Is the elastic energy of amorphous materials rotationally invariant?
S. Alexander

To cite this version:
S. Alexander. Is the elastic energy of amorphous materials rotationally invariant ?. Journal de Physique, 1984, 45 (12), pp.1939-1945. <10.1051/jphys:0198400450120193900>. <jpa-00209936>

HAL Id: jpa-00209936
https://hal.archives-ouvertes.fr/jpa-00209936
Submitted on 1 Jan 1984

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Is the elastic energy of amorphous materials rotationally invariant?

S. Alexander

The Racah Institute of Physics (*), The Hebrew University, Jerusalem, Israel and Ecole Supérieure de Physique et de Chimie Industrielles, Paris, France

(Reçu le 23 mai 1984, accepté le 24 août 1984)

Abstract. — It is shown that scalar, stress induced contributions to the elastic energy, which are not explicitly rotationally invariant can play an important role in determining the elastic properties of tenuous structures. Recent scaling results are used to show that the scalar contributions tend to be dominant at criticality. It is argued that liquid-like hydrostatic stresses are large and will tend to lead to dominant scalar elasticity in many situations as they are known to do for rubbers. Some new results on the elastic properties of a Sierpinski gasket are presented in an appendix.

1. Introduction.

In discussions of elasticity it is usually assumed that only the linear part of the (symmetric) strain tensor need be considered in the elastic energy. This is equivalent to the assumption that the harmonic elastic energy in terms of the particle displacements

$$E = \frac{1}{2} \sum_{i,j} u_i K_{ij} u_j$$

(1)

is explicitly invariant under rigid rotations. This is however not strictly correct. In a strained material one has an additional contribution to the same order in the $u_i$ which is scalar in the relative displacements and therefore not explicitly invariant under rigid rotations

$$E_s = \frac{1}{2} \sum_{i,j} A_{ij} (u_i - u_j)^2.$$  

(2)

This corresponds to the interaction of internal stresses with the quadratic terms in the local strain [1]

$$\varepsilon = \frac{1}{2} (\partial_i u_\beta + \partial_\beta u_i + \partial_\gamma u_\gamma \partial_\rho u_\rho)$$

(3)

where summation over repeated indices is implied. While this result goes back to Cauchy [2] it is not usually emphasized, and it is customary to make the Cauchy assumption that internal stresses can be neglected. It is of course well known that all real materials have internal stresses. Their contribution to the elastic properties is usually small and the isotropic elasticity they induce can be neglected.

The purpose of this note is to show that this assumption is not justified when one is trying to relate the elastic properties of amorphous materials to microscopic models. One actually expects the stress induced scalar elastic energy to be large and even dominant in many cases.

The fundamental reason for this difference is that the shear rigidity of amorphous materials is very often due to a tenuous, weakly bound, network which is superimposed on a « liquid » responsible for the compressibility. The entropy and interactions responsible for the mechanical properties of the liquid cannot be described by the rigid net responsible for shear rigidity. This is certainly true for polymeric materials, like rubbers and gels, but also we believe for most glasses near the glass point. One result of this is that internal strains become large. As a result the scalar force constants ($A_{ij}$ Eq. (2)) become comparable to the rotationally invariant ones — at the microscopic level. A
manifestation of this is the fact that standard theories of rubber elasticity [3] give a purely scalar elastic energy which does not even contain the standard rotationally invariant terms. A second reason, which is independent but certainly related to the above on a deeper level, involves scaling considerations. For a tenuously connected net one needs some renormalization or coarse graining procedure to connect the microscopic properties to macroscopic, large scale, averages. Recent investigations of this scaling behaviour on the Sierpinski gasket [4-5] and on percolation clusters [6-7] showed that the rotationally invariant force constants are irrelevant under scaling when scalar terms are present [8]. We shall argue that this is a much more general phenomenon and probably applies to all sufficiently tenuous structures.

This has several implications. One is that it justifies de Gennes [9, 10] conjecture that the shear modulus of gels should scale with conductivity indices. It also justifies the use of the diffusion index (θ) [11] and the fracton dimension [12] in describing the localization and the density of states of amorphous materials [12-15]. In this work [9-15] rotational invariance was ignored. The results of references [4-7] show that this is not necessarily justified. For rotationally invariant elasticity both indices would be different and the scaling behaviour of the modes seems to become more complex. There is also an important conceptual advantage because the liquid like compressibility, pressure, and entropic contributions enter in a natural way into the determination of internal stresses and therefore into the determination of the shear modulus.

