag{4.6}$$
$$\lambda \approx \xi_0 = \sqrt{J_R/2\,|A|} \tag{4.7}$$
$$g \approx (|A|/B)\sqrt{J_R\,|A|} . \tag{4.8}$$

Relations (4.3) to (4.7), together with the scaling relations $D\nu = 2 - \alpha = 2\,\beta + \gamma$ and $\gamma/\nu = 2 - \eta$, imply for small H,

$$|A| \approx J/\xi_0^{2-\eta} \tag{4.9}$$
$$B \approx J/\xi_0^{4-D-2\eta} \tag{4.10}$$
$$J_R \approx J\xi_0^{\eta} \tag{4.11}$$

and therefore, according to (4.8),

$$g \approx J/\xi_0^{D-1}. \tag{4.12}$$

Consistently with the assumption of an upper limit $q \approx 1/\xi_0$, the following condition should be imposed to $h(\mathbf{r})$ in (4.1)

$$\overline{h(\mathbf{r})\, h(\mathbf{r} + \mathbf{r}')} \approx H^2\, \xi_0^{-D/2} \exp(-\, r/\xi_0). \tag{4.13}$$

Finally, the relation between ξ_0 and H [25, 13] is obtained if one writes that for a « domain » of size ξ_0, the surface energy $g\xi_0^{D-1} \approx J$ has the same magnitude as the volume random field energy $Hm_0\, \xi_0^{D/2}$

$$\xi_0^{1-\eta/2} \approx J/H. \tag{4.14}$$

A more detailed derivation of this relation will be found in reference [13].

4.2 METASTABILITY. — The concept of domain is questionable at T_c, since the domain size R_0 and the domain wall thickness λ are of the same order of magnitude. However, this does not exclude the possibility of having $R_0/\lambda \approx 5$ or 10, etc. The concept of domain is useful because it provides an explanation of metastability [13], and metastability does exist at T_c. This results from the experimental observation [26-29] confirmed by computer simulations, [30-33] that three-dimensional Ising ferromagnets do not order when cooled down in constant random field. A possibility would be of course that the equilibrium state is not ordered, but this would be in contradiction with the present theoretical understanding [1-4], which is in agreement with the experimental fact that zero-field cooled samples remain ordered when a random field is switched on [27].

Mathematically, the existence of metastable states means that equation (4.2) may have several solutions for $A < 0$. This is readily seen in the non-interacting case $J_R = 0$, where at each site $\mathbf{r}$ the magnetization m satisfies

$$Am + Bm^3 = h. \tag{4.15}$$

This equation has 3 solutions for $A < 0$ and $|h| < |2A/3|\sqrt{|A|/3B}$. In the case of an infinite system, equation (4.2) is believed to have an infinite number of solutions (« metastable states ») at low temperatures, corresponding to bumps on the domain walls [13]. It is reasonable to assume that this is also true at T_c, since A is negative.

Now, the assignment of the correct Boltzmann factor to all solutions of (4.15) is a difficult, and unsolved, problem. If there is a single solution, one can eliminate the field $h(r)$, and express the correlation functions as functional integrals over $m(r)$ as shown by Parisi and Sourlas [7]. The resulting equation (N° 4 in Ref. [7]) is wrong for $A < 0$ [34]. This can easily be seen in the simple case of equation (4.15). One can easily show (for $A > 0$) that

$$\overline{m^2} = \int_{-\infty}^{\infty} \mathrm{d}\mu\, P(h(\mu))\, \mu^2(A + 3\, B\mu^2) \tag{4.16}$$

where $h(\mu) = A\mu + B\mu^3$ and $P(h)$ is the probability density. Equation (4.16) is a special case of formula (4) of reference [7], which is the basis of the ε-expansion. Equation (4.16) is exact for $A > 0$, and wrong for $A < 0$. Indeed, the 3 solutions of (4.15) should be given the respective weight 1, -1 and 1 in order to obtain (4.16). A weight cannot be negative. All three weights might be made equal to $+1$ by replacing the last factor of (4.16) by its absolute value, but it is no good solution either because the sum of the weights should be 1, not 3 !

Since the correct weighting of all solutions appears to be so difficult, the simplest attitude is therefore to *assume* that only a small number of states are important, and this is the assumption A of section 2. This assumption, as well as the next ones, is probably too clear cut. For instance, more than two states may be available, but it is important that their energy is, so to speak, quantized, and that they are far away in the phase space.

The alternative to assumptions B and C would be the usual conception of critical phenomena : weak local fluctuations with long range correlations. In this case, one might even assume that one state only (with essentially zero order parameter) is important at T_c. However, the assumption of weak local correlation is inconsistent with the experimental fact that the equilibrium, ordered state is not reached when cooling in constant random field. This effect can only be understood if the system has to go a long way in the phase space to equilibrate. This occurs in a first order transition, but also in the type of second order transition described here.

Assumptions D and E are natural consequences of scaling ideas. As noticed by Krey [12], random field renormalization can roughly be described as an effect of small domains within big domains : small domains appear at places where random field fluctuations are strong. Thus, domains erase the strongest fluctuations.

4.3 UPPER CRITICAL DIMENSION. — It is of interest to know for which space dimensions D the present picture is acceptable. In this subsection we assume $D > 4$, so that the linear response to random field is

$$m \sim \xi_0^2\, H_{\mathrm{eff}}/J$$

where ξ_0 is the correlation length of the zero-field model and $H_{\mathrm{eff}} \approx H\xi_0^{-D/2}$. On the other hand the non-linear response at T_c is $m \approx (H_{\mathrm{eff}}/J)^{1/3}$. Linear response is acceptable if

$$\xi_0^{-2} \gtrsim (H_{\mathrm{eff}}/J)^{2/3} \approx \left(\frac{H}{J}\right)^{2/3} \xi_0^{-D/3}$$

or

$$H/J \lesssim \xi_0^{(D/2-3)}. \tag{4.17}$$

For $D = 4$, this condition crosses over to (4.14). For $D < 6$, condition (4.17) is always satisfied near T_c, linear response applies and assumptions of section 2 are not acceptable.

5. A renormalization group approach.

In this section, the transition of the random field Ising model is interpreted as due to a vanishing surface tension of domain walls. An approximate recursion formula for the size-dependent surface tension $g(R)$ is derived. A single recursion formula is of course not sufficient to find the 3 exponents ν', ρ, σ, but the present treatment sheds some light on the mechanism of a possible second order transition.

A domain wall tends to decrease its free energy W by forming bumps [4]. The typical energy gain for a bump of radius r and height ζ

$$\delta W \approx -\tilde{H} r^{D/2} \zeta^{1/2}$$

where $\tilde{H} = H(R)\, m_0$. The energy loss due to the surface tension g is

$$\delta W_2 \approx g r^{D-1} (\zeta/r)^2 .$$

Minimization of $(\delta W_1 + \delta W_2)$ with respect to ζ yields [13]

$$\zeta \approx (\tilde{H}/g)^{2/3}\, r^{(5-D)/3} \tag{5.1}$$

and the resulting energy gain per unit area is

$$\delta W/r^{D-1} \approx -\tilde{H}(\tilde{H}/g)^{1/3}\, r^{-2(D-2)/3} . \tag{5.2}$$

This should be considered as a modification g of the surface tension, corresponding to the elimination of degrees of freedom of wavelength about r (say, between r and $2\,r$). In differential form one finds

$$\mathrm{d}g = -K^{4/3}\, \tilde{H}(\tilde{H}/g)^{1/3}\, r^{-2(D-2)/3}\, \mathrm{d}r/r \tag{5.3}$$

where K is a constant.

This is the required recursion formula. Neglecting the renormalization of H, integration yields

$$g^{3/4}(r) - (K\tilde{H})^{4/3}\, r^{-2(D-2)/3} = f(H, T) . \tag{5.4}$$

For a given value of r, the left-hand side is a local quantity which should have no singularity at T_c. Thus, $f(H, T)$ is analytic in T. On the other hand, T_c should be defined by $f(H, T_c) = 0$, since in this case (5.4) yields the scale-invariant behaviour

$$g(r) \approx K\tilde{H}/r^{(D-2)/2} . \tag{5.5}$$

Therefore

$$f(H, T) \sim T_c - T .$$

If T is slightly different from T_c, (5.5) is satisfied for r smaller than some value ξ, while for larger values g is constant (below T_c) or meaningless (above T_c). Equation (5.4) is therefore consistent with a continuous transition.

According to (5.4), ξ should satisfy the condition

$$\xi \approx (K\tilde{H})^{2/(D-2)}\, |\, f(H, T)\, |^{3/2(D-2)}$$

and insertion of (5.5) yields

$$\nu' = \frac{3}{2D-4} . \tag{5.6}$$

This value is probably not correct since σ was assumed to vanish.

Insertion of (5.5) into (5.1) yields $\zeta \sim r$. This means that a domain wall at T_c is fractal. The well-known Koch curve gives a two-dimensional idea of what it may look like.