In section 2 we show explicitly how the isotropic terms in the elastic energy arise from the expansion of the rotationally invariant total energy. We discuss the continuum approximation and rotational invariance. While there are no really new results in this section the emphasis is different than in standard texts on elasticity.

For rigidly bound solid materials the scalar elastic terms are usually small. In section 3 we show that they should be large for weakly bound materials when the shear modulus is much smaller than the bulk modulus. We discuss rubbers, where the scalar terms are dominant and the implications for gels and glasses.

In section 4 we discuss the scaling results of references [4-7]. We show that they imply that scalar elasticity is always dominant when renormalization is important. We argue that the results are probably applicable to all situations where coarse graining or renormalization is important. In section 5 we discuss some implications of these results. The scaling behaviour of the elastic force constants on the Sierpinski gasket is discussed in the Appendix. In particular we discuss the effect of adding a scalar term to the elastic energy.

2. The scalar elastic energy.

The mechanical energy of a solid must be invariant under rigid rotations. This is reflected in the fact that most of the terms in a standard harmonic expansion are explicitly rotationally invariant. The scalar terms in this expansion are not separately rotationally invariant. They were discovered by Cauchy [2]. These terms are usually very small and are therefore neglected. This is known as the Cauchy approximation. We want to argue that this approximation is not justified in discussing the elastic properties of tenuous, weakly bound amorphous materials. Scalar terms can be extremely important and even dominant there. Since the relevant terms are not emphasized in most texts we start with an explicit derivation. This is the purpose of this section. There are no new results. The correct reference is Cauchy [2]. Our point of view is however somewhat different.

The mechanical energy (or free energy) of a self-bound solid system can, quite generally be considered as a function of the interparticle distances (rij) [1]

\[ E = E(r_{ij}) . \] (4)

It is evident that this energy is invariant under rigid translations and rotations. Consider now the expansion around some equilibrium positions of the particles (R_i)

\[ r_i = R_i + u_i . \] (5)

To first order one has

\[ E_1 = \frac{1}{2} \sum_{i,j} \frac{\partial E}{\partial R_{ij}} (u_{ij})^2 \] (6)

where \( u_{ij} = u_i - u_j \) and \( R_{ij} = R_i - R_j \).

Expansion to second order gives an isotropic elastic energy

\[ E_2 = \frac{1}{4} \sum_{i,j} \frac{\partial E}{\partial R_{ij}} (u_{ij})^2 \frac{R_{ij}}{R_{ij}} . \] (7a)

and

\[ E_3 = -\frac{1}{4} \sum_{i,j} \frac{\partial E}{\partial R_{ij}} (u_{ij} R_{ij})^2 \frac{R_{ij}}{R_{ij}^3} . \] (7b)

One then has a fairly large number of standard elastic terms involving second derivatives. They include two body central force constants \((\partial^2 E/\partial R_{ij}^2)\) three body angular forces \((\partial^2 E/\partial R_{ij} \partial R_{jk} \partial R_{ik})\) and four body terms \((\partial^2 E/\partial R_{ij} \partial R_{jk} \partial R_{ik} \partial R_{kl})\). All these terms and \( E_2 \) (Eq. (7b)) vanish when one substitutes

\[ u_i = \omega \times R_i , \] (8)

in the relevant expressions. Thus they are explicitly rotationally invariant. The only exception is the scalar term \( E_3 \) (Eq. (7a)).