6. Fractal aspects.

It may be helpful to show an explicit picture of the domain structure, although no new result will be deduced. Figure 2 is intended to give an idea of what the domain structure may look like at T_c. It may represent a section of a 3-dimensional random field Ising model. It can also correspond to a two-dimensional Ising model with « anticorrelated » random frozen fields, such that

$$\int_{r<R} \langle H(\mathbf{r}')\, H(\mathbf{r} + \mathbf{r}') \rangle\, \mathrm{d}^2 r \sim R^{2-\mu}$$

with $\mu > 0$.

The fractal nature of domain walls at T_c has not been represented. Moreover, domain wall intersections are supposed to be forbidden. This property has no reason to be satisfied by the random field Ising model, but makes the hierarchy of domains more intuitive. Each domain is within another bigger domain. The recipe according to which figure 2 was fabricated is not interesting enough to be revealed here. It is convenient to classify domain sizes as being of order of magnitude ξ_0, $Q\xi_0$, $Q^2\, \xi_0, \ldots, Q^l\, \xi_0, \ldots$, where Q is an appropriate number ($Q = 5$ in the case of Fig. 1).

Let Γ be a given domain of size $R = Q\xi_0$. On the average, a proportion α of its volume R^D is occupied by domains of size $Q^{l-1}\, \xi_0$ (in Fig. 2, α lies between 4/25 and 1/5). A proportion α of the remaining volume is occupied by domains of size $Q^{l-2}\, \xi_0$, etc. After subtraction of all these domains of the « first generation », the total remaining volume is a fraction $(1-\alpha)^l$ of the initial volume R^D and has magnetization (or staggered magnetization) m_0. Similarly, after domains of the second generation have been subtracted, the remaining volume of the domains of the first generation is $l\alpha(1-\alpha)^{l-1}$ and has magnetization $-m_0$.

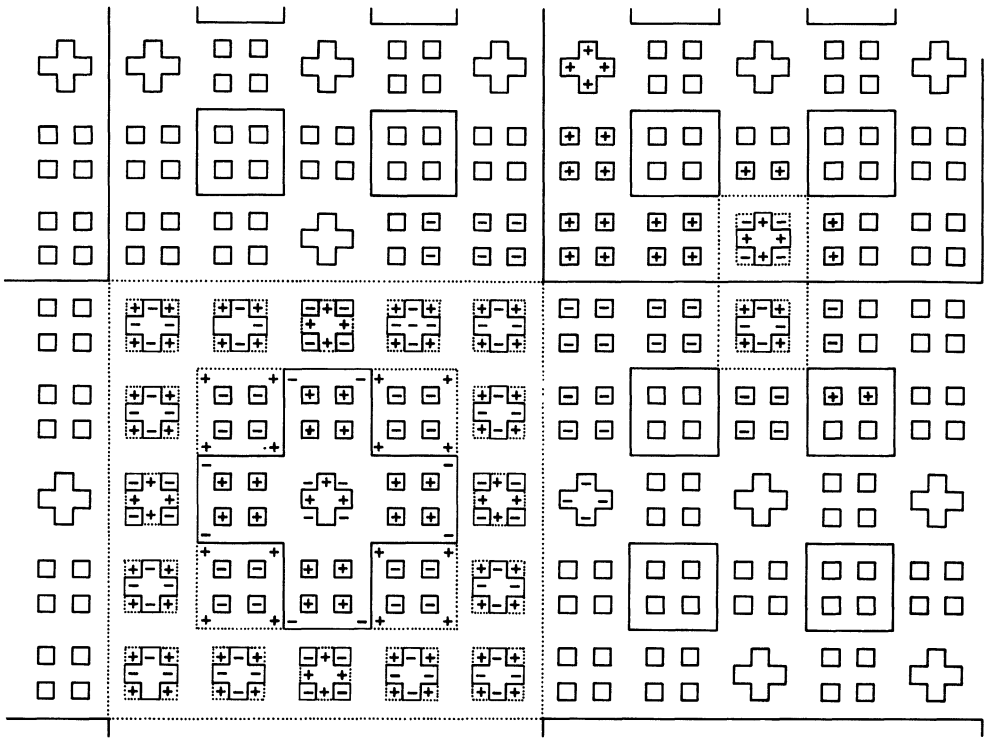

Fig. 2. — An example of what the domain structure at T_c may be. Full lines represent domain walls. Dotted lines are new domain walls which may appear slightly above T_c, implying the disappearance of some inner walls (thinner lines).

Finally, the total value of the order parameter is

$$M_R = m_0[(1-\alpha)^l - \alpha(1-\alpha)^{l-1} + \cdots + C_l^m(-\alpha)^m(1-\alpha)^{l-m} + \cdots + \alpha^l] = m_0(1-2\alpha)^l .$$

Comparison with (3.4) shows that the critical exponent ρ is related to the purely geometrical quantities α, Q by the relation

$$\rho = 2 + D - 2\,\frac{\ln(1-2\alpha)}{nQ}.$$

Similarly, the exponent δ might be calculated from the assumption that each generation of inner domains of size $Q^m\,\xi_0$ cut random field fluctuations by a certain factor. This factor is a *dynamic* property which we do not believe to be related to geometrical ones.