The three terms \( E_1 \) and \( E_2 \) and \( E_3 \) (Eqs. (6) and (7)) represent the effect of changing the two particle separation \((R_{ij})\)

\[ \delta r_{ij} = \frac{R_{ij} u_{ij}}{R_{ij}} + \frac{1}{2} (u_{ij})^2 \frac{R_{ij}}{R_{ij}} - \frac{1}{2} \frac{(R_{ij} u_{ij})^2}{R_{ij}^3} \] (9)
on the elastic energy, to first order in $\delta r_{ij}$. Thus these terms only show up when $\partial E/\partial R_{ij} \neq 0$. Mechanical equilibrium for the initial configuration ($R_i$) requires

$$\frac{\partial E}{\partial R_i} = \sum_j \frac{\partial E}{\partial R_{ij}} \frac{R_j}{R_{ij}} \equiv 0 .$$ (10)

For this one does not require that all pair forces ($\partial E/\partial R_{ij}$) vanish. One can have an equilibrium situation for which the (finite) pairwise forces balance [16]. At the microscopic level this is probably the rule, rather than the exception for realistic models. To get a contribution to the macroscopic, long range, elastic properties one requires internal stresses. We perform the gradient expansion explicitly

$$u_{ij} \sim (R_{ij} \cdot \nabla)_i u(r)$$ (11)

where $u(r)$ is the deformation field. Thus, from equation (6) the local stress tensor at $R_i$ is

$$\mathbf{S}_{ij} = \frac{1}{2} \sum_j \frac{\partial E}{\partial R_{ij}} \frac{R_j \cdot R_{ij}}{R_{ij}} .$$ (12)

Equation (6) ($E_i$) gives the coupling of the stress to the linear part of the strain ($\partial \mu_i u_i + \partial \mu_j u_j$, Eq. (3)) and $E_s$ (Eq. (7a)) the coupling of the same stress to the quadratic term in equation (3)

$$E_s \sim \frac{1}{4} \sum_{i,j} S^{\alpha \beta} \left( \sum \partial \mu_i u_i \partial \mu_j u_j \right)_i .$$ (13)

Both terms are necessary to describe the linear term in the stress strain expansion of the elastic energy.

It is of course well known that materials can have internal stresses without violating the mechanical equilibrium condition (Eq. (10)). Stresses can arise because of external forces or from dislocations and point defects. It can be seen from equation (12) that this implies that at least some $\partial E/\partial R_{ij} \neq 0$.

The scalar term becomes particularly simple when the stresses are hydrostatic

$$S^{\alpha \beta} = p \delta(\alpha, \beta)$$ (14)

$$E_s = \frac{p}{4} \sum_{\alpha, \beta} (\partial \mu_i u_i)^2 .$$ (15)

One notes that $E_s$ acts as a proper elastic energy and implies a resistance to shear even when the stresses are purely hydrostatic (Eq. (15)). Thus even with nearest neighbour central forces a strained simple cubic lattice and a strained tree will resist shear.

The term multiplying the stress in equation (13) ($\sum_j \partial \mu_i u_i \partial \mu_j$) is a component of the strain and therefore a symmetric tensor. Nevertheless it cannot be expressed in terms of the linear strains only. One has to invoke rotations ($\partial \mu_p \mu_q - \partial \mu_p \mu_q$). We have thus shown that a material with internal stresses has a scalar term in the harmonic expansion (Eq. (2)) which is not explicitly rotationally invariant and leads to a term in the elastic energy which cannot be expressed in terms of the linear symmetric strains (Eq. (13)).

It is of some interest to see explicitly how a scalar elastic term ($E_s$) can be consistent with overall rotational invariance. We have derived $E_s$ from a rotationally invariant elastic energy (Eq. (4)). Thus rotational invariance cannot be violated in the expansion. This involves a cancellation. The coefficients of $\partial E/\partial R_{ij}$ in equations (6) and (7) together represent the change in the separation between $i$ and $j$ ($\partial r_{ij}$, Eq. (9)). For a rigid rotation $R_{ij}$ is not changed ($\partial r_{ij} \equiv 0$) and the sum of the coefficients must vanish. To see this one has to go beyond equation (8)

$$u_{ij} = \omega \times R_{ij} + \frac{1}{2} \omega \times (\omega \times R_{ij}) .$$ (8a)

The second term on the right hand side of equation (8a) gives a contribution to $E$ (Eq. (6)) which exactly cancels the corresponding term in $E_s$ (Eq. (7a)) to order $\omega^2$. The inclusion of scalar elasticity ($E_s$) is therefore essential to assure overall rotational invariance to this order. Since the $R_{ij}$ do not change the contributions of a rigid rotation must vanish for each pair ($ij$) separately.

It is possible, but somewhat confusing, to use the equilibrium condition (Eq. (10)) in evaluating the contribution of a rigid rotation (Eq. (8a)) to $E_i$ (Eq. (6)). This reduces the sum to a surface term which is cancelled by $E_s$ (to order $\omega^2$). When there are no external forces $E_i = 0$ and the positive ($\partial E/\partial R > 0$) and negative ($\partial E/\partial R < 0$) terms in $E_s$ must cancel exactly for this deformation. This does not mean that the individual terms in $E_s$ vanish. It simply reflects the fact that in this case there are no stresses acting on the surface. We note that the coefficients of $(u_{ij})^2$ can be negative. For a mechanically stable configuration of stresses the overall mechanical eigenvalue spectrum is of course always essentially positive. For the hydrostatic case (Eq. (15)) the stresses have a unique sign. This is the case we shall mainly consider below.

To avoid confusion we finally note that an expression for the harmonic elastic energy which is not rotationally invariant and contains a scalar term, is occasionally used in the literature (the Born model). An example is reference [6]. As explained there this is meant to be a model to simulate the rotationally invariant energy. Explicit expressions which contain non-central forces and are rotationally invariant become complicated and inconvenient. Thus the physical origin and meaning is very different from that of equation (7). We also note that the Born model is not adequate when scaling is important. The rotationally invariant and the scalar contributions scale differently [4-7] and one cannot use one as an approximation for the other.
3. Hydrostatic stresses close to the liquid solid transition.

Quite generally one expects amorphous materials to have a high density of defects (say dislocations) and therefore relatively large internal stresses. This certainly means that the isotropic elastic energy \( E \) will always be larger than in proper crystalline materials. This is however not automatically sufficient to assure that the isotropic elastic energy is significant. It is in fact fairly easy to see that one should expect these terms to be small. We can write

\[
\frac{1}{R} \frac{dE}{dR} \sim \frac{d^2E}{dR^2} \frac{\delta R}{R}
\]

where \( \delta R \) is the deviation of the bond length from its equilibrium value \( R \). \( \delta R/R \) is, in essence, the internal strain and is expected to be small for realistic situations in tightly bound materials. The coefficient of the scalar terms \( \left( \frac{d^2E}{dR^2} \frac{\delta R}{R} \right) \) is then small compared to the central force constant \( (\frac{\partial^2 E}{\partial R^2}) \). This is the justification for neglecting \( E \) (the Cauchy approximation [2]). This certainly applies in many situations. Consider however an amorphous material close to threshold. One has a small shear modulus. The bulk modulus is taken over from the liquid and therefore relatively large. The crucial point is that, as a solid, the material is weakly bound. Shear rigidity is due to a tenuous, weak bonding structure superimposed on a liquid. An elastic energy expansion of the form of equation (1) (or Eq. (4)) only makes sense for a rigid structure. One cannot describe a liquid in this way. The elastic energy \( (E, \text{Eq. (4)}) \) is therefore defined, at least implicitly, on a « phantom » net responsible for the rigidity of the shape. The pressure and compressibility are due to all the other interactions which are still liquid like. While this description is heuristic, and therefore tentative, we find it hard to imagine a situation where it would not apply qualitatively, close to threshold. The implications are quite serious. If the bonded net does not include the interactions responsible for maintaining the density, the pressure must be taken as externally imposed. This induces a hydrostatic stress in the bonded net. Thus the bonds relevant to the harmonic expansion (Eq. (1)) should be considered as stretched. The force exerted by the stressed net on each side of any surface compensates for the missing force of the liquid pressure. Moreover, for a tenuously bound net the effective strains \( (\delta R/R) \) can be very large.

A typical example is rubber elasticity. A rubber is produced by crosslinking a polymer melt. Without crosslinks one has a liquid [17] which obviously maintains its own density. Rubber elasticity [3] is described by the crosslinked net with an additional constraint on the density of crosslinks. The forces between crosslinking points describe the stretching of the chains connecting them but these chains are already stretched by the density constraint. Also, for a random chain one can easily obtain large strains. Typically models of this type, with purely central spring constants between crosslinks, give a purely scalar elastic energy [3]. The usual, rotationally invariant, terms in the elastic energy do not even show up. To find these terms one would have to look for triangular plaquets and tetrahedra to obtain shear rigidity from the central forces or, alternatively, consider the propagation of angular forces across a long polymer coil [7]. Both effects are presumably very weak in this situation.