Below T_c, the situation is similar, but the number of domain walls decreases instead of increasing (Fig. 3).

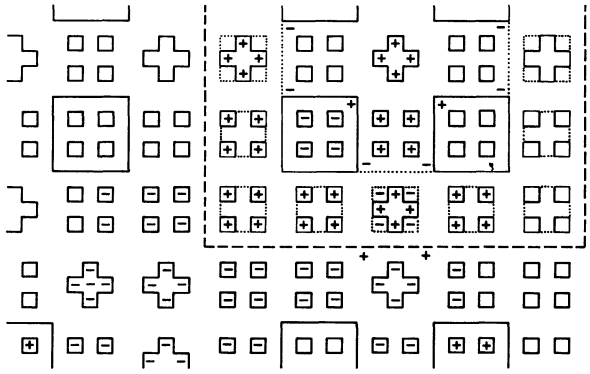

Fig. 3. — The upper right-hand side of figure 2. Below T_c, the block limited by the dashed line flips. This implies the appearance of new walls (dotted lines) and disappearance of certain inner walls (thin lines).

7. Conclusion.

We have described a speculative model of a phase transition in disordered systems, which proceeds through a complete reorganization of the structure, characterized here by the domain wall array. Our description is phenomenological. It has been preferred to the standard description of critical phenomena (which involves weak fluctuations) because, in the random field Ising model, this description is not consistent with the experimentally observed metastability. The reorganization of the domain structure is a slow process which can account for this effect. Another possible scheme is a first-order transition [11, 22] but we wished to point out here that it is not the only possibility. The Vogel-Fulcher law (3.1) implies a very steep increase of the relaxation time which may be difficult to distinguish from a first-order transition.

Since the equilibration of the system is so slow, some of the properties are just unobservable. For instance, we do not believe that the exponent η' can be experimentally determined. Exponents ν' and ρ should be measurable by elastic neutron scattering. σ might be obtained from the relaxation time (3.1), or from d.c. susceptibility measurements in the case of a ferromagnet.

Belanger *et al.* [24] have published a set of critical exponents which are in agreement with a theory of Shapir [9], but which violate an exact inequality [22]. Thus, their result cannot describe the critical behaviour, but might be appropriate for some intermediate range. On the other hand, their values $\alpha = 0$ and $\nu' = 1$ are consistent with our figure 1. Insertion into (3.13) yields a logarithmic increase of the domain size with time. This law, already predicted at low temperatures [13, 35], failed to be observed by Cowley *et al.* [27] at a temperature which they believe to be below T_c, but agrees with observations of Belanger *et al.* [29]. The question is certainly not settled, and even theoretically the derivation of (3.13) is too crude.

Finally, we wish to say a few words about dimensional reduction. The parameters ρ and δ may be replaced by 2 parameters η' and d, and then one obtains relations (3.23) and (3.24), which look like hyperscaling. This is however not sufficient to prove that η' is the exponent corresponding to the d-dimensional, zero-field Ising model. If our picture is correct, η' is related to jumps from a « valley » of the phase space to a remote one, and this mechanism has no counterpart in the zero-field Ising model. Thus, we are rather inclined to follow Krey's idea [12] that there are just 3 independent exponents ν', ρ, σ (or ν', η', d) not related by any straightforward relation. On the other hand, in contrast with Krey, we believe that the lower critical dimension is 2, and that the exponents have to comply with this requirement (Fig. 1b).

Note added in proof. — Three recent preprints by Bray and Moore (BM), D. Fisher (F) and Nattermann (N) should be compared with this work. BM and F find the same relations (3.22), (3.23), (3.24), (3.21), (3.9) which give the exponents as functions of *three* independent ones. F also finds the Vogel-Fulcher law (3.1). Nattermann gives a calculation analogous to that of our section 5, bus instead of neglecting the renormalization of $H(r)$ in (5.3), he assumes $H(r) \sim HM_r/r^D$, where M_r is given by (3.4). This assumption implies $2\,\sigma = D - 2 + \rho$. Then, N finds $\nu' = 4(D - 2 + \rho/2)/3$, which reduces to (5.6) if $\rho = 0$. BM present an expansion in powers of $\varepsilon = D - 2$, according to which $\sigma = 0$. However, they do not obtain our formula (5.6) because they do not use formula (5.2) valid for broad domain walls, but the analogous formula for narrow domain walls.

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