The physical situation is similar for gels. Shear rigidity is determined by the backbone of the infinite cluster while the (osmotic) compressibility is dominated by the overall monomer density which includes finite clusters and dead ends. Again, the « bonds » are long and tortuous and should be regarded as stretched.

Discussions of rubber or gel elasticity usually use a different terminology but this does not change the basic physics. Unbonded interactions and entropy effects cannot be incorporated directly into an elastic net picture. They do show up here through the imposed stress. This is, we believe, their main effect. We believe that similar considerations probably apply to most glasses close to the glass transition.

4. Scaling behaviour.

We have shown that scalar contributions to the elastic energy are present and expected to be fairly large. The purpose of this section is to argue that they are expected to be dominant in sufficiently tenuous bonding structures. The essence of our argument is the claim that microscopic scalar forces are more effective than rotationally invariant forces in producing macroscopic shear rigidity. We support this by some heuristic arguments and by the scaling results of references [4-7]. For self similar fractal geometries it shows up in that the rotationally invariant force constants are found to be irrelevant under scale transformations compared to the scalar terms. Self similarity is however not really essential for the argument though it obviously facilitates the calculation.

Quite generally equation (1) defines a set of linear equations on a net which is locally inhomogeneous. To obtain the large scale response one must either use decimation (i.e. retain only a dilute subset of the vertices) or some other coarse graining prescription. The essence of our claim here is that scalar contributions are more effective in creating large scale rigidity and therefore tend to become dominant under these transformations. Consider first some qualitative considerations.

For scalar elastic coupling (Eq. (2)) the equations for each Cartesian component of the \( e_i \) are separate and identical to those for a scalar quantity e.g. to the electric circuit equations with conductances \( (\sum_j) \) proportional to the force constants \( (A_{ij}, \text{Eq. (2)}) \). Thus the similarity between the equations governing elastic res-
response and the Kirchoff equations used by de Gennes [9] becomes an identity. The elastic constant is always identically proportional to the conductances at any level of scaling, renormalization or coarse graining. This is not true for the rotationally invariant couplings. The scaling indices are different and moreover depend on the detailed layout of the net.

Purely central forces require triangular loops (and tetrahedra) to give any sort of shear resistance. They are therefore extremely ineffective on a large tenuous structure. They can e.g. never give any rigidity to a tree. One therefore has to rely on explicit short range angular forces. Again it is hard to see how this can be very effective. For a tree it would imply that shear rigidity is completely dependent on microscopic angular forces at some central linkage point. This is very ineffective. Thus, as for central forces one is again dependent in practice on a sufficiently interconnected system of loops. On the other hand scalar elasticity will give shear rigidity to any connected structure, with conductivity indices.

These qualitative considerations are supported by the, admittedly rather scarce, scaling calculations of which we are aware.

Bergman [4] and Bergman and Kantor [5] have investigated the elastic properties of the Sierpinski gasket. There are three microscopic nearest neighbour force constants. One has a central force spring constant \(K_c\), an explicit angular force \(K_a\) and an isotropic constant \(K_s\). For the rotationally invariant part \((K_c; K_a)\) Bergman and Kantor [5] find the scaling relationships:

\[
K_c' = \frac{1}{2} K_c, \quad K_a' = \frac{1}{4} K_a.
\]

The isotropic constant scales like the conductivity and therefore [18]

\[
K_s' = 3/5 K_s.
\]

Thus for the mechanical properties of a large gasket \(K_s\) is always irrelevant and \(K_c\) is also irrelevant when there is a finite scalar contribution. We discuss these results in some more detail in the Appendix.

A similar situation is of course well known for the conformation of (linear or branched) polymers [19] where both the short range angular forces and the n.n. mechanical force constants play no rôle in determining the mechanical properties of a long chain. From the ends such a chain looks like a flexible chain of rigid beads.

Feng and Sen [6] investigated the appearance of a shear modulus above the percolation threshold for a nearest neighbour central force model on the triangular and fcc lattices. Their numerical results give an index \(\mu_k\) for the shear modulus

\[
K \propto (\Delta P)^{\mu_k}
\]

which is about twice that of the conductivity \((\mu)\)

\[
\mu_k \sim 2.5 \quad d = 2; \quad \mu_k \sim 4 \quad d = 3.
\]

Using standard scaling arguments [11] this means that central forces are irrelevant in the short range fractal regime \((R < \xi)\):

\[
K_c(R)/K_c(R_c) \propto R^{(\mu_k - \mu_c)/\nu}.
\]

Feng and Sen [6] also find that a shear modulus does not show up at the (conductivity) percolation threshold \((p_c)\) but only above a considerably higher shear transition \((p'_c > p_c)\). This result would imply that for \(p_c < p < p'_c\) the infinite cluster does not have enough loops for shear rigidity with purely central forces. It obviously rules out a pure central force model for a phenomenon like gellation. A model for which the divergence of the viscosity (at \(p_c\)) does not coincide with the threshold for the appearance of a shear modulus \((p'_c)\) is obviously unphysical.

As noted by Feng and Sen [6] this anomaly disappears when one introduces angular forces. They try to discuss this in terms of the Born model (i.e. introducing a small scalar term). This leads to conductivity indices but certainly does not show how really rotationally invariant angular forces scale. Kantor and Webman [7] have more recently tried to estimate the effect of properly rotationally invariant angular forces. They find an index \(\mu_4\) which is again considerably larger than the conductivity index.

Thus for the rotationally invariant forces are irrelevant if scalar elasticity is present. This is of course not conclusive. It does however strongly suggest that the rotationally invariant microscopic contributions will tend to be ineffective in producing large scale shear rigidity. One would of course like to see some additional, and more careful investigations of this question.

5. Discussion.

The essence of our argument was to show that a rigid net with rotationally invariant elastic force constants is not the only approach for describing the elasticity of tenuous objects and amorphous materials. Revising some very old results [2] we have shown that there are also scalar contributions to the elastic energy in stressed systems. These contributions are always negligibly small in crystalline materials. We have shown that this is not necessarily true when shear rigidity is due to tenuous bonding structures — as in many amorphous situations. It is also suggested by existing scaling results that stress induced scalar elasticity is more effective in establishing macroscopic shear rigidity. This was the purpose of this paper. The relative importance of the two microscopic contributions obviously depends on the specific physical situation and one cannot make any universal statements. The scaling results suggest that close enough to criticality (e.g. the gel point)
Scalar elasticity will always dominate. The width of the region for which this holds still depends on the ratio of the microscopic contributions. It can be seen from the Appendix that this can be quite complex. Assume for example that \((K_s/\kappa_c)\) is initially small. On a gasket one would then initially have

\[(K_s/K_c)_0 \ll 1 \quad (22)\]

from equations (A.9, A.10) crossing over to

\[(K_s/K_c)_0 \sim (5/4)^n \gg 1 \quad (23)\]

from (A.12) and (A.13) where the crossover stage \((n_c)\) is determined from \((K_s/K_c)n_c = 1\). One notes that this differs from the naive independent scaling ansatz

\[(K_s/K_c)_0 \sim (6/5)^n (K_s/K_c)_0 \quad (24)\]

one would obtain using equations (17) and (18).

There are some possibilities for a direct experimental check.

The most obvious is the de Gennes conjecture [9] — i.e. the scaling index of the shear modulus above the gel point. There is a considerable amount of experimental results with a fairly large scatter of measured indices. These may however be due either to the importance of rotationally invariant contributions or to the inadequacy of the percolation model for the structure. A direct check would be possible if one could measure the conductivity directly for the same gel. This may be possible in some special cases [20].

Purely scalar elasticity implies a relationship between the shear modulus and the compressibility analogous to the Cauchy relations for purely central forces. Its most striking manifestation would be the equality of the longitudinal and transverse velocities of sound. This would certainly be observable. For the models we have discussed there are however additional contributions to the compressibility which are not due to the net.

Acknowledgments.

I would like to thank D. Bergman for insisting and finally convincing me of the importance of rotational invariance in this problem. I am also grateful to P. G. de Gennes, E. Guyon, G. Grest, P. Pincus and I. Webman for very helpful discussions and to the ESPCI (Paris) and the Laboratoire de Physique des Solides (Orsay) where this work originated and to Exxon Research and Engineering (Clinton) where it was concluded, for their hospitality and support.

Appendix.

Scaling relations on the Sierpinski gasket. — Bergman and Kantor [5] use a somewhat unusual representation for the Sierpinski gasket which is related to the standard picture by a star triangle transformation on the (smallest) elementary triangles. They also do not include a scalar coupling term in their calculation. It is straightforward to obtain iterative expressions for the complete spectrum using the decimation technique of references [13, 18 and 21]. The algebra is however somewhat cumbersome and in itself not very illuminating. We only present the scaling behaviour of the zero frequency elastic coupling constants. We describe an elementary triangle by three force constants. A central force nearest neighbour spring constant \((K_c)\) an explicit angular force which only shows up in the bending modes of the triangle \((K_a)\) and a scalar force constant \((K_s)\). This means that for the symmetric stretch mode of the triangle \((A_1)\) one has

\[
\hat{K} | A_1 > = (2K_c + K_s) | A_1 > \quad (A.1)
\]

for the two degenerate bending modes \((E_i)\)

\[
\hat{K} | E_i > = (K_c + K_a + K_s) | E_i > \quad (A.2)
\]

and for a rigid rotation \((A_2)\)

\[
\hat{K} | A_2 > = K_s | A_2 > . \quad (A.3)
\]

In addition there are the two (zero force) translational modes. In the above \(\hat{K}\) is the force constant matrix for the whole triangle and we have used standard notation for the irreducible representations (of the point group \(D_3\)) to describe the modes. One can now use this to calculate the effective force constants seen at the external vertices of a (stage 1) gasket formed by joining three such triangles. One finds:

\[
K_c' = \frac{12 K_c (K_c + K_a + K_s)^2}{(4 K_c + 4 K_a + 5 K_s)(6 K_c + 4 K_a + 5 K_s)} \quad (A.4)
\]

\[
K_a' = \frac{3 K_a (K_c + K_a + K_s)}{4 K_c + 4 K_a + 5 K_s} \quad (A.5)
\]

\[
(K_c + K_a + K_s) = 3(K_c + K_a + K_s) \quad (A.6)
\]

\[
\frac{K_c^2 + K_a K_s + 4 K_c K_a + K_a K_s + 2 K_s^2}{(6 K_c^2 + 7 K_c K_a + K_a^2 + 20 K_c K_s + 7 K_a K_s + 10 K_s^2)}
\]

\[
\frac{K_s^2}{(6 K_c^2 + 7 K_c K_a + K_a^2 + 20 K_c K_s + 7 K_a K_s + 10 K_s^2)}.
\]
There are two obvious solutions. The rotationally invariant case \( (K_g = 0) \) gives from (A.4) and (A.6)
\[
K'_c = \frac{3 K_c (K_c + K_a)}{6 K_c + 4 K_a} \quad (A.7)
\]
\[
K'_a = \frac{3 K_a}{6 K_c + K_a} K'_c. \quad (A.8)
\]

So that \( K_a \) always decreases more rapidly than the central force \( K_c \). Asymptotically \( (K_a \ll K_c) \) one has the result of reference [5] :
\[
K'_c = 1/2 K_c; \quad K'_a = 1/4 K_a. \quad (A.9)
\]

It is easy to check that \( K_a \) is unstable. Introducing a small \( K_a (K_c \gg K_a; K_a) \) one finds from equation (A.5)
\[
K'_c = 3/4 K_c; \quad \frac{K_a}{K_c} = 3/2 \left( \frac{K_a}{K_c} \right) \quad (A.10)
\]
so that the ratio \( K_a/K_c \) increases. The other solution is scalar
\[
K'_c \gg K_c; \quad K_a \quad (A.11)
\]
(A.5) gives the conductivity result
\[
K'_c = 3/5 K_c \quad (A.12)
\]
and from (A.4) and (A.6)
\[
K'_c = (12/25) K_c; \quad K'_a = (9/25) K_a \quad (A.13)
\]
to leading order in \( K_a/K_c \) and \( K_a/K_c \).

References

[8] The emphasis in references [4-7] is on showing that the rotationally invariant terms do not scale like the conductivity. This point is therefore not discussed there.